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de Jongh, D.; Zhao, Z.

DOI
10.1007/978-3-662-46906-4_11
Publication date
2015
Document Version
Final published version
Published in
Logic, Language, and Computation

Positive Formulas in Intuitionistic and Minimal Logic

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Abstract. In this article we investigate the positive, i.e. $\neg, \bot$-free formulas of intuitionistic propositional and predicate logic, IPC and IQC, and minimal logic, MPC and MQC. For each formula φ of IQC we define the positive formula φ^+ that represents the positive content of φ. The formulas φ and φ^+ exhibit the same behavior on top models, models with a largest world that makes all atomic sentences true. We characterize the positive formulas of IPC and IQC as the formulas that are immune to the operation of turning a model into a top model. With the +-operation on formulas we show, using the uniform interpolation theorem for IPC, that both the positive fragment of IPC and MPC respect a revised version of uniform interpolation. In propositional logic the well-known theorem that KC is conservative over the positive fragment of IPC is shown to generalize to many logics with positive axioms. In first-order logic, we show that IQC + DNS (double negation shift) + KC is conservative over the positive fragment of IQC and similar results as for IPC.

Keywords: Intuitionistic logic · Minimal logic · Jankov's logic · Intermediate logics · Positive formulas · Interpolation · Conservativity

1 Introduction

In this paper we discuss the formulas in intuitionistic logic containing no negation or $\bot$. For propositional logic IPC this is the fragment $[\wedge, \vee, \rightarrow]$. Smaller fragments not containing both $\vee$ and $\rightarrow$ have been extensively studied. By Diego's theorem [4] they are locally finite, i.e. they do contain only finitely many equivalence classes of formulas in a fixed finite number of variables. For a discussion of the history of these studies see [15]. The fragment $[\wedge, \vee, \rightarrow]$, which we call the positive fragment, does not have this property. It has been little studied as a fragment. Its interest is to start with that it has a very close relationship to minimal logic, the logic resulting when the ex falso principle is deleted from intuitionistic logic. In fact, one can see minimal propositional logic as this fragment with one designated propositional variable (the contradiction), and this is not different in first order logic. The ex falso principle has been criticized from the start, for example by Kolmogorov [13] in the earliest partial formalization of intuitionistic

M. Aher et al. (Eds.): TbiLLC 2013, LNCS 8984, pp. 175–189, 2015.
DOI: 10.1007/978-3-662-46906-4_11

logic. Heyting, however, did accept the principle in his basic papers [10], and from then on it has been accepted as a principle of intuitionistic logic. After this, Johansson, not supporting the ex falso principle, introduced minimal logic in [12]. Some proponents of intutionistic mathematics (Griss [9], van Danzig) favored the idea of dropping negation altogether: negationless mathematics, but they had few followers. Brouwer himself thought formulas with negation to be indispensable in intuitionistic mathematics [1].

It is worth mentioning that in the pure arithmetic (of natural numbers), formalized in Heyting Arithmetic HA it makes no difference whether one accepts the ex falso principle or introduces negation, since in HA from $0 = 1$ all arithmetic sentences are derivable without the use of either (see e.g. [16], Vol. I, Proposition 3.2, p. 126). In analysis this is still true as long as one has only equations between numerical terms as atomic formulas, but no longer so when one e.g. has set variables with undecidable atomic formulas $X(t)$. A final striking fact is that first order intuitionistic logic without $\bot$ can be proved to be complete for so-called Beth-models by constructive methods whereas this is not the case for full first order logic (see [16], Vol. II, p. 685, which uses a proof by H. Friedman in an unpublished manuscript). In any case, it is good to start with logic to see how the positive fragment fits into the full logic. For that purpose we define in this paper a $+$-operation on the formulas of intuitionistic logic which we claim represents their positive content. This operation turns out be very useful in studying various properties of positive formulas in the framework of the full logic.

Minimal propositional logic MPC and minimal predicate logic MQC are obtained from the positive fragment, i.e. the $\neg, \bot$-free fragment, of intuitionistic propositional logic IPC and intuitionistic predicate logic IQC by adding a weaker negation: $\neg\varphi$ is defined as $\varphi \to f$, where the special propositional variable f is interpreted as the contradiction. Therefore, the language of minimal logic is the $\neg, \bot$-free fragment of intuitionistic logic plus f. Variable f has no specific properties, the Hilbert type system for MQC is as IQC's but without $f \to \varphi$. An alternative formulation of minimal logic, in fact the original one, in a language containing $\neg$ instead of f can be given by adding to a Hilbert type axiom system for the positive fragment the axiom $(p \to q) \to ((p \to \neg q) \to \neg p)$ (see [12]).

For the semantics of minimal logic, f is interpreted as an ordinary propositional variable, so we get the semantics of the $[\vee, \wedge, \to]$-fragment of IPC (resp. the $[\vee, \wedge, \to, \forall, \exists]$-fragment of IQC), with an additional propositional variable f.

The content of this article is the following:

In Sect. 2 we recall the syntax and semantics of intuitionistic and minimal logic. In Sect. 3 we introduce the top-model property and the $+$-operation on formulas, and show that the top-model property characterizes the positive formulas of IPC and IQC. We then use this property in Sect. 4 to show that the positive fragment of IPC has a revised form of uniform interpolation and that this transfers to MPC. In Sect. 5 we discuss the behavior of positive formulas in some extensions of IPC and IQC, taking as a starting point the theorem that Jankov's Logic KC has the same positive fragment as IPC.

2 Syntax and Semantics of MPC

In this section we recall the syntax as well as the derivation systems of IPC, IQC, MPC and MQC, and their Kripke semantics. For more details, see [2] and [17].

2.1 Syntax

The propositional language $\mathcal{L}_{\mathsf{I}}(P)$ of IPC consists of a countable or finite set P of propositional variables p_0, p_1, p_2, ..., propositional constants $\bot, \top$ and binary connectives $\wedge, \vee, \rightarrow$. A first order language $\mathcal{L}_{\mathsf{I}}(Q)$ of IQC consists of a countable or finite set Q of predicate letters and individual constants[1], propositional constants $\bot, \top$, binary connectives $\wedge, \vee, \rightarrow$ and quantifiers $\forall$ and $\exists$. In both cases $\neg\varphi$ is defined as $\varphi \rightarrow \bot$, although in practice it is often convenient to view formulas as containing both $\neg$ and $\bot$. The *positive fragment* $\mathcal{L}_{\mathsf{I}}^{+}(P)$ of IPC consists of the formulas of $\mathcal{L}_{\mathsf{I}}(P)$ that do not contain $\neg$ or $\bot$, similarly for a language $\mathcal{L}_{\mathsf{I}}(Q)$.

The propositional language $\mathcal{L}_{\mathsf{M}}(P)$ of MPC (resp. first order language $\mathcal{L}_{\mathsf{M}}(Q)$ of MQC) consists of the formulas of the positive fragment to which the special propositional variable f is added. We may drop the indices I and M and write $\mathcal{L}(P)$ etc. if the distinction is irrelevant.

We take the axioms of IPC as in [2]. The axioms for MPC are the same except that $\bot \rightarrow \varphi$ is left out. So, derivations in MPC are the same as in IPC except that no $\bot$ or $\neg$ occurs, instead f may have occurrences. To add predicate-logical axioms to obtain IQC and MQC we use the approach of Enderton [5] to classical logic with universally quantified axioms and modus ponens as the only rule. In this paper we will both proof-theoretically and semantically be only interested in sentences.

For the discussion of uniform interpolation in Sect. 4 we introduce the following notation: For any formula φ and any sequence $\boldsymbol{p} = (p_1, \ldots, p_n)$ of propositional variables (here p_i can be f, but cannot be $\bot, \top$), $\varphi(\boldsymbol{p})$ is a formula with only propositional variables in $\boldsymbol{p}$.

2.2 Kripke Semantics

In this part we give the Kripke semantics of our systems.

Definition 1. *A* propositional Kripke frame *is a pair* $\mathfrak{F} = (W, R)$ *where* W *is a non-empty set and* R *is a partial order on it.*

A propositional Kripke model *is a triple* $\mathfrak{M} = (W, R, V)$ *where* (W, R) *is a Kripke frame and* V *is a* valuation $V : P \cup \{f\} \rightarrow \mathscr{P}(W)$ *(where* $\mathscr{P}(W)$ *is the powerset of* W*) such that for any* $q \in P \cup \{f\}$, $V(q)$ *is an* upset*: for any* $w, w' \in W$, $w \in V(q)$ *and* wRw' *imply* $w' \in V(q)$.

To be able to treat propositional and predicate logic uniformly we define first-order models in a similar way. For a language $\mathcal{L}(Q)$, we write At_Q or At for the set of atomic sentences.

[1] We do not consider identity and functional symbols, but our results will surely hold for the extension with such symbols.

Definition 2. *A* predicate Kripke frame *for a language* $\mathcal{L}(Q)$ *is a triple* $\mathfrak{F} = (W, R, \{D_w \mid w \in W\})$ *where* W *is a non-empty set,* R *is a partial order on* W*, and* $\{D_w \mid w \in W\}$ *a set of non-empty domains such that for any* $w, w' \in W$*,* wRw' *implies* $D_w \subseteq D_{w'}$*.*

A predicate Kripke model *for a language* $\mathcal{L}(Q)$ *is a quadruple* $\mathfrak{M} = (W, R, \{D_w \mid w \in W\}, V)$ *where* $(W, R, \{D_w \mid w \in W\})$ *is a Kripke frame and* V *is a* valuation $V : At \cup \{f\} \to \mathscr{P}(W)$ *such that for any* $Ad_1 \ldots d_k$ *in* At*,* $V(Ad_1 \ldots d_k) \subseteq \{w \in W \mid (d_1, \ldots, d_k) \in (D_w)^k\}$*, and* $w, w' \in W$*,* $w \in V(Ad_1 \ldots d_k)$ *and* wRw' *imply* $w' \in V(Ad_1 \ldots d_k)$*, similarly for* f*.*

For propositional formulas, the satisfaction relation is defined as usual with clauses for p, f, $\bot$, $\top$, $\vee, \wedge, \to$, where the semantics of f is the same as for the other propositional variables. For predicate logic only sentences will be evaluated with clauses for $\forall, \exists$ as e.g. in van Dalen [19]. In the first order case $w \models \varphi$ (and hence $w \not\models \varphi$) is only defined if the individual constants in φ are in D_w. If we define V on P or At and omit the clause for f, then we get the Kripke semantics of IPC or IQC; if we omit the clause for $\bot$, then we get the Kripke semantics of MPC or MQC. We use $\models_{\mathsf{I}}$ and $\models_{\mathsf{M}}$ to distinguish the satisfaction relation of IQC and MQC, and omit the index when it is not important or clear from the context.

For IQC, we have the following completeness theorem (see e.g. [2]):

Theorem 1 (Strong Completeness of IQC**)**
For any set of IQC*-sentences* Γ *and* φ*,* $\Gamma \vdash_{\mathsf{IQC}} \varphi$ *iff* $\Gamma \models_{\mathsf{I}} \varphi$*.*

By a standard Henkin type completeness proof, we have that MQC is strongly complete with respect to Kripke models, i.e. for any Γ and φ, $\Gamma \vdash_{\mathsf{MPC}} \varphi$ iff $\Gamma \models_{\mathsf{M}} \varphi$. The proof procedure is essentially the same as the proof for IQC with respect to Kripke frames, just leave out $\bot$ and the accompanying condition that the members of the model have to be consistent sets (which of course they are).

Theorem 2 (Strong Completeness of MQC**)**
For any MQC*-formulas* Γ *and* φ*,* $\Gamma \vdash_{\mathsf{MQC}} \varphi$ *iff* $\Gamma \models_{\mathsf{M}} \varphi$*.*

By a completeness-via-canonicity proof using adequate sets, we have the finite model property for IPC (again see [2]) and thereby for MPC:

Theorem 3 (Finite Model Property of MPC**)**
For any MPC*-formula* φ*, if* $\nvdash_{\mathsf{MPC}} \varphi$*, then there is a rooted finite Kripke model* $\mathfrak{M}$ *falsifying* φ*.*

By the completeness theorem for MQC and IQC, since the semantic behavior of MQC in the language $\mathcal{L}_{\mathsf{M}}(Q)$ is exactly the same as that of IQC in the language $\mathcal{L}_{\mathsf{I}}(Q \cup \{f\})$ without $\bot$ (i.e. the positive $[\vee, \wedge, \to, \top, \forall, \exists]$-fragment $\mathcal{L}_{\mathsf{I}}^{+}(Q \cup \{f\})$ of $\mathcal{L}_{\mathsf{I}}(Q \cup \{f\})$), we can regard MQC as the positive fragment of IQC, and we have the following lemma:

Lemma 1. *For any sentences* Γ *and* φ *in* $\mathcal{L}_{\mathsf{M}}(Q) = \mathcal{L}_{\mathsf{I}}^{+}(Q \cup (\{f\})$*,* $\Gamma \vdash_{\mathsf{MQC}} \varphi$ *iff* $\Gamma \vdash_{\mathsf{IQC}} \varphi$*.*

This allows us to write $\vdash \varphi$ if the index does not matter.

For intermediate logics we sometimes need descriptive frames.

Definition 3. *A* general frame *is a triple* $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$, *where* $\langle W, R\rangle$ *is a Kripke frame and* $\mathcal{P}$ *is a family of upward closed sets containing* $\emptyset$ *and closed under* $\cap$, $\cup$ *and the following operation* $\supset$*: for every* $X, Y \subseteq W$,

$$X \supset Y = \{x \in W \mid \forall y \in W(xRy \wedge y \in X \to y \in Y)\}$$

Elements of the set $\mathcal{P}$ *are called* admissible sets.

Definition 4. *A general frame* $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ *is called* refined *if for any* $x, y \in W$,

$$\forall X \in \mathcal{P}\,(x \in X \to y \in X) \Rightarrow xRy.$$

$\mathfrak{F}$ *is called* compact, *if for any family* $\mathcal{Z} \subseteq \mathcal{P} \cup \{W \setminus X \mid X \in \mathcal{P}\}$ *with the finite intersection property,* $\bigcap(\mathcal{Z}) \neq \emptyset$.

Definition 5. *A general frame* $\mathfrak{F}$ *is called a* descriptive frame *iff it is refined and compact.*

Intermediate propositional logics are complete with respect to descriptive frames (see [2]):

Theorem 4. *If* L *is an intermediate propositional logic, then, for all formulas* φ, $\vdash_L \varphi$ *iff* φ *is valid in all descriptive frames* $\mathfrak{F}$ *that satisfy* L.

3 The Top-Model Property

We give a characterization of the $\neg, \bot$-free formulas of IPC by means of the following property:

Definition 6 (Top-Model Property)

1. *A propositional or predicate Kripke model* $\mathfrak{M} = (W, R, V)$ *is a* top model *if it has a largest point* t, *the* top *of the model, in which all formulas in* P *or* At *are satisfied.*
2. *Any model* $\mathfrak{M} = (W, R, V)$ *can be turned into its top model* $\mathfrak{M}^+ = (W^+, R^+, V^+)$ *by adding a node* t *at the top of the model, connecting all worlds* w *to* t*, and making all atomic sentences true in* t*. In case of first order logic,* $D_t = \bigcup_{w \in W} D_w$.
3. *A formula* φ *has the* top-model property, *if for all Kripke models* $\mathfrak{M} = (W, R, V)$*, all* $w \in W$*,* $\mathfrak{M}, w \models \varphi$ *iff* $\mathfrak{M}^+, w \models \varphi$.

Analogously to 1,2 of the above definition we talk about *top frames*.

Lemma 2. *Let* t *be the top of any top model, and let* φ *be a positive formula without free variables. Then* $t \models \varphi$.

Proof. Trivial, by induction on the length of φ. □

For the top-model property we have the following theorem. It was first proved in [18,21] (see also [15]). We write $\varphi \sim \psi$ for $\vdash \varphi \leftrightarrow \psi$.

Theorem 5. *1. Every formula in $\mathcal{L}_{\mathsf{I}}^{+}(P)$, $\mathcal{L}_{\mathsf{I}}^{+}(Q)$, $\mathcal{L}_{\mathsf{M}}(P)$ and $\mathcal{L}_{\mathsf{M}}(Q)$ has the top-model property, and so has $\bot$.*

2. *For any formula φ in $\mathcal{L}_{\mathsf{I}}(P)$, there exists a procedure to produce a formula φ^{+} in $\mathcal{L}_{\mathsf{I}}^{+}(P)$ or $\varphi^{+} = \bot$ such that for any top model $\mathfrak{M}$ and any node w in $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi \leftrightarrow \varphi^{+}$.*
3. *For any formula φ in $\mathcal{L}_{\mathsf{I}}(Q)$, there exists a procedure to produce a formula φ^{+} in $\mathcal{L}_{\mathsf{I}}^{+}(Q)$ or $\varphi^{+} = \bot$ such that for any top model $\mathfrak{M}$ and any node w in $\mathfrak{M}$, we have $\mathfrak{M}, w \models \varphi \leftrightarrow \varphi^{+}$.*
4. *For any set of formulas Γ in $\mathcal{L}_{\mathsf{I}}(P)$ or $\mathcal{L}_{\mathsf{I}}(Q)$, any top model $\mathfrak{M}$ and any node w in $\mathfrak{M}$, we have $\mathfrak{M}, w \models \Gamma$ iff $\mathfrak{M}, w \models \Gamma^{+}$, where $\Gamma^{+} = \{\gamma^{+} \mid \gamma \in \Gamma\}$.*

Proof. 1. By induction on the length of the formula φ. We just give the inductive steps for $\to$ and $\forall$. Let t denote the top element of $\mathfrak{M}$.

- $\mathfrak{M}, w \models \psi \to \chi \iff$ in all w' such that wRw', if $\mathfrak{M}, w' \models \psi$ then $\mathfrak{M}, w' \models \chi$ $\iff_{IH}$ in all $w' \in W \backslash \{t\}$ such that wRw', if $\mathfrak{M}^{+}, w' \models \psi$ then $\mathfrak{M}^{+}, w' \models \chi$ [Now note that since φ is positive, and χ is a subformula of φ, it must be the case that χ is positive. Therefore, by Lemma 2, $t \models \chi$] $\iff$ in all $w' \in W$ such that wRw', if $\mathfrak{M}^{+}, w' \models \psi$ then $\mathfrak{M}^{+}, w' \models \chi \iff \mathfrak{M}^{+}, w \models \psi \to \chi$.
- $\mathfrak{M}, w \models \forall z \psi(z) \iff$ if wRw' then $\mathfrak{M}, w' \models \psi(d)$ for all $d \in D_{w'}$ [Now note that by Lemma 2, $t \models \psi(d)$ for all $d \in D_t$.] $\iff_{IH}$ if wRw' then $\mathfrak{M}^{+}, w' \models \psi(d)$ for all $d \in D_{w'} \iff \mathfrak{M}^{+}, w \models \forall z \psi(z)$.

2 and 3. We obtain φ^{+} from φ in stages. That is, $\varphi = \varphi^{0} \dashrightarrow \varphi^{1} \dashrightarrow \cdots \dashrightarrow \varphi^{n} = \varphi^{+}$. Each stage m starts off with φ^{m} and produces φ^{m+1}. The procedure starts at $n = 0$.

Stage $2n$. Remove all $\top$ and $\bot$ using the following equivalences:

Remove$\bot$	Remove$\top$
$\bot \wedge \varphi \sim \varphi \wedge \bot \sim \bot$	$\top \wedge \varphi \sim \varphi \wedge \top \sim \varphi$
$\bot \vee \varphi \sim \varphi \vee \bot \sim \varphi$	$\top \vee \varphi \sim \varphi \vee \top \sim \top$
$\bot \to \varphi \sim \top$	$\top \to \varphi \sim \varphi$
$\varphi \to \bot \sim \neg\varphi$	$\varphi \to \top \sim \top$
$\neg\bot \sim \top$	$\neg\top \sim \bot$

This procedure may produce a formula φ^{2n+1} containing neither $\top$ nor $\bot$. However, it is also possible that it ends by producing $\top$ or $\bot$. In the latter two cases, the theorem is trivial, since in any model $\mathfrak{M}$ and any world w, $\mathfrak{M}, w \models \top$ and $\mathfrak{M}, w \not\models \bot$, and therefore $\iff$ holds. So, in the remainder of this proof we assume that not $\varphi^{2n+1} = \bot$ and not $\varphi^{2n+1} = \top$. Note the special feature of the procedure: a new negation may be produced.

Stage $2n + 1$**.** Consider the first $\neg$ in φ^{2n+1} such that $\neg\psi$ is a subformula of φ^{2n+1} and ψ is positive: that is, ψ does not contain $\neg, \bot$. This can be done since all $\bot$ were removed in the previous stage. Replace $\neg\psi$ by $\bot$. This results in $\varphi^{2n+2} = \varphi^{2n+1}[\bot/\neg\psi]$, which contains less symbols than φ^{2n+1}.

The even stages use logical equivalences, so by definition $\mathfrak{M}^+, w \models \varphi^{2n} \iff \mathfrak{M}^+, w \models \varphi^{2n+1}$ (valuations on $\mathfrak{M}^+$ are preserved), since for equivalent formulas this holds for any model.

Next, it has to be shown that also the odd stages preserve valuations on $\mathfrak{M}^+$, that is: $\mathfrak{M}^+, w \models \varphi^{2n+1} \iff \mathfrak{M}^+, w \models \varphi^{2n+2} = \varphi^{2n+1}[\bot/\neg\psi]$ for all $n \in \mathbb{N}$. Let $\psi = \psi(x_1, \ldots, x_k)$ and $d_1, \ldots, d_k \in D_w$. Consider the valuation of $\psi(d_1, \ldots, d_k)$ in top world t. We have chosen ψ positive. Therefore, by Lemma 2, $t \models \psi(d_1, \ldots, d_k)$. By definition of $\mathfrak{M}^+$, wRt for all $w \in W$, so for all $w \in W$, there is a w' such that wRw' and $w' \models \psi(d_1, \ldots, d_k)$ (namely $w' = t$). Therefore, for all $w \in W$, it must be the case that $\mathfrak{M}^+, w \not\models \neg\psi(d_1, \ldots, d_k)$. It can be concluded by a trivial induction that φ^{2n+1} is equivalent to $\varphi^{2n+1}[\bot/\neg\psi]$.

The described procedure will come to an end, since all steps reduce the number of symbols in the formula. Therefore, there is a final stage, say stage m, which produces a φ^{m+1} that no longer contains $\bot$ or $\neg$. Now define $\varphi^{m+1} = \varphi^+$. Since both the odd and even stages preserve valuations on $\mathfrak{M}^+$, we know that $\mathfrak{M}^+, w \models \varphi^{n-1} \iff \mathfrak{M}^+, w \models \varphi^n$ for all n. By induction, this implies that $\mathfrak{M}^+, w \models \varphi \iff \mathfrak{M}^+, w \models \varphi^+$.

4 follows immediately from 2 and 3. □

And this theorem leads to the following characterization.

Theorem 6. *A formula φ of* IPC *or* IQC *has the top-model property iff φ is equivalent to a $\neg, \bot$-free formula (in fact to φ^+) or to $\bot$.*

Proof. The direction from right to left is Theorem 5.1, so let us prove the other direction and assume that φ has the top-model property, but is not equivalent to φ^+. Then there is a model $\mathfrak{M}$ with a world w so that φ and φ^+ have different truth values in $\mathfrak{M}, w$. Then, because both have the top-model property, φ and φ^+ have different truth values in $\mathfrak{M}^+, w$ as well. But that contradicts the fact given by Theorem 5 that φ and φ^+ behave identically on top models. □

Theorem 7. *1. If $\vdash_{\mathsf{IPC}} \varphi$, then $\vdash_{\mathsf{IPC}} \varphi^+$. If $\vdash_{\mathsf{IQC}} \varphi$, then $\vdash_{\mathsf{IQC}} \varphi^+$.*
2. *Not always $\vdash_{\mathsf{IPC}} \varphi \to \varphi^+$ and not always $\vdash_{\mathsf{IPC}} \varphi^+ \to \varphi$.*
3. *If $\varphi(\psi_1, \ldots, \psi_k)$ arises from the simultaneous substitution of $\psi_1, \ldots, \psi_k$ for $p_1, \ldots, p_k$ in $\varphi(p_1, \ldots, p_k)$, then $(\varphi(\psi_1, \ldots, \psi_k))^+ = (\varphi(\psi_1^+, \ldots, \psi_k^+))^+$.*
4. *If $\vdash_{\mathsf{IPC}} \varphi \to \psi$, then $\vdash_{\mathsf{IPC}} \varphi^+ \to \psi^+$. If $\vdash_{\mathsf{IQC}} \varphi \to \psi$, then $\vdash_{\mathsf{IQC}} \varphi^+ \to \psi^+$.*
5. *φ^+ is unique up to provable equivalence.*
6. *If $\vdash_{\mathsf{IPC}} \varphi \to \psi$ and ψ is positive, then $\vdash_{\mathsf{IPC}} \varphi^+ \to \psi$. If $\vdash_{\mathsf{IQC}} \varphi \to \psi$ and ψ is positive, then $\vdash_{\mathsf{IQC}} \varphi^+ \to \psi$. If $\vdash_{\mathsf{IPC}} \psi \to \varphi$ and ψ is positive, then $\vdash_{\mathsf{IPC}} \psi \to \varphi^+$. If $\vdash_{\mathsf{IQC}} \psi \to \varphi$ and ψ is positive, then $\vdash_{\mathsf{IQC}} \psi \to \varphi^+$.*
7. *If $\Gamma \vdash_{\mathsf{IPC}} \varphi$ and φ is positive, then $\Gamma^+ \vdash_{\mathsf{IPC}} \varphi$, where $\Gamma^+ = \{\gamma^+ \mid \gamma \in \Gamma\}$.*

Proof. 1. Assume not $\vdash_{\mathsf{IPC}} \varphi^+$. Then $\mathfrak{M}, w$ exist such that $\mathfrak{M}, w \not\models \varphi^+$. By Theorem 5.1 also $\mathfrak{M}^+, w \not\models \varphi^+$. But then by Theorem 5.2, $\mathfrak{M}^+, w \not\models \varphi$, so not $\vdash_{\mathsf{IPC}} \varphi$. Same for IQC.

2. For $\varphi = p \vee \neg p$, $\varphi^+ = p$, so $\nvdash_{\mathsf{IPC}} \varphi \to \varphi^+$. For $\varphi = \neg\neg p$, $\varphi^+ = \top$, so $\nvdash_{\mathsf{IPC}} \varphi^+ \to \varphi$.
3. By the fact that the construction of the +-formula in Theorem 5 is inside-out. We can construct $(\varphi(\psi_1, \ldots, \psi_k))^+$ by first applying the +-operation to the formulas $\psi_1, \ldots, \psi_k$ in $\varphi(\psi_1, \ldots, \psi_k)$ to obtain $\varphi(\psi_1^+, \ldots, \psi_k^+)$, and then continue to work on the remainder to obtain $(\varphi(\psi_1^+, \ldots, \psi_k^+))^+$.
4. Suppose $\vdash_{\mathsf{IPC}} \varphi \to \psi$ and $\nvdash_{\mathsf{IPC}} \varphi^+ \to \psi^+$, then by the completeness of IPC, there is a rooted model $\mathfrak{M}$ with root w such that $\mathfrak{M}, w \vDash \varphi^+$ and $\mathfrak{M}, w \nvDash \psi^+$. By Theorem 5.1, $\mathfrak{M}^+, w \vDash \varphi^+$ and $\mathfrak{M}^+, w \nvDash \psi^+$. By Theorem 5.2, $\mathfrak{M}^+, w \vDash \varphi$ and $\mathfrak{M}^+, w \nvDash \psi$, a contradiction to $\vdash_{\mathsf{IPC}} \varphi \to \psi$. For IQC, the proof is similar.
5. Immediate from 4.
6. From 4.
7. Similar to 4, where the strong completeness is used. □

Items 5 and 6 give us the right to say that φ^+ represents the positive content of φ. Item 3 will be used to obtain results on positive formulas proved by intermediate logics in Sect. 5.

We finally sketch another approach to get to Theorem 7.1 the advantage of which is that it can be transformed into a full proof-theoretic proof. We do not fully execute this here because of lack of space. The first step is the next theorem for which we provide here only a semantic proof.

Theorem 8. *If $\varphi(p_1, \ldots, p_k)$ is positive and $\vdash_{\mathsf{IPC}} \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \varphi$, then $\vdash_{\mathsf{IPC}} \varphi$.*

Proof. Asume, φ positive, $\nvdash_{\mathsf{IPC}} \varphi$. Then for some model $\mathfrak{M}$ with root r, $\mathfrak{M}, r \not\models \varphi$. Hence, by Theorem 5.1, $\mathfrak{M}^+, r \not\models \varphi$. But also, $\mathfrak{M}^+, r \models \neg\neg(p_1 \wedge \cdots \wedge p_k)$, so $\mathfrak{M}^+, r \not\models \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \varphi$, and finally, $\nvdash_{\mathsf{IPC}} \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \varphi$. □

The next step (which replaces Lemma 2 in this approach) is trivial:

Lemma 3. *If $\psi(p_1, \ldots, p_k)$ is positive, then $\vdash_{\mathsf{IPC}} \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \neg\neg\psi$.*

After this one proceeds to prove Theorem 7.1 as follows. If $\vdash_{\mathsf{IPC}} \varphi$, then also $\vdash_{\mathsf{IPC}} \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \varphi$, after which $\vdash_{\mathsf{IPC}} \neg\neg(p_1 \wedge \cdots \wedge p_k) \to \varphi^+$ follows, since under the assumption $\neg\neg(p_1 \wedge \cdots \wedge p_k)$, φ and φ^+ are equivalent by the same procedure as used in the proof of Theorem 5.2, using the just stated lemma on the way when we replace $\neg\psi$ by $\bot$. Finally, we can conclude $\vdash \varphi^+$ by Theorem 8. For first order logic this approach works as well when one replaces $\neg\neg(p_1 \wedge \cdots \wedge p_k)$ by $\neg\neg\forall \boldsymbol{x}(A_1 \wedge \cdots \wedge A_k)$.

4 Uniform Interpolation

In this section we prove a revised version of the uniform interpolation theorem for the positive fragment of IPC and for MPC, using the uniform interpolation theorem of IPC and the top-model property.

First of all we state the uniform interpolation theorem of IPC. We formulate the theorem for formulas $\varphi(\boldsymbol{p}, q)$ and $\psi(\boldsymbol{p}, r)$ with one variable q and r in addition to the common ones $\boldsymbol{p}$; the more general case with $\boldsymbol{q}$ and $\boldsymbol{r}$ then follows by repeated application.

Theorem 9 (Uniform Interpolation Theorem of IPC)

1. *For any formula $\varphi(\boldsymbol{p}, q)$ in which q is not a member of $\boldsymbol{p}$, there is a formula $\chi(\boldsymbol{p})$, the* uniform post-interpolant *for $\varphi(\boldsymbol{p}, q)$, such that*
   (a) $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to \chi(\boldsymbol{p})$,
   (b) *For any $\psi(\boldsymbol{p}, \boldsymbol{r})$ where $\boldsymbol{r}$ and $\boldsymbol{p}, q$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to \psi(\boldsymbol{p}, \boldsymbol{r})$, then $\vdash_{\mathsf{IPC}} \chi(\boldsymbol{p}) \to \psi(\boldsymbol{p}, \boldsymbol{r})$.*
2. *For any formula $\psi(\boldsymbol{p}, r)$ in which r is not a member of $\boldsymbol{p}$, there is a formula $\chi(\boldsymbol{p})$, the* uniform pre-interpolant *for $\psi(\boldsymbol{p}, r)$, such that*
   (a) $\vdash_{\mathsf{IPC}} \chi(\boldsymbol{p}) \to \psi(\boldsymbol{p}, r)$,
   (b) *For any $\varphi(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{q}$ and $\boldsymbol{p}, r$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$, then $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \chi(\boldsymbol{p})$.*

This theorem is proved in [14] by a proof-theoretical method and in [8] by the bisimulation quantifier method. In accordance with the latter we write $\exists q\, \varphi(\boldsymbol{p}, q)$ for the post-interpolant and $\forall r\, \psi(\boldsymbol{p}, r)$ for the pre-interpolant.

For the positive fragment, we first treat the post-interpolant. There is a complication in the case of the pre-interpolant.

Theorem 10 (Uniform Interpolation Theorem for the positive fragment of IPC, post-interpolant)
For any positive formula $\varphi(\boldsymbol{p}, q)$ in which q is not a member of $\boldsymbol{p}$, there is a positive formula $\theta(\boldsymbol{p})$ such that

1. $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to \theta(\boldsymbol{p})$,
2. *For any positive $\psi(\boldsymbol{p}, \boldsymbol{r})$ where $\boldsymbol{r}$ and $\boldsymbol{p}, q$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to \psi(\boldsymbol{p}, \boldsymbol{r})$, then $\vdash_{\mathsf{IPC}} \theta(\boldsymbol{p}) \to \psi(\boldsymbol{p}, \boldsymbol{r})$. Moreover, $\theta(\boldsymbol{p})$ is $(\exists q\, \varphi)^+$, where $\exists q\, \varphi$ is the uniform post-interpolant for φ in full* IPC.

Proof. 1. By Theorem 9.1(a), $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to \exists q\, \varphi(\boldsymbol{p}, q)$. As $\varphi(\boldsymbol{p}, q)$ is positive, by Theorem 7.6, $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, q) \to (\exists q\, \varphi(\boldsymbol{p}, q))^+$. Note that, since $\varphi(\boldsymbol{p}, q)$ is satisfiable (it is positive!), $(\exists q\, \varphi(\boldsymbol{p}, q))^+$ cannot be $\bot$ and hence is positive.
2. By Theorem 9.1(b), $\vdash_{\mathsf{IPC}} \exists q\, \varphi(\boldsymbol{p}, q) \to \psi(\boldsymbol{p}, \boldsymbol{r})$. As $\psi(\boldsymbol{p}, \boldsymbol{r})$ is positive, by Theorem 7.6, $\vdash_{\mathsf{IPC}} (\exists q\, \varphi(\boldsymbol{p}, q))^+ \to \psi(\boldsymbol{p}, \boldsymbol{r})$. □

This result is not trivial. The post-interpolant of $(p \to q) \to p$ in full IPC is $\neg\neg p$. In the positive fragment it is $(\neg\neg p)^+ = \top$.

For the pre-interpolant the situation is more complex. For example, $\forall r.\, p \to r$ is $\neg p$ and that is (up to equivalence) the only formula in p without r to imply $p \to r$, and therefore no pre-interpolant for $p \to r$ exists in the positive fragment. Actually, this is not a real surprise since in classical propositional logic the situation is the same. However, in a way this is the only failure of the theorem; pre-interpolants exist as long as we just consider positive formulas that are implied by at least one positive one not containing the quantified variables.

Theorem 11. (Uniform Interpolation Theorem for the positive fragment of IPC, pre-interpolant)
For any positive formula $\psi(\boldsymbol{p}, r)$ in which r is not in $\boldsymbol{p}$, one of the following two cases holds:

1. *There is a positive formula $\theta(\boldsymbol{p})$, the uniform pre-interpolant for $\psi(\boldsymbol{p}, r)$, such that*
   (a) $\vdash_{\mathsf{IPC}} \theta(\boldsymbol{p}) \to \psi(\boldsymbol{p}, r)$,
   (b) *For any $\varphi(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{q}$ and $\boldsymbol{p}, r$ are disjoint, if $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$, then $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \theta(\boldsymbol{p})$. Moreover, $\theta(\boldsymbol{p})$ is $(\forall r\, \psi)^+$.*
2. *For any positive $\theta(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{q}$ and $\boldsymbol{p}, r$ are disjoint, $\nvdash_{\mathsf{IPC}} \theta(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$.*

Proof. 1(a). By Theorem 9.2(a), $\vdash_{\mathsf{IPC}} \forall r\, \psi(\boldsymbol{p}, r) \to \psi(\boldsymbol{p}, r)$. As $\psi(\boldsymbol{p}, r)$ is positive, by Theorem 7.6, $\vdash_{\mathsf{IPC}} (\forall r\, \psi(\boldsymbol{p}, r))^+ \to \psi(\boldsymbol{p}, r)$. The case that $(\forall r\, \psi(\boldsymbol{p}, r))^+ = \bot$ will be treated under 2. In the other cases, we are done.

1(b). By Theorem 9.2(b), $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \forall r\, \psi(\boldsymbol{p}, r)$. As $\varphi(\boldsymbol{p}, \boldsymbol{q})$ is positive, by Theorem 7.6, $\vdash_{\mathsf{IPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to (\forall r\, \psi(\boldsymbol{p}, r))^+$.

2. If $\vdash_{\mathsf{IPC}} \theta(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$, then, by 1(b), $\vdash_{\mathsf{IPC}} \theta(\boldsymbol{p}, \boldsymbol{q}) \to (\forall r\, \psi(\boldsymbol{p}, r))^+$. This means that, if $(\forall r\, \psi(\boldsymbol{p}, r))^+ = \bot$, $\theta(\boldsymbol{p}, \boldsymbol{q})$ cannot be positive, since positive formulas are satisfiable. □

Again, the result is not trivial. The pre-interpolant of $((p \to q) \to p) \to p$ in the full logic is $\neg\neg p \to p$. In the positive fragment it is $(\neg\neg p \to p)^+ = p$. Uniform interpolation for MPC immediately follows.

Corollary 1 (Uniform Interpolation Theorem for MPC)

1. *For any formula $\varphi(\boldsymbol{p}, q)$ of MPC in which q is not a member of $\boldsymbol{p}$, and $\boldsymbol{p}, q$ may contain f, $\vdash_{\mathsf{MPC}} \varphi(\boldsymbol{p}, q) \to (\exists q\, \varphi(\boldsymbol{p}, q))^+$, and for any positive $\psi(\boldsymbol{p}, \boldsymbol{r})$ where $\boldsymbol{r}$ and $\boldsymbol{p}, q$ are disjoint, if $\vdash_{\mathsf{MPC}} \varphi(\boldsymbol{p}, q) \to \psi(\boldsymbol{p}, \boldsymbol{r})$, then $\vdash_{\mathsf{MPC}} (\exists q\, \varphi(\boldsymbol{p}, q))^+ \to \psi(\boldsymbol{p}, \boldsymbol{r})$.*
2. *For MPC-formula $\psi(\boldsymbol{p}, r)$ in which r is not a member of $\boldsymbol{p}$ one of the following two cases holds:*
   (a) *$(\forall r \varphi(\boldsymbol{p}, r))^+$ is an MPC-formula, $\vdash_{\mathsf{MPC}} (\forall r\, \varphi(\boldsymbol{p}, r))^+ \to \psi(\boldsymbol{p}, r)$, and for any $\varphi(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{q}$ and $\boldsymbol{p}$, r are disjoint, if $\vdash_{\mathsf{MPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$, then $\vdash_{\mathsf{MPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to (\forall r\, \psi(\boldsymbol{p}, r))^+$.*
   (b) *For any MPC-formula $\varphi(\boldsymbol{p}, \boldsymbol{q})$ where $\boldsymbol{q}$ and $\boldsymbol{p}$, r are disjoint, $\nvdash_{\mathsf{MPC}} \varphi(\boldsymbol{p}, \boldsymbol{q}) \to \psi(\boldsymbol{p}, r)$.*

This means that in MPC the uniform post-interpolant exists for any formula, and the uniform pre-interpolant exists for any formula that is implied by at least one formula with the right variables. The result stands if instead of the formulation of the syntax with the additional variable f one chooses to formulate MPC with $\neg$. In itself this is not remarkable, but there is a stark contrast with full IPC, in which as we have seen, uniform interpolants of positive formulas may need $\neg$.

We do not obtain uniform interpolation for the positive fragment of IQC since it does not even hold for IQC itself (see e.g. [20]). But simple interpolation for the positive fragment of IQC immediately follows from the usual proofs of simple interpolation in IQC itself.

5 Relationship with KC and Other Logics

5.1 Propositional Case

We consider intermediate propositional and predicate logics, logics between IPC and classical logic. We assume they are given by axiomatizations plus the rules of substitution and modus ponens. We first show that to derive positive formulas just positive substitutions in the axioms and the +-operation nearly suffice. This is the basic theorem of this section.

Theorem 12. *If L is an intermediate logic, φ is positive and $L \vdash \varphi$, then there are axioms $\alpha_0(p_0, \ldots, p_{n_0}), \ldots, \alpha_k(p_0, \ldots, p_{n_k})$ of L and formulas $\psi_{00}, \ldots, \psi_{0n_0}, \ldots, \psi_{k0}, \ldots, \psi_{kn_k}$, which are positive or $\bot$, such that φ is derivable in* IPC*, resp.* IQC*, from $(\alpha_0(\psi_{00}, \ldots, \psi_{0n_0}))^+, \ldots, (\alpha_k(\psi_{k0}, \ldots, \psi_{kn_k}))^+$.*

Proof. If $L \vdash \varphi$, then there are axioms $\alpha_0(p_0, \ldots, p_{n_0}), \ldots, \alpha_k(p_0, \ldots, p_{n_k})$ of L and formulas $\theta_{00}, \ldots, \theta_{0n_0}, \ldots, \theta_{k0}, \ldots, \theta_{kn_k}$ such that φ is derivable in IPC or IQC from $\alpha_0(\theta_{00}, \ldots, \theta_{0n_0}), \ldots, \alpha_k(\theta_{k0}, \ldots, \theta_{kn_k})$. By Theorem 7.7, φ is derivable in IPC or IQC from $(\alpha_0(\theta_{00}, \ldots, \theta_{0n_0}))^+, \ldots, (\alpha_k(\theta_{k0}, \ldots, \theta_{kn_k}))^+$. Then, by Theorem 7.3, φ is derivable in IPC or IQC from $(\alpha_0(\theta_{00}^+, \ldots, \theta_{0n_0}^+))^+, \ldots, (\alpha_k(\theta_{k0}^+, \ldots, \theta_{kn_k}^+))^+$. Now $\psi_{00}, \ldots, \psi_{0n_0}, \ldots, \psi_{k0}, \ldots, \psi_{kn_k}$ can be taken to be $\theta_{00}^+, \ldots, \theta_{0n_0}^+, \ldots, \theta_{k0}^+, \ldots, \theta_{kn_k}^+$. □

The reader should note that in the above proof the formulas $(\alpha_0(\theta_{00}^+, \ldots, \theta_{0n_0}^+))^+, \ldots, (\alpha_k(\theta_{k0}^+, \ldots, \theta_{kn_k}^+))^+$ may not be derivable in L itself. Nevertheless, the theorem turns out to be very useful.

It is well-known that KC is conservative over the positive fragment of IPC (see [2]). This now follows directly.

Theorem 13. *If φ is positive, then $\vdash_{\mathsf{IPC}} \varphi \iff \vdash_{\mathsf{KC}} \varphi$.*

Proof. Let us just prove the non-trivial direction. Assume $\vdash_{\mathsf{KC}} \varphi$ and φ is positive. Then, by Theorem 12, φ is a consequence in IPC of some formulas of the form $(\neg\psi \vee \neg\neg\psi)^+$ with ψ positive or $\bot$. Since $(\neg\psi \vee \neg\neg\psi)^+ \sim \bot \vee \top \sim \top$ or $\sim \top \vee \bot \sim \top$ (depending on whether ψ is positive or $\bot$) , this implies that $\vdash_{\mathsf{IPC}} \varphi$. □

An immediate consequence is:

Corollary 2. *If φ and ψ are positive and $\vdash_{\mathsf{KC}} \varphi \vee \psi$, then $\vdash_{\mathsf{KC}} \varphi$ or $\vdash_{\mathsf{KC}} \psi$.*

By a slightly more complicated argument, using that KC can be axiomatized by $\neg p \vee \neg\neg p$ for all atoms p, uniform interpolation for KC follows.

Theorem 13 can be generalized in three directions. In the first place, Jankov's Theorem [11] states that KC is the strongest intermediate logic with this property. A frame-theoretic proof was given in [3], followed by a simpler approach in [18]. Secondly, there are generalizations to predicate logic, which we will discuss in the next subsection. Finally, as discussed to a certain extent in [3], the corollary can be strengthened by considering the relationship of KC with other intermediate logics. It turns out that for many such logics Theorem 13 generalizes. So, we turn to the question for which intermediate logics L, $\mathsf{KC} + L$ is conservative over L with respect to positive formulas. The next example shows that this is not so for all such logics.

Example 1. $\mathsf{BD}_2 + \mathsf{KC}$ is not conservative over the positive fragment of BD_2, the logic of the frames bounded to depth 2 (see [2])[2].

Proof. The logic BD_2 is often axiomatized by $p \vee (p \to q \vee \neg q)$, but can be axiomatized positively e.g. by $((p \to (((q \to r) \to q) \to q)) \to p) \to p$. $\mathsf{BD}_2 + \mathsf{KC}$ contains LC, Dummett's logic. This logic is axiomatized by the positive formula $(p \to q) \vee (q \to p)$ (expressing linearity of frames), which is not provable in BD_2. □

Definition 7. *An intermediate logic L has the* $+$ -property*, if, whenever* $\vdash_L \varphi$*, also* $\vdash_L \varphi^+$*.*

Theorem 14. *If L is an intermediate propositional logic axiomatized over* IPC *that has the $+$-property and φ is positive, then $\vdash_{\mathsf{IPC}+L} \varphi$ iff $\vdash_{\mathsf{KC}+L} \varphi$.*

Proof. Assume $\vdash_{\mathsf{KC}+L} \varphi$ and φ is positive. Then, by Theorem 12, φ is a consequence in IPC from some formulas of the form $(\neg\psi \vee \neg\neg\psi)^+$ and some formulas $\alpha_0^+, \ldots, \alpha_k^+$, where $\alpha_0, \ldots, \alpha_k$ are L-axioms. The formulas $(\neg\psi \vee \neg\neg\psi)^+$ can be treated as in the proof of Theorem 13. The L-axioms are provable in L, and by the $+$-property, so are their $+$-formulas. □

Theorem 15. *If L is an intermediate propositional logic that is complete with respect to a class of frames that is closed under the operation that turns a frame into its top frame, then L has the $+$ -property.*

Proof. Repeat the proof of Theorem 7.1. □

The last two theorems immediately lead to

Theorem 16. *If L is an intermediate propositional logic that is complete with respect to a class of frames that is closed under the operation that turns a frame into its top frame, then, for positive φ, $\vdash_{\mathsf{IPC}+L} \varphi$ iff $\vdash_{\mathsf{KC}+L} \varphi$.*

To give a semantic characterization of the $+$-property of logics we need descriptive frames. First we give a lemma.

Lemma 4. *If $\mathfrak{F} = \langle W, R, \mathcal{P}\rangle$ is a descriptive frame, then so is $\mathfrak{F}^+ = \langle W \cup \{t\}, R^+, \mathcal{P}^+\rangle$, if $\mathcal{P}^+ = \{X \cup \{t\} \mid X \in \mathcal{P}\} \cup \{\emptyset\}$.*

Proof. Straightforward. □

A semantic characterization of the $+$-operation for intermediate logics can then be given as follows (simultaneously strengthening Theorem 15).

[2] A Kripke frame is of depth n if the largest chain contains n nodes.

Theorem 17. *An intermediate logic L has the $+$-property iff, for each descriptive frame $\mathfrak{F}$ of L, $\mathfrak{F}^+$ is a descriptive L-frame as well.*

Proof. $\Leftarrow$: Again like Theorem 7.1.

$\Rightarrow$: Assume $\mathfrak{F}$ is a descriptive L-frame, but $\mathfrak{F}^+$ is not. Then, for some φ, $\vdash_L \varphi$ but there exists a model $\mathfrak{N}$ on $\mathfrak{F}^+$ that falsifies φ. If this is not a top model, then some propositional variables are false in the top node. This means that they are false in the whole model and can be replaced by $\bot$ without influencing the truth value of any relevant formula. So, the formula $\varphi^\bot$ resulting from the substitution of $\bot$ for the propositional variables in question is still falsified. Moreover, $\varphi^\bot$ is provable in L as well.

So, w.l.o.g. we can assume that $\mathfrak{N}$ is a top model $\mathfrak{M}^+$ falsifying φ. Then $\mathfrak{M}^+$ falsifies φ^+ as well, and hence also $\mathfrak{M}$ falsifies φ^+. But that means that $\nvdash_L \varphi^+$, and hence that L does not have the $+$-property. $\square$

Unfortunately, the theorem has not yet been of much practical value to determine for which logics L, $\mathsf{IPC}{+}L$ and $\mathsf{KC}{+}L$ prove the same positive formulas. But it does enable us to see that the $+$-property is not necessary.

Example 2. The finite Gödel-Dummett logics LC_n with linear orders of length n as their characteristic frames, extend KC, and therefore satisfy $\mathsf{LC}_n \vdash \varphi \;\Leftrightarrow\; \mathsf{KC}{+}\mathsf{LC}_n \vdash \varphi$ for even all formulas. But by Theorem 17, they lack the $+$-property because, clearly, their class of frames is not closed under the $+$-operation.

We could conclude here by applying Theorem 15 that the tree logics T_n of [6] do satisfy the $+$-property, but we prefer to give a more satisfying proof applicable to first-order logic in the next section.

5.2 First Order Case

Let QKC be IQC plus KC. Theorem 13 can be directly, with the same proof, generalized to

Theorem 18. *If φ is positive, then $\vdash_{\mathsf{IQC}} \varphi$ iff $\vdash_{\mathsf{QKC}} \varphi$.*

This can further be strengthened by adding DNS (Double Negation Shift), axiomatized by $\forall x\, \neg\neg Ax \to \neg\neg \forall x\, Ax$, to QKC. Just as QKC the logic DNS is always valid on top models, and, in the proof of Theorem 13, applying the $+$-operation in the same way turns this axiom into $\top$ when a positive formula or $\bot$ is substituted for Ax. So, we get

Theorem 19. *If φ is positive, then $\vdash_{\mathsf{IQC}} \varphi \iff \vdash_{\mathsf{QKC+DNS}} \varphi$.*

In predicate logic we have of course the same propositional intermediate logics with positive axioms to strengthen IQC. Let us take a look at the $\mathsf{T_n}$.

Lemma 5. *$\mathsf{IQC} + \mathsf{T_n}$ has the $+$-property.*

Proof. We can apply Theorem 12. It is easy to check that the form of the $\mathsf{T_n}$-axioms, $\bigwedge_{i=0}^{n}((p_i \to \bigvee_{j\neq i} p_j) \to \bigvee_{j\neq i} p_j) \to \bigvee_{i=0}^{n} p_i$, is such that substitution of $\bot$ for an atom in one of these axioms gives a formula provable in IPC itself. □

We can now immediately conclude:

Corollary 3. $\mathsf{QKC} + \mathsf{T_n}$ *is conservative over the positive fragment of* $\mathsf{IQC} + \mathsf{T_n}$.

Proof. Assisted by the proof of the last lemma we can follow the line of the proof of Theorem 13. □

There is another very important logic with positive axioms, the logic CD, axiomatized by $\forall x(A \vee B(x)) \to A \vee \forall x\, B(x)$ and known to be complete with respect to Kripke models with constant domains (see [7]). Results apply here because, if $\mathfrak{M} \models \mathsf{CD}$, then $\mathfrak{M}^+ \models \mathsf{CD}$, since the domain of the top point is the union of all the domains of $\mathfrak{M}$, and thus the same domain as the other worlds of $\mathfrak{M}$.

Corollary 4. *Assume φ is positive. Then* $\vdash_{\mathsf{IQC+CD}} \varphi \iff \vdash_{\mathsf{QKC+CD+DNS}} \varphi$.

The same results as for $\mathsf{IQC} + \mathsf{CD}$ hold for the logic axiomatized by $\forall x, y\, (Px \to Py)$, the logic for constant domains consisting of a single element. Actually, this is not an intermediate logic of course, it is not contained in classical logic, and more properly called a superintuitionistic logic.

Acknowledgement. We thank Albert Visser, Nick Bezhanishvili, Rosalie Iemhoff, Grisha Mints and Anne Troelstra for informative discussions on the subject. We thank the referees for their corrections and Linde Frölke for her preparatory work.

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