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A new multivariate product growth model*

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Abstract
To examine cross-country diffusion of new products, marketing researchers have to rely on a multivariate product growth model. We put forward such a model, and show that it is a natural extension of the original Bass (1969) model. We contrast our model with currently in use multivariate models and we show that inference is much easier and interpretation is straightforward. Especially if the number of countries is larger than two. In fact, parameter estimation can be done using standard commercially available software. We illustrate the benefits of our model relative to other models in simulation experiments. These experiments show that in the competing models the cross-country effects are actually very difficult to identify from the data. An application to a three-country CD sales series shows the merits of our model in practice.

Keywords: Diffusion, international marketing, econometric models

JEL: M31, C33

*We thank Ronald Bewley for allowing us to use the data on CD sales.

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1 Introduction

The diffusion of innovations and new products is a key research topic in economics, marketing, and operations research. Early references are Bain (1963), Mansfield (1961), and Tanner (1978) and they already cover a variety of application areas. At present, the analysis of diffusion data is most prominent in marketing and in management as it is widely recognized that diffusion data share common properties across products and across countries, which can be instrumental for marketing purposes. These properties are associated with the product life cycle, which is perhaps best documented in Rogers (2003). Rogers proposes that five types of adopters of a new product can be categorized. These are, first, innovators, then early adopters, then the early majority, then the late majority, and finally the laggards. Given these five types, the pattern of adoption data (or sales) mimics a hump-shaped curve, which could be symmetric but not necessarily. The cumulative sales data would then obey an S-shaped curve. The literature on fitting models to such S-shaped data for single innovations is large, and many models do exist, see Meade and Islam (2006) for a recent survey. In this paper we also propose such a model, although we now focus not on just a single series of sales data but on a multivariate series.

One way to deal with multivariate diffusion data is to consider models for the individual series and to allow for cross-equation restrictions. In this paper we start from another angle by allowing all series to react on common and idiosyncratic shocks, and in a sense this resembles the notion of a vector autoregression [VAR] in time series analysis. The economic motivation is that firms issue marketing campaigns in more than one country at the same time. For example, price cuts of durable products may be introduced in all western European countries at the same time. Also, newer versions of such products may be launched about simultaneously. And, new product sales may also respond similarly to common economic conditions. For example, think of the recent sequential introduction of the PlayStation 3 in Japan, the US and Europe. It is of interest to see if the adoption process in these areas is similar in the sense that deviations from a typical diffusion path may be correlated. For example, if the economies in Japan, the US and Europe would suffer from the same current global downturn, one may expect that PlayStation 3 sales will be affected in a similar way for all countries. When the effects of shocks do not occur at the same time, a multivariate diffusion model can also be beneficial in terms of forecasting. If economic shocks hit the US economy first and with some delay the European economy, one can use the diffusion patterns in the US to tell what might happen in Europe.

In sum, to describe and understand new product diffusion, when the product is launched in various areas, it makes sense to incorporate the diffusion in these areas into one single model. This model can then be used to see if some areas adopt faster or slower, if some countries lead or lag, how shocks to one country influence others, and even to see which strategy would be better in the future. This paper deals with such a model.

The (3) model is often used in marketing research to describe and to forecast the empirical adoption curve of new products and technological innovations. It basically has three unknown parameters, which characterize the important features of a variety of S-shaped curves that are typical for the diffusion process of new products. As the number of parameters is small, the Bass model is usually considered for a single annual series, where the time span concerns 10 to 20 years.

Ever since its inception, the Bass model witnessed a large amount of modifications and extensions in various directions. These model versions aim to capture more dynamics, additional explanatory variables, sequences of generations, and also, more than one diffusion variable. In this paper we address this last case, that is, the extension of the basic Bass model to allow for two or more diffusion series, which somehow may be correlated with each other. Hence, we propose a model for multivariate new product growth. In this paper we present the model in terms of a cross-country diffusion analysis, however the model can also be applied to other multivariate diffusion processes, such as the (joint) diffusion of multiple products in the same country.

There are many multi-national diffusion studies available in the marketing literature. Broadly speaking, we can divide the literature into two more or less separate streams. On the one hand we have the studies that focus on measuring and explaining differences in characteristics of the diffusion process across countries. Good examples of such an approach are (17) and (18). A relatively recent overview is given in (8). Important questions in this field are: which countries lead and which countries lag in the adoption process; and is the diffusion process faster in countries where the product is introduced at a later stage (the so-called learning effect)? The other stream of research focusses on the influence of the diffusion in one country on the diffusion in another country. In this research stream models are developed to *jointly* capture the diffusion in two (or more) countries. The present paper fits this latter research stream.

There are various multivariate versions of the Bass model around, see (15) and (11) for example. In Section 2, we review these models and some related models presented in the literature. We also show that these models suffer from the critique summarized in (4). This critique basically says that if the diffusion of adoptions may depend on an explanatory variable, which itself is strongly correlated with time, then the Bass model without any such variables would fit about equally well. Hence, if country A's diffusion would be correlated with that in country B, then adding country B's diffusion to the model for country A would not contribute much to the fit. In fact, a single-country Bass model would fit equally well. In econometric language, the explanatory variables for country B's diffusion process are too collinear with the explanatory variables for the diffusion process in country A, so that proper parameter estimates are difficult or even impossible to obtain. In (20) this problem is solved by adopting a Bayesian approach and using informative priors on the model parameters.

In the same Section 2 we put forward a new multivariate product growth model, that does not have the above-mentioned problems. Our model extends in a natural way the empirical Bass-type model as it is proposed in (6), and hence it builds naturally on the original Bass model. The basic idea in (6) is that a univariate diffusion series follows an S-shaped path, from which short-lived deviations are possible. The speed of adjustment towards this path is given by an adjustment parameter, which in the final empirical model implies the inclusion of an additional lagged adoption variable. Additionally, (6) propose to modify the original Bass model with heteroskedastic error terms. Such an extension is also rather straightforward for our new multivariate model, as we will demonstrate in the empirical illustration.

Our multivariate extension of the (6) model and hence the Bass model allows for cross-equation adjustments to country-specific paths. In other words, deviations from the path in country A have an effect on the deviations from the underlying adoption process in country B. We also allow for contemporaneous correlation. After discretizing, our model is a system of equations with parameter constraints. This system can easily be estimated by (nonlinear) Generalized Least Squares.[1]

In Section 3, we report on the outcomes of a limited simulation experiment in which we compare the various multivariate Bass models. For each model we generate diffusion processes and calculate the root mean squared error for all the multivariate diffusion models. We show that diffusion paths generated by the earlier proposed multivariate diffusion models can indeed be replicated almost perfectly by simple univariate Bass models. Even if the data is generated with a particular dependence structure, these

[1]We have estimated the parameters in EViews, Quantitative Micro Software, www.eviews.com. The code is available from the authors.

models do not fit the data better than the simple Bass model. A possible explanation for this follows a multicollinearity argument. The basic Bass model fits an S-shaped curve to the penetration data in a country. The existing multivariate models attempt to model the cross-country dependence by adding additional explanatory variables that follow the same type of S-shape. The same reasoning holds for trying to use price data in the Bass model, see Bass et al. (1994). In any case, the above gives a clear sign that there may be (empirical) identification issues with these models. The model of (6), and by extension our model, allows for the case of a stochastic diffusion process. In a second experiment we consider the influence of the relative size of the stochastic component on the performance of the different models.

In Section 4, we illustrate the usefulness of the various models on data for CD adoption in the US, Canada and Japan. We give the data in Appendix 1. We show that our model yields valuable insights in the relation between different countries. For example, we show that cross-country diffusion dependence may not be symmetric. Finally, in Section 5 we summarize limitations of the current paper and we suggest further research topics.

2 Multivariate product growth models

In this section we discuss four variants of multivariate product growth models, where we consider the case of two countries to save notation. Extensions to more than two countries follow straightforwardly, at least in theory (as we will show below, in practice matters can become every different). We first consider a simple extension of the basic Bass model to the case of two countries. Then we consider two models that have been proposed recently, that is the model of Putsis et al. (1997), which we assign the acronym PBKS, and the model of Kumar and Krishnan (2002), which we will label the KK model. Finally, we introduce our model, for which we use the acronym MBF. This acronym is short for Multivariate Boswijk/Franses. At the start, we will use the commonly used notation in continuous time, while towards the end we will discuss the discretized versions of the models. We will write all models in a unifying notation, such that, one can easily see the similarities and the differences between the models.

We denote $n_i(t)$ as the continuous time increments of the product growth process of country i and $N_i(t)$ as the level of adoption at time t, that is, $n_i(t) = dN_i(t)/dt$. The basic Bass model reads as

$$\frac{n_i(t)}{1 - N_i(t)} = p_i + q_i N_i(t), \tag{1}$$

where it is assumed here that 1 marks the maturity level of the adoption process. The parameter p_i is called the innovation parameter, and q_i is the imitation parameter. For the sake of notation, to become useful below we rewrite this equation as

$$n_i(t) = (1 - N_i(t))(p_i + q_i N_i(t)) := n_i^*(t). \tag{2}$$

Note that the left hand side of the equation now contains the variable of interest that one wants to explain, that is, the growth, while the right hand side only depends on the current installed base at time t. Throughout this paper we will denote the implied growth rates at time t, conditional on $N_i(t)$, according to the Bass model by $n_i^*(t)$. Note that $n_i^*(t)$ depends on $N_i(t)$, which of course can differ across models. Modifying the expressions to allow for a maturity level m_i is well known to be straightforward, but for ease of notation we abstain from this for the moment. Allowing for an unknown maturity level requires replacing $n_i(t)$ by $n_i(t)/m_i$ and $N_i(t)$ by $N_i(t)/m_i$ in (2). Of course, in the empirical application in Section 4 we will take into account the maturity level.

2.1 Currently available models

The easiest multivariate diffusion model simply specifies independent Bass models for each country, that is, we specify (2) for $i = 1$ and $i = 2$. countries $i = 1$ and $i = 2$. To provide a consistent notation with the other models to come, we write these two independent Bass models in one multivariate system, that is,

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} (1 - N_1(t))(p_1 + q_1 N_1(t)) \\ (1 - N_2(t))(p_2 + q_2 N_2(t)) \end{pmatrix} := \begin{pmatrix} n_1^*(t) \\ n_2^*(t) \end{pmatrix}. \tag{3}$$

After some manipulation one can also write

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} 1 - N_1(t) & 0 \\ 0 & 1 - N_2(t) \end{pmatrix} \left[\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} \right]. \tag{4}$$

We label this model the diagonal multivariate Bass [DMB] model. This model assumes that each of the two countries has its own adoption process and that these two processes are everywhere independent. We take this model as a benchmark model when we study in the simulation experiments whether the same issues as noted in (4) extend to the multivariate case.

However, the diffusion processes in countries 1 and 2 may interact somehow. A proposal for such a multivariate model is done in Putsis et al. (1997), which in our general notation can be written as

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} 1 - N_1(t) & 0 \\ 0 & 1 - N_2(t) \end{pmatrix} \left[\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} \right]. \tag{5}$$

We label this the PBKS model. Clearly, the key difference between the diagonal multivariate Bass model and this one is in the 2×2 matrix with the imitation parameters. This can be seen even more clearly by writing the PBKS model as

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} n_1^*(t) \\ n_2^*(t) \end{pmatrix} + \begin{pmatrix} 1 - N_1(t) & 0 \\ 0 & 1 - N_2(t) \end{pmatrix} \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix}. \tag{6}$$

From a theoretical point of view, this extension makes much sense. Indeed, it can be the case that the level of the diffusion in a neighboring country exercises an effect on the own country diffusion. Restricting one of the q_{ij}'s to zero leads to a model in which one country leads and the other country lags. On the other hand, note that in theory this model only allows for a positive effect of one country on another. The cross-country influence is modeled through the number of contacts in one country with another country. Furthermore, as the number of *cumulative* adopters $N_i(t)$ cannot decrease by definition, $n_i(t)$ must be positive. In order to maintain logical consistency of the model we must therefore restrict $q_{ij} > 0$.

(15) were not the first to present this model. The same type of specification can be found in (14), (9) and (10). Although historically it may be more appropriate to use the names of Peterson and Mahajan to label this model, we will stick to the acronym PBKS, as (15) further developed the model in terms of mixing of populations. In (15) the parameters q_{ij} are further specified to measure the type and extent of cross-country influences.

From a practical point of view, the PBKS model in (6) might not be easy to handle. The main reason is that it suffers from exactly the same collinearity problems as the generalized Bass model does, as noted convincingly in Bass et al. (1994). Indeed, it is most likely that both $N_1(t)$ and $N_2(t)$ have an S-shaped pattern that is strongly correlated with time, and with each other. Hence adding say $N_2(t)$ to an equation for $N_1(t)$ makes it difficult to estimate the corresponding parameters q_{11} and q_{12}. As an extreme example, consider the case where the diffusion patterns in two countries follow (almost) the same S-shape, but where the two adoption processes are in fact unrelated. In other words, the adoption in one country does not influence the adoption in the other. However, as the basis S-curve is the same in both countries, the correlation between *the sample paths* of

$N_1(t)$ and $N_2(t)$ will be almost perfect. In this example, it is impossible to estimate the q_{ij} parameters. Furthermore, in case the diffusion curves are different and interrelated, the additional explanatory power of the cumulative number of adoptors in country 2 ($N_2(t)$) to explain the diffusion in country 1 may be very limited. This point can be further motivated by noting that the fit of the Bass model, for example in terms of R^2, is usually very high. In empirical applications, there tends to be little room for improvement. Multicollinearity will in such cases be even more troublesome. In our simulation experiments below, we will also demonstrate this phenomenon.

Ignoring these possible multicollinearity problems, estimation of the PBKS model parameters is rather straightforward. After discretizing the continuous time model in (6), one can apply least squares to obtain estimates of the model parameters, see Section 2.3 for more details.

A second multivariate model that has been proposed recently in the marketing literature is given in (11). In their notation it reads as

$$\begin{aligned} \frac{n_1(t)}{1 - N_1(t)} &= (p_1 + q_1 N_1(t))(1 + b_{21} n_2(t)) \\ \frac{n_2(t)}{1 - N_2(t)} &= (p_2 + q_2 N_2(t))(1 + b_{12} n_1(t)), \end{aligned} \tag{7}$$

which in fact seems very close to a multivariate version of the generalized Bass model. To make the model comparable with the other models, we rewrite this model first as

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} n_1^*(t) & 0 \\ 0 & n_2^*(t) \end{pmatrix} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & b_{21} \\ b_{12} & 0 \end{pmatrix} \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} \right], \tag{8}$$

where $n_i^*(t)$ is defined in (2). The model implies that product growth $n_i(t)$ of country i may (permanently) deviate from the underlying path ($n_i^*(t)$) by a factor that is linear in the other country's product growth $n_j(t)$. We can rewrite (8) as

$$\left[\mathbf{I} - \begin{pmatrix} n_1^*(t) & 0 \\ 0 & n_2^*(t) \end{pmatrix} \begin{pmatrix} 0 & b_{21} \\ b_{12} & 0 \end{pmatrix} \right] \begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \begin{pmatrix} n_1^*(t) \\ n_2^*(t) \end{pmatrix}, \tag{9}$$

where $\mathbf{I}$ denotes a 2×2 identity matrix. Solving this equation for the 2×1 vector $n(t)$ containing the country-specific diffusion series gives

$$\begin{pmatrix} n_1(t) \\ n_2(t) \end{pmatrix} = \left[\mathbf{I} - \begin{pmatrix} n_1^*(t) & 0 \\ 0 & n_2^*(t) \end{pmatrix} \begin{pmatrix} 0 & b_{21} \\ b_{12} & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} n_1^*(t) \\ n_2^*(t) \end{pmatrix}, \tag{10}$$

provided that the inverse exists. It is clear that the KK model nests the diagonal multivariate Bass model as setting $b_{12} = 0$ and $b_{21} = 0$ in the KK model yields the other model. An empirical comparison of KK versus DMB can simply be done using likelihood ratio tests, say. Comparing the KK model with the PBKS model is not straightforward, however.

The expression in (10) shows that this multivariate model is highly non-linear in variables as well as in parameters, as it contains the inverse of a matrix with elements that contain parameters and variables. This may make the theoretical interpretation of this model not easy. Also, at first sight one may expect that this should complicate parameter estimation, and this is confirmed in Kumar and Krishnan (2002). Indeed, these authors describe a rather complicated estimation routine, which also seems to have problems to deliver standard errors (as these are not reported in their tables). The absence of standard errors makes it difficult to test any hypothesis on cross-country influences.

For parameter estimation in the simulation experiments below, we consider an alternative procedure to the one proposed in Kumar and Krishnan (2002). Instead of estimating the "structural" model (7), we consider the "reduced form" (10), where the words strucural and reduced form should be seen in the typical multivariate time series sense. We apply the common discretization techniques and estimate the model parameters using least squares, see also Section 2.3.

2.2 A new multivariate model

To overcome the potential problems raised above concerning currently available multivariate models, we intend to propose a model that is easy to interpret and of which the parameters are easy to estimate. The main idea is to extend the error-correction type expression of (6) to the multivariate setting. (6) consider the basic univariate Bass model and address the issue of the stochastic nature of the diffusion process. The Bass model can be seen as the result of aggregating the purchase timing behavior over a large homogeneous population. Although at the individual level the behavior is stochastic, after aggregation a deterministic model remains. To allow for estimation, the original Bass equation in continuous time is discretized and than simply an error term is added, which is assumed to have mean zero and common variance. (6) argue that this approach of adding an error term does not match with the original notions behind the Bass theory. Also incorporating a multiplicative error term as in (18) does not solve the problem. Although a multiplicative specification reduces the effect of heteroscedasticity, this idea also raises problems with the interpretation of the parameters. For example, using simple simulations, one can show that the maturity level in this model is no longer equal to the parameter m (or in their notation, α).

In fact, it makes more sense to incorporate the uncertainty from first principles. Boswijk and Franses assume that the underlying theoretical diffusion follows a deterministic S-shaped path, around which the actually observed diffusion process fluctuates. These fluctuations are caused by random events, individual-specific characteristics, or by marketing-mix effects. These fluctuations are however such that there always is a tendency to return to the theoretical underlying deterministic S-shape. This implies that there is, say, a target or attractor-like diffusion process, around which the actual diffusion fluctuates around while preserving a tendency to return to that target level.

In the notation of the current paper, the key equation in (6) for a single country i is

$$dn_i(t) = \alpha_i(n_i^*(t) - n_i(t))dt + \sigma_i n_i(t)_i^\gamma dW_i(t), \tag{11}$$

where $W_i(t)$ is a standard Brownian motion. The actual diffusion series $n_i(t)$ wanders around the target diffusion $n_i^*(t)$, where deviations from this target are caused by random shocks driven by $W_i(t)$. The size of the random shocks are proportional to $n_i(t)^\gamma$. Over time the $n_i(t)$ should return to the target path, the speed of this return is determined by α_i. Note that the diffusion process returns to the target path by definition. If it would not return to the target path, we would not have estimated the target path correctly. As we only have one observation of the diffusion process in a country, we cannot distinguish between a target path that changes over time and a fixed target path with a certain shape. The target path in this model is exactly the path according to the Bass diffusion model (2). For the same reason we cannot allow the target paths of the two countries to be dependent on each other. Note that the model is now specified in terms of changes in the growth rate, that is, $dn_i(t)$.

The BF model explicitly allows for random events to influence the diffusion. Diffusion curves that correspond with this model do not necessarily show a perfect S-shape. This is an important feature of this model, as in practice one also does not always encounter perfect curves.

A multivariate version of this error correction type model, to be labeled as MBF, is given by the system of equations

$$\begin{pmatrix} dn_1(t) \\ dn_2(t) \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} n_1^*(t) - n_1(t) \\ n_2^*(t) - n_2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 n_1(t)^\gamma dW_1(t) \\ \sigma_2 n_2(t)^\gamma dW_2(t) \end{pmatrix}, \tag{12}$$

where $W_1(t)$ and $W_2(t)$ are possibly correlated Brownian motions. The off-diagonal elements of the matrix α, that is, α_{12} and α_{21}, have a straightforward interpretation. They can be interpreted as the effect that deviations in one country have on the deviations from the underlying diffusion path in another country. For example, suppose that $\alpha_{12} < 0$ and $n_2(t) < n_2^*(t)$, that is, the actual diffusion in country 2 is below its target path. The product $\alpha_{12}(n_2^*(t) - n_2(t))$ is then also negative, meaning that the diffusion in country 1 will slow down. Hence, deviations from the target paths in each of the countries also have an effect on the changes in the diffusion in the other country. As in the univariate case, these deviations can be due to marketing-mix effects. By using this specification we do not encounter the collinearity issues that complicate the other models. The deviations from the path in a neighboring country are not likely to be strongly correlated with time. Even if we consider more than two countries we are not likely to run into multicollinearity problems.

When we restrict $\alpha_{12} = \alpha_{21} = 0$ we end up with two stacked BF models, where the random shocks may be correlated. One could call this model a diagonal MBF model. Below we refer to this model as the BF model.

Estimation of and inference on the parameters of the MBF model is based on a discretization of the stochastic differential equations in (12), discussed in more detail in the next sub-section. Because the additional variables in this discretized MBF model do not have patterns that come close to trends or sigmoid shape trends, the MBF model does not have the problems noted in Bass et al. (1994).

In sum, the currently available multivariate models seem to suffer from potential estimation problems, while the MBF model does not. Additionally, the MBF model has easy to interpret parameters from a marketing point of view. To illustrate the estimation issues mentioned above, we turn to a report of simulation experiments, and we postpone an empirical illustration to Section 4. Before doing so, we say a few words about discretization and estimation.

2.3 Discretization and estimation

All models discussed above are in continuous time. In practice we need to fit these models to, say, annual data, that is, data measured at discrete intervals. The diffusion models therefore have to be transformed to discrete time. The equations to use for parameter estimation follow naturally from these transformations. In general we index the observations using a subscript k, where observation k corresponds to time $t_k = k\delta,\ k = 0, \ldots, T/\delta$, where δ denotes the length of the time interval between two observations. In most practical cases we may set $\delta = 1$, so that the observations occur at times $t = 0, 1, \ldots, T$. To keep the presentation as general as possible, we use the subscript k to denote the observation times. For example, $N_{i,k} = N_i(\delta k)$ denotes the cumulative adoption in country i at time $t_k = k\delta$.

Following Boswijk and Franses (2005), we propose to use Euler approximations[2] to discretize the continuous-time models. The deterministic models (DMB, PBKS, and KK) may all be written as a system of ordinary differential equations

$$dN_i(t) = n_i(t)dt = f_i(N_1(t), N_2(t))dt, \qquad i = 1, 2,$$

for some suitble choice of f_i. They are discretized as

$$X_i = \Delta N_{i,k} = N_{i,k} - N_{i,k-1} = f_i(N_{1,k-1}, N_{2,k-1})\delta, \qquad i = 1, 2.$$

Note that in the KK models, $n_i(t)$ depends on $N_1(t)$ and $N_2(t)$ via $n_1^*(t)$ and $n_2^*(t)$. For the discretization of the stochastic components in the BF and MBF model, we first apply the Euler approximation to approximate (12) by

$$\begin{aligned}\begin{pmatrix} n_1(t_k) - n_1(t_{k-1}) \\ n_2(t_t) - n_2(t_{k-1}) \end{pmatrix} &= \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} n_1^*(t_{k-1}) - n_1(t_{k-1}) \\ n_2^*(t_{k-1}) - n_2(t_{k-1}) \end{pmatrix} \delta \\ &\quad + \begin{pmatrix} \sigma_1 n_1(t_{k-1})^\gamma [W_1(t_k) - W_1(t_{k-1})] \\ \sigma_2 n_2(t_{k-1})^\gamma [W_1(t_k) - W_1(t_{k-1})] \end{pmatrix}. \end{aligned} \tag{13}$$

Next, we approximate $n_i(t) = dN_i(t)/dt$ at $t = t_k$ by $X_{i,k}/\delta = \Delta N_{i,k}/\Delta t_k$. This yields

$$\begin{pmatrix} \Delta X_{1,k} \\ \Delta X_{2,k} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_{1,k-1}^* - X_{1,k-1} \\ X_{2,k-1}^* - X_{2,k-1} \end{pmatrix} \delta + \begin{pmatrix} (X_{1,k-1})^\gamma \varepsilon_{1,k} \\ (X_{2,k-1})^\gamma \varepsilon_{2,k} \end{pmatrix}, \tag{14}$$

where

$$X_{i,k-1}^* = n_i^*(t_{k-1}) = (1 - N_i(t_{k-1}))(p_i + q_i N_i(t_{k-1})),$$

and $\varepsilon_{i,k} = \sigma_i \delta^{1-\gamma}[W_i(t_k) - W_i(t_{k-1})] \sim$ i.i.d. $N(0, \sigma_i^2 \delta^{3-2\gamma})$. If the two Brownian motions W_1 and W_2 have a correlation ρ, then this implies $\text{cov}(\varepsilon_{1,k}, \varepsilon_{2,k}) = \rho\sigma_1\sigma_2\delta^{3-2\gamma}$.

For parameter estimation in the DMB, PBKS and KK models one can minimize the sum of squared errors in the discretized models. Note that this estimation procedure

[2]Other methods are available, but these lead to more complicated discrete time models. Furthermore, by choosing the Euler method we stay close to the current practice in the literature.

for the KK model is far more straightforward than that proposed in Kumar and Krishnan (2002). For the BF and MBF models, the discretized model (14) can be estimated by maximum likelihood. Assuming that the parameter γ is known (typical choices are $\gamma = \frac{1}{2}$ or $\gamma = 1$), the analysis is facilitated by dividing the left-and right-hand-side of the equations by $(X_{i,k-1})^{\gamma}$, yielding a system of non-linear regression equations with homoskedastic normal disturbances. If the contemporaneous covariance between the two disturbances is ignored, as well as possible differences in the variance parameters, then the parameters can be estimated simply by minimizing the sum of squared errors over all observations. We opt for this procedure in the simulation experiments below. For the empirical section we take the stochastic component seriously and estimate the parameters by applying (nonlinear) GLS to (14), based on the covariance matrix (taking $\gamma = 1$ for convenience):

$$\Sigma = \mathrm{var}\begin{pmatrix} \varepsilon_{1,k} \\ \varepsilon_{2,k} \end{pmatrix} = \delta \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Boswijk and Franses (2005) prove consistency and derive the asymptotic distribution of the parameter estimators in the univariate BF model. They show that consistency can only be proved assuming continuous record asymptotics, where the time span $[0, T]$ over which we observe the process is fixed, but the number of observations T/δ tends to infinity, so that the time interval δ converges to 0. Such an asymptotic scheme is needed, first, to make the Euler discretization errors vanish, and second, to let the statistical information on the parameters increase with the sample size. Given that the MBF is a natural generalization of the BF model, the same techniques as in Boswijk and Franses (2005) may be used to prove consistency, and to derive the (non-standard) asymptotic distributions of the parameter estimators.

2.4 Forecasting

All the discretized models that we consider can be written in the same general form, that is,

$$X_k = f(N_{k-1}, X_{k-1}, \theta) + \eta_k, \text{ where } \eta \sim N(0, \Omega), \tag{15}$$

where $X_k = (X_{1k}, X_{2k})'$, $N_k = (N_{1k}, N_{2k})'$, θ denotes a parameter vector and Ω the variance matrix of the error terms. The form of the function $f()$ depends on the particular model specification. Although the dependent variable in the BF and MBF is ΔX_{ik} one can rewrite $\Delta X_{ik} = X_{ik} - X_{ik-1}$ and move X_{ik-1} to the right-hand side of the equation.

Given estimated parameters $\hat{\theta}$, one-step ahead forecasting is straightforward. Suppose we want to forecast the growth rate at time $k\delta$, that is, X_k. The available information contains $N_{i,k-1}$ and $X_{i,k-1}$ for $i = 1, 2$. Ignoring parameter uncertainty, the unbiased forecast is now given by

$$\hat{X}_k = \mathrm{E}[X_k | N_{k-1}, X_{k-1}] = f(N_{k-1}, X_{k-1}, \hat{\theta}). \tag{16}$$

Using the forecast of the growth, the forecasted level of adoption at time $k\delta$ equals $\hat{N}_k = N_{k-1} + \hat{X}_k$. As the function $f()$ is non-linear, multi-step ahead forecasting is more complicated. The unbiased forecast $\mathrm{E}[X_{k+1}|N_{k-1}, X_{k-1}]$ is not equal to $f(\hat{N}_k, \hat{X}_k, \hat{\theta})$. To obtain unbiased multi-step forecasts we have to rely on simulation. The general simulation schema is as follows. We start forecasting at observation $k-1$. First we generate L pseudo realizations of X_k using

$$X_k^{(l)} = f(N_{k-1}, X_{k-1}, \hat{\theta}) + \eta_k^{(l)},\ l = 1, \ldots, L \tag{17}$$

where $\eta_k^{(l)}$ is a draw from $N(0, \hat{\Omega})$. Simulated realizations of the adoption rate are given by $N_k^{(l)} = N_{k-1} + X_k^{(l)}$. The law of large numbers states that if L approaches infinity, $\frac{1}{L}\sum X_k^{(l)}$ converges to the unbiased forecast in (16). To obtain a $n+1$-step ahead forecast we recursively generate

$$X_{k+n}^{(l)} = f(N_{k+n-1}^{(l)}, X_{k+n-1}^{(l)}, \hat{\theta}) + \eta_{k+n}^{(l)},\ l = 1, \ldots, L, \tag{18}$$

where $N_{k+n}^{(l)} = N_{k+n-1}^{(l)} + X_{k+n}^{(l)}$ for $n = 1, 2, 3, \ldots$. The unbiased $n+1$-step ahead forecast is now given by

$$\mathrm{E}[X_{k+n}|N_{k-1}, X_{k-1}] \approx \frac{1}{L}\sum X_{k+n}^{(l)}. \tag{19}$$

In practice L should be set to a relatively large number to avoid simulation error in the forecasts. In the simulations below we set $L = 10,000$.

3 Simulation experiments

In this section we illustrate the performance of the different multivariate models discussed above. The ultimate comparison of the competing models would of course consist of a forecasting comparison based on a large collection of real world data sets. However, a necessary condition for a model to be useful is that feature one is looking for is identified from the data in the optimal situation. In the experiment below, we will explicitly test this condition.

The setup of the experiment is as follows. We select one of the models and generate data according to that model. Next, each of the five models is fitted to the generated data. Given the estimated parameters the models are used to generate forecasts of the growth figures. We generate one-step, two-step, and five-step ahead forecasts. For each of the forecasting horizons, we measure the forecasting performance by the median root mean squared error [RMSE] of the forecasts over all generated datasets. Comparing the RMSE of the different models gives us insight into how much the models differ from each other in the implied diffusion curves. For example, when for data generated with a given model, the DMB yields roughly the same RMSE as the true model, one can conclude that the latter model does not add much to the DMB model. Or in other words, the cross-country dependence implied by that model is badly identified from the data. Possible explanations for findings such as this one are the problems first noted in Bass et al. (1994).

Except for the BF and the MBF model, the (multivariate) diffusion models do not have a stochastic component. That is, given the model parameters, the diffusion curve is fixed. Therefore we first consider the deterministic case, where for BF and MBF we set $\sigma_i = 0$. Afterwards we consider what happens when the BF or the MBF model is used as data generating process with $\sigma_i \neq 0$.

Deterministic models

For each data generating process [DGP], that is, for DMB, PBKS, KK, BF and MBF, we generate data for $T = 20$ periods. To generate data we use the discretizations as discussed in Section 2.3 but now with $\delta = 0.001$, that is, we very closely approximate continuous time. For model estimation and diffusion forecasting we of course only use the observations for $t = 0, 1, 2, \ldots, T$. Next, each of the models is fitted to these data. For example, we generate data from (2), and fit models (2), (6), (10), and (12), and so on. Next we generate one-step, two-step, and five-step ahead forecasts starting at $t = 1$. To allow for a comparison across forecast horizons we evaluate the root mean squared error over the periods $t = 6, \ldots, T$, such that the same observations used for every forecasting horizon. Finally, for each case we calculate the median root mean squared error over 2000 replications.

To generate data that look like the typical data obtained in practice, we have to set proper parameter values. The illustrations in the various studies that put forward the models under scrutiny provided a source of inspiration. For DMB model we draw values for the imitation and innovation parameters in country 1 and 2 from two uniform distributions, that is

$$\begin{aligned} p_i &\sim \mathrm{U}[0.005, 0.055], \\ q_i &\sim \mathrm{U}[0.15, 0.55], \end{aligned} \tag{20}$$

where i is 1 or 2. For the PBKS model we interpret q_i from (20) as the total external influence, we divide this influence over the within-country influence (q_{ii}) and the across-country influence (q_{ij}) using

$$\begin{aligned} f_i &\sim \mathrm{U}[0, 0.5], \\ q_{ii} &= (1 - f_i)q_i, \\ q_{ij} &= f_i q_i. \end{aligned} \tag{21}$$

To make sure that the resultant bivariate series make sense we restrict the cross-country influence to be smaller than the within-country influence. Unreported graphs of the series substantiate this claim. Next, for the KK model we use the same way to generate p_i and q_i as in the DMB model, where we additionally draw the b_{ij} parameters according to

$$b_{ij} \sim \mathrm{U}[0, 0.75]. \tag{22}$$

For the univariate BF model we additionally need the error correction parameters, and we draw these according to

$$\alpha_i \sim \mathrm{U}[2, 10]. \tag{23}$$

Finally, the MBF requires error correction parameters across countries, and these we generate according to

$$\begin{aligned} r_i &\sim \mathrm{U}[-0.9, 0.9] \\ \alpha_{ij} &= r_i \alpha_{ii}. \end{aligned} \tag{24}$$

Again we restrict the size of cross-country effect to be smaller than the within-country effect. Note that the MBF model also allows for a negative correlation between countries, which is in contrast to the PBKS model. In sum, we have 5 sets of DGPs, and in the first round of experiments, we also fit 5 models for each DGP. Although we have set $m = 1$ in the DGPs and in the discussion above, we do treat m as an unknown parameter in these simulation experiments. To allow for a fair forecast comparison, we fit all models by minimizing the sum of the squared residuals.

The simulation results for the deterministic case are given in Table 2. The first panel gives the median root mean squared errors over 2000 replications. Note that the DGPs in this case are deterministic, that is, conditional on the parameters the diffusion figures are fixed. In principle, the model of the DGP would fit the data perfectly. The reason that we do not find zero RMSE in Table 2 is that there are discretization errors. The data are generated in (almost) continuous time, while we estimate the model for discrete time.

The second panel gives these figures again, but now scaled towards each of the DGPs. Hence, the value 1 should appear when the same model is fitted to the data generated by that model. The first panel already indicates that differences can be quite large, and this can even be better seen from the second panel. For example, when the data are generated by an DMB model, then the root mean squared error of the PBKS model is 0.92 times that of the DMB model itself. And, when the data are generated by an MBF model, the RMSE of the KK model is 3.4 times as large that of the MBF model. Clearly, the results in Table 2 indicate that it is best to fit the MBF model when the data are generated by any of the other models. This improvement in fit is of course partly because the MBF model contains one additional parameters per country compared to the PBKS and KK models. Note however that the BF model also outperforms the KK and PBKS models, while they have an equal number of parameters.

The differences in performance of the PBKS and KK models versus the DMB model are quite small, even when the data are generated using PBKS or KK. This is mainly due to the problems noted in Bass et al. (1994). The additional regressors in the PBKS and KK models do not add much in explanatory power. In fact the generated diffusion curves very closely resemble curves that can be fitted using the standard Bass model. This implies that it will in practice be difficult to find cross-country effects using one of these two models. The poor fit of the DMB model in case BF and MBF are the DGPs is due to the fact that the DMB lacks relevant variables.

Nonzero error variance

Our next set of simulations concerns the cases where the BF and MBF models are the DGP, where we now allow an error term to enter the model with a nonzero variance. Note that the main difference in the BF model versus the original Bass model lies in the fact that the BF model allows for stochastic variation in the diffusion. In the multivariate case it is exactly the stochastic variation that helps us to identify cross-country influences. Note that we impose zero contemporaneous correlation between the error terms.

In Table 3 we report similar results as we did in the second panel of Table 2. The results in Table 3 are easy to interpret. When the DGP is BF, and the error variance is small, then the MBF model improves the fit on all forecasting horizons. This is of course due to the fact that the MBF model contains more parameters. Qualitatively similar results appear when the MBF model is the DGP. A second important result from these experiments is that when the error variance becomes larger, the differences in fit become smaller. This especially holds for a larger forecasting horizon. Indeed, it might be expected that more variation in the data leads to decreased performance for all models. Once the signal to noise ratio in the data becomes very low, the differences between the models disappear as there is only noise to fit.

In sum, the simulation results in this section clearly show that the new multivariate product growth model outperforms other models in terms of fit, even when it is not the data generating process.

4 Cross country effects of CD diffusions

We now turn to an illustration of the new multivariate model for new product growth, also to demonstrate its relevance for actually observed data.[3] We consider annual time series running from 1983 to 1996 concerning the penetration of CDs in the US, Canada and Japan. The data are taken from (5). The series give the CD sales as a percentage of the total music sales. By looking at percentages we do not have to worry about matching the populations on their population size (7). The data are given in Appendix 1, and the graphs of these three series appear in Figure 1. When we compute the correlation between these series, we get 0.996 for the US with Canada, 0.953 for the US with Japan, and 0.929 for Canada with Japan. These high values already suggest that the PBKS and KK models would run into estimation problems, so these models are not considered here.

We fit the MBF model while allowing for heteroskedasticity. We follow the suggestions in Boswijk and Franses (2005) that dividing the left-hand side and the right-hand side of the equations trough $X_{j,k-1}$ can take care of it, hence we set $\gamma = 1$ in (12) and we do

[3]We also fitted the PBKS and KK models for these data. Detailed results can be obtained from the authors. We do mention though that we were not able to obtain plausible parameter estimates for the key parameters. The relevant Eviews codes and output are also available upon request. The results clearly show that q_{ij} is not estimated as larger than zero. Imposing this nonnegativity restriction further deteriorates the estimates of the parameters.

not follow a formal empirical testing strategy for γ. Hence, the model that we actually estimate is

$$\begin{pmatrix} \frac{\Delta X_{1,k}}{X_{1,k-1}} \\ \frac{\Delta X_{2,k}}{X_{2,k-1}} \\ \frac{\Delta X_{3,k}}{X_{3,k-1}} \end{pmatrix} = \begin{pmatrix} \frac{1}{X_{1,k-1}} & & \\ & \frac{1}{X_{2,k-1}} & \\ & & \frac{1}{X_{3,k-1}} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \times$$

$$\begin{pmatrix} (m_1 - N_{1,k-1})(p_1 + q_1 N_{1,k-1}/m_1) - X_{1,k-1} \\ (m_2 - N_{2,k-1})(p_2 + q_2 N_{2,k-1}/m_2) - X_{2,k-1} \\ (m_3 - N_{3,k-1})(p_3 + q_3 N_{3,k-1}/m_3) - X_{3,k-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1,k} \\ \varepsilon_{2,k} \\ \varepsilon_{3,k} \end{pmatrix}, \quad (25)$$

where $X_{i,k} = N_{i,k} - N_{i,k-1}$. Note that in this model we now do not restrict the maturity level m_i to one.

The estimation results for the three-country MBF model are given in Table 4. The estimation results in Table 4 show that there are effects running from the US to Canada and Japan, as the estimated parameters α_{21} and α_{31} are significant at the 5 per cent level. Both α_{21} as α_{31} are negative, this implies that when the development of the diffusion in the US falls behind the target path the same will happen in Canada and in Japan. The same holds for the reverse case, if the diffusion in the US goes faster than the target path, the diffusion in Canada and Japan will also speed up. Other cross-country effects are insignificant, see Table 4. The implied p, q and m parameters of the standard Bass model also seem reasonable, see the top panel of Table 4.

In sum, we see that the new multivariate version of a Bass model can be fitted quite easily to a trivariate series, that it delivers interpretable and meaningful parameters, and that it has a high in-sample fit.

5 Conclusion

In this paper we have put forward a new multivariate product growth model, and we contrasted it with other such multivariate models using theoretical and simulation-based arguments. Given the outcomes of the simulations, we are tempted to conclude that our new model outperforms its rivals on various dimensions. For the competing models of (15) and (11) we have shown in simulations that the cross-country effects specified in these models are difficult to identify even in the ideal case where simulated data is used.

The empirical illustration shows that interesting conclusions on cross-country influences can be obtained using our new model. To name one conclusion, we have found that there exist asymmetric effects. The USA is independent of other countries, but does influence Japan and Canada.

Further applications should substantiate our claims, and also out-of-sample forecasting contests should even further do so. Together with designing a specification strategy for the best way to incorporate heteroskedasticity, we leave these issues for further research.

Appendix 1: CD diffusion in three countries

Table 1: CD penetration in the USA, Canada and Japan.

Year	USA	Canada	Japan
1983	0.001763	0.000000	0.011117
1984	0.010578	0.007250	0.040811
1985	0.042465	0.019174	0.134696
1986	0.101068	0.048887	0.284236
1987	0.163569	0.119618	0.391979
1988	0.222702	0.173152	0.497629
1989	0.301163	0.230174	0.653189
1990	0.386953	0.336152	0.747037
1991	0.477370	0.451939	0.895274
1992	0.524994	0.527244	0.914487
1993	0.592513	0.578241	0.929264
1994	0.655934	0.663102	0.934292
1995	0.725858	0.747712	0.935567
1996	0.773400	0.787030	0.946600

Source: (5)

Figure 1: Graphical representation of the diffusion data

Other tables

Table 2: Simulation results for the deterministic case ($\sigma = 0$), where DMB denotes the diagonal multivariate Bass model, PBKS is the model in Putsis et al. (1997), KK is the model of Kumar and Krishnan (2002), BF is the univariate model in Boswijk and Franses (2005) and MBF is the new multivariate model.

	Estimated model				
DGP	DMB	PBKS	KK	BF	MBF
	Root mean squared errors				
DMB	0.219	0.201	0.166	0.044	0.039
PBKS	0.223	0.190	0.155	0.082	0.074
KK	0.217	0.201	0.160	0.041	0.037
BF	0.131	0.125	0.099	0.032	0.028
MBF	0.146	0.140	0.118	0.053	0.035
	Root mean squared errors relative to DGP				
DMB	1	0.917	0.757	0.201	0.178
PBKS	1.173	1	0.818	0.431	0.388
KK	1.356	1.257	1	0.254	0.231
BF	4.088	3.899	3.095	1	0.872
MBF	4.162	3.990	3.376	1.527	1

Table 3: Simulation results for the cases with nonzero va
PBKS is the model in Putsis et al. (1997), KK is the mo
Boswijk and Franses (2005) and MBF is the new multivar
numbers give the median RMSE relative to the median R

	One-step ahead					T	
σ	DMB	PBKSD	KK	BF	MBF	DMB	PBK
DGP: BF							
0.00	4.21	4.22	2.38	1	0.81	5.07	5.
0.01	3.93	3.84	2.18	1	0.83	4.28	4.
0.50	1.12	1.12	1.07	1	0.95	1.01	1.
1.00	1.06	1.03	1.03	1	0.96	0.99	0.
2.00	1.04	1.01	1.03	1	0.97	0.98	0.
DGP: MBF							

Table 4: Estimation results for a three-country MBF model, standard errors in parentheses.

	USA	CAN	JPN
	Diffusion characteristics		
p	0.0366	0.0389	0.0935
	(0.0195)	(0.0172)	(0.0335)
q	0.3004	0.3916	0.5141
	(0.0887)	(0.0862)	(0.1016)
m	0.9048	0.8537	0.9411
	(0.1235)	(0.0707)	(0.0117)
	....depends on		
Diffusion in....	USA	CAN	JPN
USA	0.156	0.326	0.135
	(0.253)	(0.217)	(0.107)
CAN	-1.068	1.254	-0.036
	(0.37)	(0.268)	(0.160)
JPN	-0.479	0.048	1.002
	(0.216)	(0.128)	(0.356)

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