Exact solution for the interior of a black hole

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Within the Relativistic Theory of Gravitation it is shown that the equation of state 
\( p = \rho \) holds near the center of a black hole. For the stiff equation of state 
\( p = \rho - \rho_c \) the interior metric is solved exactly. It is matched with the Schwarzschild metric, which 
is deformed in a narrow range beyond the horizon. The solution is regular everywhere, 
with a specific shape at the origin. The gravitational redshift at the horizon remains 
finite but is large, \( z \sim 10^{23} M_{\odot}/M \). Time keeps its standard role also in the interior. The 
energy of the Schwarzschild metric, shown to be minus infinity in the General Theory of 
Relativity, is regularized in this setup, resulting in \( E = M c^2 \).

Keywords: Black hole interior, stiff equation of state, vacuum equation of state, Relativistic Theory of 
Gravitation, bimetric theory, exact solution

1. Introduction

Black holes (BH’s) have fascinated mankind because of their predicted properties: at 
the classical level not even light can escape; the mass is located in the center; the 
role of time in the interior is played by space and vice versa; on the quantum 
level they radiate as a thermal body via the Hawking mechanism.

The structure of galaxies is related to astrophysical BH’s, galaxies are believed 
to have a supermassive central black hole with mass equal to 0.12% of the bulge 
mass. [1] The BH in the center of our own Galaxy has “only” four million solar 
masses. Some believe that our Universe is actually the inside of a giant black hole. [2]

As the above mentioned theoretical aspects are physically difficult to understand, 
an important question is: Are they perhaps unrealistic, and should the behavior 
of physical BH’s not be sought in a modified version of the General Theory of 
Relativity (GTR)? One such candidate is the Relativistic Theory of Gravitation 
(RTG) proposed by Logunov. [3, 4] Following idea’s of Rosen [5] and others, e.g. 
[6, 7], it writes the Hilbert-Einstein Lagrangian in terms of a field in Minkowski 
space-time, extends it with the cosmological term \(-\rho_\Lambda = -8\pi G \Lambda/c^2\) and with a
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Bimetric coupling $\frac{1}{2} \rho_{bi} \gamma_{\mu\nu} \phi^{\mu\nu}$ between the Minkowski metric $\gamma$ and the Riemann metric $g$. [3, 8] The bimetric term breaks the general coordinate invariance of GTR, and allows only the harmonic gauge for the metric or the Lorentz gauge for the field. Since it will be cosmologically small, [4] RTG will have the same content as GTR for all standard observable effects in the solar system and galactic problems. Differences may arise near singularities, in particular, for black holes and in cosmology.

An important question is the value and, in particular, the sign of the new parameter $\rho_{bi}$. Logunov pointed out that the choice $\rho_{bi} = \rho_{\Lambda}$ cancels the zero point energies of both terms, allowing to have far away from matter just a Minkowski space. When RTG was formulated, good data for $\rho_{\Lambda}$ were not available, so it was natural to choose $\rho_{bi} < 0$, with a graviton mass $m_g = \sqrt{-\rho_{bi}/16\pi}$. To make up for the observed positive value of the cosmological constant, an inflaton field can be added. [4] In [8] we have considered the situation where $\rho_{bi} = \rho_{\Lambda} > 0$, its value being set by the present cosmological data, making an inflaton field obsolete. This leads to a tachyonic graviton, but its tachyonic nature sets in only at the Hubble scale, where not individual gravitons but the whole Universe is relevant.

The bimetric coupling regularizes the infinite redshift at the horizon of black holes [3, 4]. In a previous approach we presented a scaling behavior near the horizon: coming from the outside, the time-time and radial-radial components of the metric tensor follow the Schwarzschild shape, but they cross over to an exponential decay in the interior. [8] They do not change sign, and thus leave for time its standard role, which is physically appealing: BH’s are then extreme objects, but still behave similar to normal ones. Indeed, with the Killing vector remaining time-like in the interior, the Hawking mechanism does not work, as it neither does for Newton stars.

In [8] the solution in the interior was considered on the basis of present consensus: all matter in or very near the origin. We modeled this by a very-low-pressure equation of state. In our follow up studies, we have realized that this approach is inconsistent. Here we report about the opposite case: matter spreads throughout the BH. It is modeled by the stiff equation of state $p = \rho - \rho_c$. In GTR this shape is known to have simplifying features [10] and this appears to carry over to RTG.

2. Setup of the problem

Static spherically symmetric bodies have a metric

$$d{s}^2 = U(r)c^2 dt^2 - V(r)dr^2 - W^2(r)\left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$

and lead in RTG to the 00 and 11 Einstein equations

$$\frac{1}{W^2} - \frac{W'^2}{VW^2} + \frac{2W''}{VW} - \frac{V'W'}{V^2W} = \frac{8\pi G}{c^4} \rho_{tot},$$

$$-\frac{1}{W^2} + \frac{W'^2}{VW^2} + \frac{U'W'}{UVW} = \frac{8\pi G}{c^4} p_{tot},$$

where the total density and pressure have the form

$$\rho_{tot} = \rho + \rho_{\Lambda} + \frac{\rho_{bi}}{2U} - \frac{\rho_{bi}r^2}{2V},$$

$$p_{tot} = p - \rho_{\Lambda} + \frac{\rho_{bi}}{2U} + \frac{\rho_{bi}r^2}{2V}.$$
The harmonic constraint reads
\[ \frac{U'}{U} - \frac{V'}{V} + \frac{4W'}{W} = \frac{4rV}{W^2}. \]
\[ (4) \]

Taking \( G = c = 1 \) we define the mass function \( M(r) \) by
\[ V = \frac{W'^2}{1 - 2M/W}. \]
\[ (5) \]

The Einstein equations can now be written as
\[ M' = 4\pi W'W^2 \rho_{\text{tot}}, \quad \frac{W - 2M}{2UW^2} - \frac{M}{W^3} = 4\pi p_{\text{tot}}. \]
\[ (6) \]

We shall neglect \( \rho_{\Lambda} \) and, because \( U \ll \min(V,1) \), only keep the \( \rho_{bi}/U \) terms. Indeed, they are responsible for the scaling behavior near the horizon. [3, 8] The fundamental assumption of the present Letter is that they also determine the shape in the interior of the BH.

To start, let us neglect matter and suppose that \( U \sim W^2 \). Eq. (6) first brings \( M \sim W \) and then sets \( U = 8\pi\rho_{bi}W^2 \). The energy conservation condition reads
\[ (\rho + p)U' + 2p'U = 0. \]
\[ (7) \]

If we assume that \( p \) and \( \rho \) are bounded at small \( W \), then the result \( p' \sim -W'/W \) leads to a logarithmic divergence, in conflict with the assumed boundedness. The case \( p = \kappa \rho \) implies \( \rho \sim W^{-1-1/\kappa} \). For every \( \kappa \) (0 \( \leq \kappa \leq 1 \)) this is more singular than the presumed leading term \( \rho_{bi}/(2U) \sim W^{-2} \). Thus for a singular RTG solution one cannot treat matter as a perturbation, rather one must take \( p \approx \rho \) at large \( \rho \), that is, near the origin.

Let us therefore consider the stiff equation of state [9]
\[ p = \rho - \rho_c, \quad r < R; \quad \rho = p = 0, \quad r > R, \]
\[ (8) \]

From energy conservation there arises the shape
\[ p = \frac{1}{2} \rho_c \left( \frac{U_c}{U} - 1 \right), \quad \rho = \frac{1}{2} \rho_c \left( \frac{U_c}{U} + 1 \right), \]
\[ (9) \]
where the subscript \( c \) denotes the point \( r = R \).

Notice that the vacuum equation of state, \( p = -\rho \) is covered in case they are homogeneous. It appears in the limit \( U_c \to 0, x_c \to 0 \).

3. Exact solution for the interior

The problem now leads to a similar exact solution. In terms of
\[ x = \frac{W}{W_1}, \quad W_1 = \sqrt{\frac{3}{8\pi\rho_c}}, \]
\[ (10) \]
the result \( U = 8\pi(\rho_c U_c + \rho_{bi})W^2 \) may be written as
\[ U = U_c x^2, \quad x_c = \sqrt{\frac{\rho_c U_c}{3(\rho_c U_c + \rho_{bi})}}. \]
\[ (11) \]
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while density and pressure read

\[ \rho = \frac{1}{2} \rho_c \left( \frac{x^2}{x^2} + 1 \right), \quad p = \frac{1}{2} \rho_c \left( \frac{x^2}{x^2} - 1 \right). \]  

(12)

We may define at the point \( x = 1 \)

\[ \kappa_1 = \frac{p(x = 1)}{\rho(x = 1)} = \frac{x^2 - 1}{x^2 + 1}. \]  

(13)

This allows to express

\[ \rho_c U_c = 3x^2_\rho \rho_{bi} = \frac{3(1 + \kappa_1)}{2(1 + 2\kappa_1)} \rho_{bi}. \]  

(14)

Both factors in the left hand side being positive (\( \rho_c \) because \( p \leq \rho \); \( U_c \) to avoid a horizon \( U = 0 \)), it is seen that the sign of \( \rho_{bi} \) is set by the physically allowed value of \( \kappa_1 \). For “classical” matter, it is natural to assume that negative pressures do not occur, and that \( p = 0 \) at the horizon, so \( \kappa_1 = 0 \). This is the case to be considered from now on. (Quantum matter may have \( \kappa_1 \) down to \(-1\); we come back to this in the discussion.) The bimetric and cosmological coupling constants are the assumed to be negative. We now have

\[ M = \frac{W_1}{4} x(1 + x^2), \quad V = \frac{2W_1^2 x^2}{1 - x^2}. \]  

(15)

The harmonic constraint thus brings

\[ 2x' \frac{x'}{x} - 2x'' \frac{x'}{1 - x^2} + 4 \frac{x'}{x} - \sqrt{5} y \frac{1 + y}{y} \sqrt{5} = 4x. \]  

(16)

Going to the inverse function \( r(x) \) makes it linear,

\[ x^2(1 - x^2)r'' + x(3 - 4x^2)r' = 4r. \]  

(17)

Let us define the conjugate variable

\[ y = \sqrt{1 - x^2}, \]  

(18)

The solution is then remarkably simple,

\[ r = r_1 (1 - \frac{y}{\sqrt{5}}) x^{\sqrt{5} - 1}(1 + y)^{-\sqrt{5}}, \]  

(19)

where \( r_1 \) is the value at \( x = 1 \). Near that point one has \( r = r_1 (1 - \frac{4}{\sqrt{5}} y) \). We can now derive from (15) and (19),

\[ W' = \frac{W_1 \sqrt{5}}{4r_1} x^{2 - \sqrt{5}} y(1 + y)^{\sqrt{5}}, \]

\[ V = \frac{5W_1^2}{8r_1^2} x^{4 - 2\sqrt{5}} (1 + y)^{2\sqrt{5}}. \]  

(20)
Our solution is thus completely explicit. At the origin it exhibits the singularities known for the stiff equation of state, \([4, 10]\)

\[
\begin{align*}
U &= \bar{U} r^{\gamma_\mu}, \\
V &= \frac{1}{2} \sqrt{\frac{2}{\gamma}} W_1^2 r^{-\gamma_\mu}, \\
W &= \bar{W} r^{\frac{1}{\gamma_\mu}},
\end{align*}
\]

where \(\gamma_\mu = \frac{1}{2}(\sqrt{5} + 1)\) is the golden mean. But if we take \(W\) as the coordinate, we have the Riemann metric

\[
ds^2 = \frac{U_c}{W_1 x_c^2} W^2 dt^2 - \frac{2 dW^2}{1 - W^2 / 4M^2} - W^2 d\Omega^2,
\]

in the interior of the BH. It clearly is regular at its origin, with the factor 2 coding the above singularities, and a coordinate infinity (but not a change in signs) at the horizon.

4. Matching with the exterior

Well away from matter, the harmonic constraint brings for \(W\) the Schwarzschild shape \(W_S = r + M\), where \(M = M(R)\) is the mass, basically equal to the mass as observed at infinity. This implies via Eq. (11)

\[
V_S = \frac{1}{1 - 2M/W_S} = \frac{r + M}{r - M} = \frac{1}{U_S}.
\]

In GTR one has \(\rho_{bi} = 0\), so Eq. (11) gives \(x_c = 1/\sqrt{3}\). Taking \(W\) as coordinate sets \(W' \rightarrow 1\) in (5), yielding \(V_c = 3 = 1/U_c, W_c = 3M\). This solution is seen as a limit model for neutron stars. \([10]\) But let us consider the matching problem in the present setup at some \(r = R, x = X\). Since we have to match \(W' = 1\), it follows that \(V = 2/(1 - X^2)\), implying \(X^2 = \frac{1}{4}(R + 3M)/(R + M)\). On the other hand, equating \(M = \frac{1}{4}(1 + X^2)W_S\) to \(M\), yields as only solution \(R = M, X = 1\), exceeding the presumed maximum \(1/\sqrt{3}\). Thus, in the harmonic gauge the set \((V, W, W')\) cannot match the Schwarzschild values, so this limit model of GTR must be distrusted.

In general, we consider as regular any solution for which \(M(R) < \frac{1}{2} W(R)\). We define a black hole as a solution for which \(M(R) \approx \frac{1}{2} W(R)\). For a black hole Eq. (15) sets \(x_c = 1\) and Eq. (11) shows that this is possible within RTG, provided \(\rho_{bi}\) is negative, which is the Logunov situation with a massive graviton. Together with Eqs. (10), (19) this amounts to

\[
U_c = -\frac{3\rho_{bi}}{2\rho_c}, \quad W_1 = 2M, \quad \rho_c = \frac{3}{32\pi M^2}, \quad r_1 = R.
\]

Contrary the Schwarzschild philosophy, we demand that \(V\) remains finite, as it occurs in (38) at \(y = 0\). Eq. (1) then offers an equivalent manner to characterize a BH,

\[
\text{Criterion for black hole horizon:} \quad W'(R) \approx 0.
\]

It is handy to introduce the inverse length \(m_g\) ("graviton mass") and the dimensionless small parameter \(m_g\),

\[
\rho = \frac{m_g^2}{16\pi}, \quad m_g = m_g M \leq 1.5 \times 10^{-23} \frac{M}{M_\odot},
\]
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where we took the estimate from Ref. [4]. We shall therefore have the values at the horizon $R$

$$U_c = \frac{m_g^2}{2}, \quad V_c = \frac{5 M^2}{2 R^2}, \quad W_c = 2M, \quad W'_c = 0.$$  \(27\)

These values look worrying, as they are far from Schwarzschild’s, even when $r$ is near $M$ (e.g., $W'_S = 1$). The problem nevertheless appears to be consistent. Beyond $R$ we need the deformation of the Schwarzschild metric near the horizon, caused by the bimetric coupling. It was studied by Logunov and coworkers [3, 4], and an elegant scaling form for small $m_g$ was presented by us, [8]

$$r = M \frac{1 + \eta(e^\xi + \xi + r_0)}{1 - \eta(e^\xi + \xi + r_0)}, \quad U = \eta e^\xi,$$

$$V = \frac{e^\xi}{\eta(1 + e^\xi)^2}, \quad W = \frac{2M}{1 - \eta(e^\xi + w_0) + m_g^2 \xi}. \quad (28)$$

Here $\xi$ is the running variable, $\eta$ a small scale and $r_0$ and $w_0$ parameters. Coming from the outside, $e^\xi = \mathcal{O}(1/\eta)$, the functions follow the Schwarzschild shapes, with small corrections, but they branch off for $e^\xi = \mathcal{O}(1)$. At $\xi = 0$ the function $V$ has a maximum $1/(4 \eta)$, while $U$ has already gone down to $\eta$. Going further to the inside, for $\xi \ll -1$, $U$ and $V$ both decay exponentially over a very short distance, $\delta r = 2\eta M$. It is this exponential decay that will provide the opportunity to match the seemingly very different behaviors near the horizon. Matching $U$, $V$ and $W$ with the boundary values (27) of the interior solution, we find that

$$\eta = \sqrt{\frac{2}{5}} m_g, \quad e^\xi = \sqrt{\frac{5}{2}} m_g,$$

$$w_0 = \sqrt{\frac{5}{2}} m_g \left( \log m_g + \frac{1}{2} \log \frac{5}{2} - 1 \right). \quad \text{(29)}$$

In this regime Eq. (28) yields

$$W'(r) = \frac{\eta e^\xi - m_g^2}{\eta(e^\xi + 1)}. \quad \text{(30)}$$

The values (29) confirm that $W'(R) = 0$, at the considered order $m_g$. This property nicely settles a subtlety. The scaling shape of $W'$ becomes negative below a certain $r$, [8] a fact erroneously interpreted as self-repulsion. [3, 4] If matter is taken into account, it induces a $W' > 0$ in the interior, which goes to zero at the horizon. This matches the zero coming from the outside. All by all, one thus has $W' > 0$, except for $W'(R) = 0$, see Fig. 1.

To fix the parameters $r_0$ and $w_0$ of Eq. (28), we need to see which effects they bring at finite $r$. Around the Schwarzschild solution (neglecting for now $\rho_{bi}$ and $\rho_\Lambda$) there are four perturbative modes. The first,

$$\delta U_1 = \frac{(rL - M)M}{(r + M)^2}, \quad \delta V_1 = \frac{rL - M}{r - M} - \frac{LM^2}{(r - M)^2},$$

$$\delta W_1 = \frac{1}{2} (rL - M), \quad L = \frac{1}{2} \log \frac{r + M}{r - M}. \quad \text{(31)}$$
involves the logarithm of $V_S = 1/U_S$. The second mode, $(\delta U_2, \delta V_2, \delta W_2) = \partial (U_S, V_S, W_S)/\partial M$, relates to a shift in the mass; the third, $(U_S, 0, 0)$, rescales $U_S$, while the fourth, $(0, V_S, \frac{1}{2} W_S)$, rescales $V_S$ and $W_S^2$. We require that the second is absent at order $\mathcal{M}_g$ and, to keep the proper behavior at infinity, that the third and fourth are absent to all orders. At order $\mathcal{M}_g \sim \eta$ this may be imposed by analyzing the behaviors of $U$, $V$ and $W$ of Eqs. (28) for $r > M$, with $r - M$ small but fixed, yielding

$$r_0 = 2 + \log \eta, \quad w_0 = \mathcal{O}(\mathcal{M}_g),$$

the latter being in agreement with (29). The logarithmic mode (31) remains with prefactor $8\eta$, and it is coded in the terms linear in $\xi$ of Eq. (28). It is a finite distance, nonperturbative effect of order $\mathcal{M}_g \sim M\sqrt{-\rho_{bh}}$. Near the horizon, for $r-M \sim \mathcal{M}_g M \ln(1/\mathcal{M}_g)$, it signifies the onset of the deformation of the Schwarzschild metric.

**Fig 11.** The metric function $W(r)$ has value $2M$ and slope zero at the horizon of a black hole. $\mathcal{M} = 0.01$.

**Fig 12.** The metric function $U$ starts as a powerlaw and becomes equal to $\mathcal{M}_g^2$ at the horizon, beyond which it grows exponentially towards the Schwarzschild shape. $\mathcal{M}_g = 1.5 \times 10^{-23}$ corresponds to a one solar mass black hole.
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Fig 13. The metric function $V$ is of order unity in the interior, with a powerlaw divergence at the origin. Beyond the horizon it grows exponentially towards a maximum of order $1/m_g$, after which it joins the Schwarzschild shape. $m_g = 10^{-3}$.

From Eqs. (28) and (29) we get the horizon radius,

$$R = M \left[1 + 4\sqrt{\frac{2}{5}}m_g (\log m_g + 1)\right] < M.$$  \hfill (33)

With the metric completely specified, we present plots of $U$, $V$ and $W$ in Figs. 1, 2, 3, respectively.

Having determined the leading scaling approach, we may go to next order in $\eta$ in the peak regime $e^\xi = \mathcal{O}(1)$ and in the horizon regime where it is $\mathcal{O}(m_g)$. This rather painful analysis will not be reported here. We just mention the confirmation of our black hole condition $W''(R) = 0$ at second order in $m_g$.

All by all, we may now rewrite the scaling form by eliminating $\xi$ in favor of $U$,

$$r = M \frac{1 + U + m_g \sqrt{2/5}(\log U + 2)}{1 - U - m_g \sqrt{2/5}(\log U + 2)},$$

$$V = \frac{U}{(U + m_g \sqrt{2/5})^2},$$

$$W = \frac{2M}{1 - U + m_g^2 + m_g \log(U/m_g^2)}.$$  \hfill (34)

This describes the free space region $r \geq R$, where $m_g^2 \leq U \leq 1 + \mathcal{O}(m_g)$. At scale $r \sim 1/m$ Newton’s law is picks up the Yukawa factor $\exp(-mr)$, due to the massive nature of gravitation in RTG. [3]

The interior shape can also be expressed in $U$ as running variable, where it lies in the range $(0, m_g^2)$. Due to Eqs. (11) and (18) it also holds that

$$x = \frac{\sqrt{U}}{m_g}, \quad y = \sqrt{1 - \frac{U}{m_g^2}}.$$  \hfill (35)
The density and pressure read
\[
\rho = \frac{1}{2} \rho_\odot \left( \frac{r^2}{U} + 1 \right), \quad p = \frac{1}{2} \rho_\odot \left( \frac{r^2}{U} - 1 \right),
\]
(36)
With \( R \) given by Eq. (33), the locus is
\[
r = R \left( 1 + \frac{y}{\sqrt{5}} \right) (1 - y)^{\frac{1}{2} (\sqrt{5} - 1)} (1 + y)^{-\frac{1}{2} (\sqrt{5} + 1)},
\]
(37)
and the other two metric functions read
\[
V = \frac{5 M^2}{2 R^2} (1 - y)^{2 - \sqrt{5}} (1 + y)^{2 + \sqrt{5}}, \quad W = 2 M \frac{\sqrt{U \over m_g}}{U}.
\]
(38)

The behaviors of the metric functions \( U, V \) and \( W \) are plotted in Figs. 1, 2 and 3, respectively.

5. Properties of the solution

First of all, with \( \rho \) and \( \rho' \), the functions \( U', V', W'' \) are discontinuous at horizon. For \( U' \) this is possible in the Eq. (2), because \( W'(R) = 0 \).

The characteristic size of the deformation range of the Schwarzschild solution, \( \ell_{\text{deform}} = r(\xi = 1) - R \), is small,
\[
\ell_{\text{deform}} \leq \sqrt{\frac{8}{9}} M_g \ln \frac{e^{1+e} \sqrt{2}}{m_g \sqrt{5}} \approx 1.6 \times 10^{-18} M_g. \quad (39)
\]
For one solar mass BH’s this is comparable to the Compton radius of the \( W \) and \( Z \) bosons, \( \ell_W = 2.45 \times 10^{-18} \text{m} \).

The gravitational energy density was discussed elsewhere. [8, 11] For the metric \( t^{00} \) it takes the form
\[
t^{00} = \frac{c W^2}{8 \pi G r^6} \left( \frac{r^2 V' W W'}{V} + r^3 V' - 5r^2 W'^2 \right.
\]
\[
\left. + \frac{2r^3 V' W'}{W} + 8r WW' - 2r^2 V - 3W^2 \right).
\]
(40)
At the origin it diverges as \( r^{\sqrt{5} - 5} \), which is integrable. \(^1\) The total energy density reads \( \Theta^{00} = t^{00} + VW^4 \rho_{\text{tot}} / r^4 \). \(^2\) Its separate contributions are depicted in Fig. 2.

\(^1\) In static, spherically symmetric RTG the gravitational energy density is unique, “gravitational energy can be localized”. This is implied by the harmonic constraint. The residual gauge transformations allow only shapes that are unbounded at 0 or \( \infty \). As they would make the energy infinite, they have to be discarded. This holds both for regular solutions (stars) and for our BH solution.

\(^2\) A different energy momentum tensor was proposed in Ref. [7]. Its energy density diverges at the origin as \( r^{(\sqrt{5} - 9) / 2} \), too singular to give a finite integral. For this reason we shall abandon it.
We can calculate the material energy. A partial integration is needed to numerically tame the divergent behavior near \( r = 0 \), that is exposed in Fig. 4. This brings

\[
U_{\text{mat}} = 4\pi \int_0^R dr r^2 \frac{VW^4}{r^4} \rho = 2228.830945 \, M,
\]

which is pretty large. Gravitational terms are negative and subtract 2227.830945 \( M \) from this. The bimetric term in \( \rho_{\text{tot}} \) contributes as

\[
U_{\text{bi}} = 4\pi \int_0^R dr r^2 \frac{VW^4}{r^4} \rho_{\text{hi}} = -1372.93286 \, M.
\]

The gravitational energy inside the BH is

\[
U_{\text{grav, int}} = \int_0^R dr 4\pi r^2 \Theta^{00} = -842.898079 \, M.
\]

Together they make up for \( U_{\text{interior}} = 13 \, M \).

The gravitational energy density in the skin layer first has a large positive and then a large negative part, due to the term \( r^3 V' \), see Fig. 4 in the region around \( r/R = 1 \). The integrated effect is obtained easily since the formulation of the Einstein equations in Minkowski space implies that the total energy density is a total derivative,

\[
\Theta^{00} = \frac{1}{4\pi r^2} \frac{d}{dr} \left( \frac{VW^2}{2r} + \frac{W^4}{2r^3} - \frac{W^3W'}{r^2} \right).
\]

where we have set \( c = G = 1 \). In the Schwarzschild regime\(^3\) so its this combines into

\[
\Theta^{00} = \frac{1}{4\pi r^2} \frac{d}{dr} M(r + M)^3(2r + M) \frac{2r^3(r - M)}{2r^3(r - M)}.
\]

From \( \Theta^{00} \), \( U_{\text{exterior}} = M(1 - 5 - 8 + 0) = -12M \). Together with the interior it makes up the total BH energy \( U = Mc^2 \).

6. Conclusion

We have considered a black hole in the Relativistic Theory of Gravitation. Previous findings that the bimetric coupling regulates the divergencies of the Schwarzschild singularity, are extended to show that it sets the behavior in the interior. Near the center, consistency requires that \( p \approx \rho \) is unbounded. For the case of the stiff equation of state an exact and rather elegant solution is provided for the interior. It matches the deformed Schwarzschild solution of the exterior. For a BH of one solar mass, the deformation range is of the order of the Compton length of the W and Z bosons.

\(^3\)For the Schwarzschild problem (matter-free Einstein equations in GTR), in the standard gauge \( W_S = r, U_S = 1/V_S = 1 - 2M/r \), we obtain \( \Theta^{00} = -M^2/2\pi^2(r - 2M)^2 \). Like \( \Theta^{00} \), it has a quadratic divergence at the horizon, which is non-integrable and poses a so far overlooked problem for the Schwarzschild metric of GTR.
Powerlaw singularities occur at the origin. This has been a reason to discard the problem, [4] but they disappear when the standard radial coordinate $W(r)$ is employed, rather than $r$ itself. Away from the origin, the solution is regular, and in particular also at the horizon, for any observer. The redshift at the horizon is finite, though of the order $1/m_g \geq 10^{23} M_\odot/M$. In the interior, time keeps it standard role. Hawking radiation is absent, and Bekenstein-Hawking entropy has no bearing.

Open problems are to derive the radiation and to treat, for a given type of matter, the equation of state self-consistently with the metric, as was done here near the origin for any type of matter. Next, quantization of the field theoretic approach can be considered. One may also extend the approach to the Kerr-Newman black hole.

Though we have elegantly described the interior of a black hole, we have not been able to settle definitively the question of the sign of the bimetric coupling $\rho_{bi}$. Indeed, while Eq. (14) definitely leads to $\rho_{bi} < 0$ for non-negative pressures ($\kappa_1 \geq 0$), we cannot yet exclude the regime $-1 \leq \kappa_1 < -\frac{1}{2}$, where a positive $\rho_{bi}$ would be required. Let us mention in this connection that the situation with the vacuum equation of state $p = -\rho$ ($\kappa_1 = -1$) holding in the interior was recently analyzed and stated to describe a "gravastar" in its Bose-Einstein condensed ground state. [12] We plan to investigate the connection between RTG black holes and Bose-Einstein condensation in the near future. In that situation values of $\kappa_1$ near $-1$
may occur, in principle. Let us mention that our exact metric remains valid for the vacuum equation of state, that is, in the limit $\kappa_1 \to -1$, $U_c \to 0$, and $x_c \to 0$.

Having a complete solution at hand, we could also verify some general aspects of the gravitational energy momentum tensor. We realized that the energy density of the standard Schwarzschild metric of GTR has at the horizon a quadratic singularity. The related infinite gravitational energy is not apparent in the Riemann approach, the singularity then being viewed as a coordinate singularity. [13] When making the step from GTR to RTG, the divergence at the horizon gets regularized [8], and here we have seen that the total energy is finite, and equal to $Me^2$, as it should. We also verified that for static, spherically symmetric bodies, residual gauge transformations within this sector would lead to infinite energies. So they are forbidden, making the local energy density uniquely defined: energy and, in particular, gravitational energy, can be localized in RTG, probably under more general conditions than reported here.

The fact our black hole has no true horizon would justify as name: grey hole. However, if observed black holes in the cosmos have a huge but finite redshift at their horizon, as described here, it is better to stick to the standard name. The resolution of the singular behavior at the horizon arises from the bimetric coupling, which acts as a mass-type term. This breaks the general coordinate invariance of GTR. Since $\rho_b$ is cosmologically small, this could normally play a role only in cosmology – but it still allows the Λ Cold Dark Matter model. Indeed, it brings no change of general relativistic effects in the solar system or for gravitational radiation of binaries. However, we have seen that the bimetric term does play a role at large redshifts, that is to say, near the horizon of Schwarzschild black holes and inside it. This resolution of a singularity may be more general.

Returning to the black hole problem: It is sometimes argued that our Universe may actually be the inside of a giant black hole. [2] For that application, the Schwarzschild metric with all its matter in the center is not realistic, and our setup with matter spread throughout the interior looks more natural.

References
[13] A. N. Petrov, Found. Phys. Lett. 18 (2005) 477, shows that in this case the complete gravitational energy density can be gauged towards the origin by going from a static to a stationary metric: This situation needs further consideration.