Bifurcations from robust homoclinic cycles

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Chapter 1

Introduction

This thesis combines two topics from dynamical systems: global bifurcation theory and equivariant dynamics. The purpose of this introduction is to give some background for these two topics. In the first part of the introduction (sections 1.1–1.2) we discuss global dynamics and bifurcation theory. In particular, we examine the existence of a homoclinic connection as a codimension one phenomenon in parameter space. In the second part (sections 1.3–1.5) we discuss symmetric (or equivariant) systems. In this setting entire networks of heteroclinic connections exist. More specifically, we shall focus on the necessary and sufficient conditions for stability of heteroclinic networks. Furthermore we discuss bifurcations from heteroclinic networks in families of ordinary differential equations. We give an overview of results from the existing literature on global dynamics with symmetry and give a short introduction to our bifurcation results stated in the rest of this thesis.

1.1 Global dynamics

Consider the ordinary differential equation

\[ \frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n. \]  

Assume the differential equation has (at least) two equilibria \( p \) and \( q \). We say that a solution \( h(t) \) of (1.1.1) is a heteroclinic orbit joining \( p \) to \( q \), if \( h(t) \to p \) as \( t \to -\infty \) and \( h(t) \to q \) as \( t \to \infty \). If \( p = q \), we say that \( h(t) \) is a homoclinic orbit to \( p \), assuming that \( h(t) \) is not the equilibrium itself. We also consider heteroclinic cycles which, by definition, consist of heteroclinic orbits \( h_j(t) \), \( j = 1, \ldots, l \), so that

\[ \lim_{t \to -\infty} h_{j-1}(t) = p_j = \lim_{t \to -\infty} h_j(t), \quad j = 1, \ldots, l \]
for distinct equilibria \( p_j \), with the understanding that \( h_0 := h_l \). Here \( l \) is called the length of the cycle.

For a typical vector field \( f \), however, homoclinic orbits and heteroclinic cycles do not exist. This is a direct consequence of the Kupka-Smale theorem. Before giving details about this theorem, we will first define common objects and definitions from dynamical systems. The flow of a point \( x \in \mathbb{R}^n \) after time \( t \) is denoted by \( \phi(x,t) \), which is the (unique) solution of system (1.1.1) that satisfies \( \phi(x,0) = x \). Define the stable set of an equilibrium \( p \), denoted by \( W^s(p) \), as the union of all points for which the forward flow converges to \( p \), i.e.

\[
W^s(p) = \{ x | \lim_{t \to \infty} \phi(x,t) = p \}.
\]  

Likewise we define the unstable set of an equilibrium \( p \), denoted by \( W^u(p) \), as the union of all points for which the backward flow converges to \( p \), i.e.

\[
W^u(p) = \{ x | \lim_{t \to -\infty} \phi(x,t) = p \}.
\]

From the stable manifold theorem it follows that the sets \( W^s(p) \) and \( W^u(p) \) are manifolds if \( p \) is a hyperbolic fixed point:

**Hypothesis 1.1.1 (Hyperbolicity).** The equilibrium \( p \) is hyperbolic if the linearization \( f_x(p) \) has no eigenvalues on the imaginary axis.

**Theorem 1.1.2 (Stable manifold theorem).** Let \( f : U \subset \mathbb{R}^n \to \mathbb{R}^n \) be a smooth map with a hyperbolic fixed point \( p \). Denote by \( W^s(p) \) the stable set and by \( W^u(p) \) the unstable set of \( p \) as defined in equations (1.1.2) and (1.1.3). Then

(i) \( W^s(p) \) is a smooth immersed submanifold and its tangent space at \( p \) is the stable space of the linearization of \( f \) at \( p \).

(ii) \( W^u(p) \) is a smooth immersed submanifold and its tangent space at \( p \) is the unstable space of the linearization of \( f \) at \( p \).

Accordingly \( W^s(p) \) is a stable manifold and \( W^u(p) \) is an unstable manifold.

Note that for a homoclinic orbit \( \gamma(t) \) at \( p \) it holds that \( \gamma(t) \subset W^u(p) \cap W^s(p) \). For a heteroclinic orbit \( \gamma(t) \) joining \( p \) to \( q \) it holds that \( \gamma(t) \subset W^u(p) \cap W^s(q) \). If \( W^u(p) \cap W^s(q) \) is nonempty, then there is a point \( x \) both in the stable and unstable manifold and by definition \( \lim_{t \to -\infty} \phi(x,t) = p \) and \( \lim_{t \to \infty} \phi(x,t) = q \) and thus \( \gamma(t) = \{ \phi(x,t), t \in \mathbb{R} \} \) is a heteroclinic connection between \( p \) and \( q \). It is also clear that if \( W^u(p) \cap W^s(q) = \emptyset \), then there is no heteroclinic connection between \( p \) and \( q \). Moreover, there exists a heteroclinic connection joining \( p \) with \( q \) only if \( W^u(p) \) and \( W^s(q) \) intersect. A discussion of robustness of a heteroclinic orbit is facilitated by the notion of transversality.
1.2. Bifurcation theory

Definition (Transversality). Two sub-manifolds $X$ and $Y$ of a manifold $M$ are said to intersect transversally if for all $q \in X \cap Y$

$$T_qX + T_qY = T_qM.$$ 

In this thesis $M$ is equal to $\mathbb{R}^n$ and thus $T_qM = \mathbb{R}^n$. If $X$ and $Y$ intersect transversally we write $X \triangleleft Y$.

If $W^u(p)$ and $W^s(q)$ intersect transversally, then the heteroclinic orbit persists after a small perturbation. This is because these manifolds will then still intersect.

The Kupka-Smale theorem asserts that for generic $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, stable and unstable manifolds of a hyperbolic equilibrium intersect transversally, if they intersect at all. From this it follows directly that for generic $f$ no homoclinic orbits exist. Here generic means that the theorem only fails for a subset $F \subset C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, which is of the first Baire category in the weak $C^r$ topology. A set is said to be of the first category (in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$) if it is a union of countably many nowhere dense subsets, we refer to Hirsch [1976]. In the weak $C^r$ topology two maps $f$ and $g$ are close to each other if for a finite family of charts the first $r$ derivatives of $f$ and $g$ are close to each other in the supremum norm. For more on the weak $C^r$ topology we refer to Field [2007] and Hirsch [1976]. In this thesis we are only interested in the dynamics on a compact subset and thus the weak Whitney topology is sufficient to consider. In particular, the assertion of the Kupka-Smale theorem holds for a dense subset of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. See Abraham & Robbin [1967] for more details. Another consequence of the Kupka-Smale theorem is that for generic $f$ no heteroclinic cycles exist.

1.2 Bifurcation theory

Consider the ODE (1.1.1) and assume that $p$ is a hyperbolic fixed point and $h(t)$ is a homoclinic orbit to $p$. Suppose we perturb the system while we assume that $p$ remains a hyperbolic fixed point. For an open and locally dense set of small perturbations the homoclinic connection breaks. Because the homoclinic orbit breaks for almost every small perturbation it is more useful (and interesting) to look at families of ODE’s and consider bifurcations of homoclinic orbits. Therefore we consider a family of ordinary differential equations:

$$\frac{dx}{dt} = f(x, \mu), \quad (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m.$$ 

Without loss of generality we assume that the fixed point $p$ does not depend on the parameter $\mu$ and that there is a homoclinic orbit $h(t)$ to $p$ for $\mu = 0$. We also assume hyperbolicity of $p$ for $\mu = 0$ ensuring that $p$ itself does not undergo a bifurcation at $\mu = 0$. 

The homoclinic orbit is contained in both the stable and the unstable manifold of 
\( p \) for \( \mu = 0 \). That is, \( h(t) \) is contained in the intersection of \( W^s(p,0) \) and \( W^u(p,0) \). If this intersection is more dimensional there would be infinitely many homoclinic orbits. This does not happen in the generic case. Consider for some point on \( h(t) \), say \( h(0) \), the tangent spaces to the stable and unstable manifold, i.e. \( T_{h(0)}W^s(p,0) \) and \( T_{h(0)}W^u(p,0) \). The intersection of these tangent spaces contains the line \( \mathbb{R}h(0) \) and typically it is exactly equal to \( \mathbb{R}h(0) \). We will assume this condition and that at \( \mu = 0 \) the homoclinic orbit breaks with nonzero speed in \( \mu \). This is summarized in the following nondegeneracy conditions:

**Hypothesis 1.2.1 (Nondegeneracy).**

(i) **Stable and unstable manifolds intersect as transversally as possible:**

\[
T_{h(0)}W^s(p,0) \cap T_{h(0)}W^u(p,0) = \mathbb{R}h(0).
\]

(ii) **Stable and unstable manifolds unfold generically with respect to the parameter \( \mu \):**

\[
\bigcup_{\mu}(W^s_{\mu}(p),\mu) \cap \bigcup_{\mu}(W^u_{\mu}(p),\mu)
\]

along \( h \times \{0\} \) in \( \mathbb{R}^n \times \mathbb{R}^m \).

When the family of ODE’s is perturbed a little, the above transversality conditions still hold, where 0 is replaced by some value \( \mu_0 \) close to zero. And thus for small perturbations there still exists a homoclinic orbit near \( h(t) \) at an isolated value of \( \mu \) close to zero. Because of this, the bifurcation can be observed in physical systems, although we cannot observe the homoclinic orbit.

We avoid giving a precise definition of the codimension of a bifurcation. But we loosely define the codimension of a bifurcation as the minimal number of parameters necessary for the bifurcation to occur. This corresponds to the codimension of the parameter set for which the bifurcation occurs within the full space of parameters. In this introduction we only consider codimension one and two bifurcations.

The asymptotics of \( h(t) \) for \( |t| \gg 1 \) are, to a large extent, determined by the linearization \( \dot{x}_t = f_x(p,0)x_t \). In particular, the exponential decay rate of \( |h(t) - p| \) will be given by the real part of one of the eigenvalues of the Jacobian \( f_x(p,0) \). The eigenvalues closest to the imaginary axis will therefore typically dominate the asymptotics, as they give the slowest exponential rates. Denote the eigenvalues of \( f_x(p,0) \) by \( \lambda_j \) with \( j = 1, \ldots, n \), repeated with multiplicity and ordered by increasing real part so that

\[
\text{Re } \lambda_1 \leq \text{Re } \lambda_2 \leq \ldots \leq \text{Re } \lambda_k < 0 < \text{Re } \lambda_{k+1} \leq \ldots \leq \text{Re } \lambda_{n-1} \leq \text{Re } \lambda_n.
\]
The eigenvalues $\lambda_j$ with $\text{Re} \lambda_j = \text{Re} \lambda_k$ are called the leading stable eigenvalues, while the leading unstable eigenvalues $\lambda_j$ are those that satisfy $\text{Re} \lambda_j = \text{Re} \lambda_{k+1}$. Note that the leading stable and unstable eigenvalues are not necessarily unique (up to complex conjugation). But generically they will be unique (up to complex conjugation) and simple. Often one of the following conditions on the eigenvalues is assumed:

**Hypothesis 1.2.2 (Leading eigenvalues).** Consider one of the following eigenvalue conditions:

(i) The unique leading unstable eigenvalue $\lambda^u$ is real and simple, and we have $|\text{Re}\lambda^s| > \lambda^u$.

(ii) The leading stable and unstable eigenvalues are unique, real and simple.

(iii) The leading unstable eigenvalue is unique, real and simple. There are precisely two (non-real) leading stable eigenvalues $\lambda^s$ and $\overline{\lambda}^s$, and these are simple.

(iv) The leading stable and unstable eigenvalues are unique (up to complex conjugation) and simple.

We also want to assume conditions on the asymptotic behavior of the homoclinic orbit. Orbits in the stable manifold of $p$, except for orbits in the strong stable manifold, are tangent to the weak stable direction, i.e. the leading stable eigen direction. Generically homoclinic orbits $h(t)$ do not lie in the strong stable or unstable manifolds $W^{ss}(p,0)$ and $W^{uu}(p,0)$, see Homburg & Sandstede [2009]. We use this for the definition of an orbit-flip:

**Definition (Orbit-flip).** The homoclinic orbit is in an orbit-flip configuration inside the stable (resp. unstable) manifold if the homoclinic orbit is contained in the strong stable (resp. unstable) manifold of $p$.

An orbit-flip configuration is illustrated in figure 1.2.1. Another important geometric property associated with a homoclinic orbit $h(t)$ is the inclination property of the stable and unstable manifolds when they are transported along the homoclinic orbit. We define the extended unstable manifold $W^{ls,u}$ (and in a similar way $W^{s,lu}$) as the invariant manifold which is tangent at $p$ to the unstable manifold and the leading stable manifold. We expect the stable manifold and extended unstable manifold to intersect transversally. See figure 1.2.2 for the generic non inclination-flip configuration.

**Definition (Inclination-flip).** The stable manifold along the homoclinic orbit is in an inclination-flip configuration if $W^{ls,u}(p)$ and $W^s(p)$ intersect non-transversally along the homoclinic orbit. The unstable manifold along the homoclinic orbit is in an inclination-flip configuration precisely if $W^{s,lu}(p)$ and $W^u(p)$ intersect non-transversally along the homoclinic orbit.
Figure 1.2.1: On the left side the homoclinic orbit is in an orbit-flip configuration inside the stable manifold. On the right the homoclinic orbit is tangent to the weak stable direction, which is generic.

For a more analytic approach to describe an inclination-flip see Homburg & Sandstede [2009]. An inclination-flip configuration of the stable manifold for $n = 3$ is sketched in figure 1.2.3. The orbit-flip and inclination-flip configurations are non-generic and we therefore assume the following hypothesis:

**Hypothesis 1.2.3 (Inclination and orbit properties).** The following conditions exclude inclination-flip and orbit-flip configurations.

(i) The stable manifold along the homoclinic orbit is not in an inclination-flip configuration, i.e. $W^{ls,u}(p) \nsubseteq W^s(p)$.

(ii) The homoclinic orbit is not in an orbit-flip configuration within the stable manifold, that is, it does not lie in the strong-stable manifold $W^{ss}(p,0)$.

(iii) The unstable manifold along the homoclinic orbit is not in an inclination-flip configuration, i.e. $W^{s,lu}(p) \nsubseteq W^u(p)$.

(iv) The homoclinic orbit is not in an orbit-flip configuration within the unstable manifold, that is, it does not lie in the strong-unstable manifold $W^{uu}(p,0)$.

Because a homoclinic orbit is a codimension one phenomenon we first discuss the case that $\mu \in \mathbb{R}$. For generic one parameter families a homoclinic orbit exists at an isolated parameter value. For $n = 3$ we describe the dynamics close to a homoclinic orbit for two different eigenvalue conditions. In the case that all eigenvalues are real, simple and unique we provide heuristic arguments for a bifurcation theorem.
1.2. Bifurcation theory

Figure 1.2.2: Generically the stable manifold and the extended unstable manifold intersect transversally.

existence of infinitely many periodic orbits and chaotic attractors can be observed in the case of imaginary stable eigenvalues. The homoclinic connection is then called a Shil’nikov saddle focus homoclinic orbit.

1.2.1 Generic one parameter vector fields

In this section we discuss a homoclinic bifurcation in a generic one parameter vector field. It turns out that on one side of the bifurcation there is a periodic orbit close to the homoclinic orbit. We will first define the first return map for a cross section transverse to a periodic orbit. In a similar way we can then define first hit maps along homoclinic orbits.

Consider the system (1.2.4) for $\mu \in \mathbb{R}$. Suppose the system has a periodic orbit $\rho(t)$ for some $\mu$ fixed. Define a cross section $U_q$ transverse to the periodic orbit at some point $q \in \rho(t)$, so that $U_q \cap \rho(t) = q$. Let $\phi(t, x)$ denote the local flow of (1.2.4). There is a neighborhood $V_q \subseteq U_q$ where we can define a map $\tau_q(x) : V_q \rightarrow \mathbb{R}$ such that:

1. $\phi(\tau_q(x), x) \in U_q$ and $\phi(s, x) \notin U_q$ for $0 < s < \tau_q(x)$.

2. $\tau_q$ is bounded from below; i.e., there is a constant $c > 0$ such that $\tau_q(x) > c$ for all $x \in V_q$. 
The function $\tau_q$ is called the first return time to $U_q$. Given such a $V_q$ and $\tau_q$ as above there is a map $\psi_q(x) = \phi(\tau_q(x), x)$ which is called the first return map of (1.2.4) on $V_q$. The eigenvalues of the derivative of $\psi_q(x)$ at $q$ are independent of the choice of $q$ and are called the characteristic multipliers of $\rho(t)$.

**Definition (Hyperbolic periodic orbit).** A periodic orbit $\rho(t)$ is called hyperbolic if the characteristic multipliers of $\rho(t)$ all have norm different from 1.

Note that the periodic orbit $\rho(t)$ is attracting if and only if all characteristic multipliers of $\rho(t)$ have norm smaller than 1.

Because a homoclinic orbit is generically a codimension one phenomenon homoclinic orbits appear for isolated parameter values, say $\mu = 0$. For small parameter values $\mu \neq 0$ the following description holds generically for $m = 1$, see Shil’nikov [1992, 1968]. On one side of the homoclinic orbit, say for $\mu < 0$, there is a periodic orbit $\rho_\mu(t)$. For $\mu \to 0$ the minimal period tends to $\infty$ and the periodic orbit converges to $p \cup h(t)$. This periodic orbit is hyperbolic. For $\mu > 0$ both the periodic orbit and the homoclinic orbit disappear. This is called a ”blue sky catastrophe”. If the unstable manifold is one dimensional the periodic orbit can be stable or unstable depending on the sign of $\lambda^s + \lambda^u$. This is summarized in the following theorem:

**Theorem 1.2.4 (Shil’nikov et al. [2001]).** Let $\dot{x} = f(x, \mu)$ be a one parameter family of ODEs with a homoclinic solution $h(t)$ to a hyperbolic equilibrium $p$ at $\mu = 0$. Assume hypothesis 1.2.1 and 1.2.2(i) are met.
Then there is a unique hyperbolic periodic orbit \( \rho(t) \) existing on one side of \( \mu = 0 \), converging to \( h \) as \( \mu \to 0 \). The dimension of the stable manifold of \( \rho(t) \) is one more than the dimension of the stable manifold of \( p \).

See Shil’nikov et al. [2001] for a proof. We discuss two cases for \( n = 3 \), where we distinguish between real and complex leading eigenvalues. We first discuss the case where the leading stable eigenvalue is real. We assume that the unstable manifold of \( p \) is one dimensional. It follows that there are three real eigenvalues, i.e. \( \lambda^{ss} < \lambda^s < 0 < \lambda^u \). Here \( \lambda^{ss} \) is called the strong stable eigenvalue.

The eigenvalues depend continuously on \( \mu \) and so for \( \mu \) small enough \( \lambda^{ss}(\mu) < \lambda^s(\mu) < 0 < \lambda^u(\mu) \). We will assume that \( \lambda^s(0) + \lambda^u(0) \neq 0 \) so that in a neighborhood of \( \mu = 0 \) we have \( \lambda^s(\mu) + \lambda^u(\mu) \neq 0 \).

In this section we perform model computations assuming that the system is locally linear, i.e. we assume that in a small neighborhood of the equilibrium the system is linear. Without providing full details a first return map is constructed. A periodic orbit near the homoclinic orbit corresponds to a fixed point of the first return map. A reduced bifurcation equation for fixed points will be derived. From this it follows that there is a blue sky catastrophe, as described above, at \( \mu = 0 \).

In case \( \lambda^s(0) + \lambda^u(0) < 0 \) the periodic orbit created at bifurcation is attracting.

For each parameter value of \( \mu \) we can choose coordinates \( x = (x_{ss}, x_u, x_s) \) tangent to the eigenspaces of respectively \( \lambda^{ss} \), \( \lambda^u \) and \( \lambda^s \). Note that we omit writing the \( \mu \) dependence in the eigenvalues and solutions. These coordinates are chosen so that for all \( t \) large enough \( h(t) \) has positive \( x_s \)-coordinate and for \( -t \) large enough \( h(t) \) has positive \( x_u \)-coordinate. We can now define two cross sections in a neighborhood of \( p \)

\[
\Sigma^{in} = \{ x_s = \delta, |x_u| < \delta, |x_{ss}| < \delta \}, \\
\Sigma^{out} = \{ x_u = \delta, |x_s| < \delta, |x_{ss}| < \delta \},
\]

such that the homoclinic orbit intersects both. We scale the system by choosing coordinates \( \tilde{x} = x/\delta \). This does not change the eigenvalues. The cross sections are equal to

\[
\Sigma^{in} = \{ x_s = 1, |x_u| < 1, |x_{ss}| < 1 \}, \\
\Sigma^{out} = \{ x_u = 1, |x_s| < 1, |x_{ss}| < 1 \}.
\]

We define the local first hit map \( \Pi^{loc} : \Sigma^{in} \mapsto \Sigma^{out} \) in a similar way as the first return map at the beginning of this section. The only difference is that we now have two different sections. The time it takes for an initial point on \( \Sigma^{in} \) to flow to \( \Sigma^{out} \) is called the transition time and is denoted by \( \tau \). If we assume that \( t = 0 \) for an initial value \( x_0 \in \Sigma^{in} \) then the transition time is solved from \( x_u(\tau) = 1 \). This gives
\[ \tau = -\frac{1}{\lambda^s} \log |x_u|. \] The local first hit map \( \Pi^{\text{loc}} : \Sigma^{\text{in}} \mapsto \Sigma^{\text{out}} \) is then given by:

\[
\begin{pmatrix}
  x_{ss} \\
  x_u \\
  1
\end{pmatrix}
\mapsto
\begin{pmatrix}
  x_{ss} x_u^{-\lambda^s} \\
  1 \\
  x_u^{-\lambda^s}
\end{pmatrix}.
\]

We will compute the global first hit map \( \Pi^{\text{far}} : \Sigma^{\text{out}} \mapsto \Sigma^{\text{in}} \) and compose it with the local first hit map, which gives the first return map. From the flow box theorem, see Palis & de Melo [1982], it follows that the global first hit map is a diffeomorphism which is at lowest order given by:

\[
\begin{pmatrix}
  x_{ss} \\
  x_u \\
  1
\end{pmatrix}
\mapsto
\begin{pmatrix}
  C_1(\mu) + A_1 x_{ss} x_u^{-\lambda^s} + B_1 x_u^{-\lambda^s} \\
  C_2(\mu) + A_2 x_{ss} x_u^{-\lambda^s} + B_2 x_u^{-\lambda^s}
\end{pmatrix}.
\] (1.2.5)

An inclination-flip of the stable manifold corresponds to \( B_2(0) = 0 \). By assumption of the nonexistence of an inclination-flip we thus have \( B_2(\mu) \neq 0 \) for small \( \mu \). The coordinate \((C_1(\mu), C_2(\mu), 1)\) is the intersection of the unstable manifold of \( p \) with the cross section \( \Sigma^{\text{in}} \). Just considering the two nonconstant components the composition yields the first return map \( \Pi : \Sigma^{\text{in}} \mapsto \Sigma^{\text{in}} \) at lowest order:

\[
\begin{pmatrix}
  x_{ss} \\
  x_u \\
  1
\end{pmatrix}
\mapsto
\begin{pmatrix}
  C_1(\mu) + A_1 x_{ss} x_u^{-\lambda^s} + B_1 x_u^{-\lambda^s} \\
  C_2(\mu) + A_2 x_{ss} x_u^{-\lambda^s} + B_2 x_u^{-\lambda^s}
\end{pmatrix}.
\] (1.2.6)

The generic unfolding condition 1.2.1(ii) for the stable and unstable manifold with respect to \( \mu \) implies that after a possible reparameterization we may assume that \( C_2(\mu) = \mu \). Note that \( \lambda^{ss} < \lambda^s \) implies that \( x_{ss} x_u^{-\lambda^s} \) is higher order compared to \( x_u^{-\lambda^s} \). We now want to solve the fixed point problem \( \Pi(x_{ss}, x_u) = (x_{ss}, x_u) \):

\[
x_{ss} = C_1(\mu) + B_1 x_u^{-\lambda^s} + O(x_{ss} x_u^{-\lambda^s}) + O(x_u^{-2\lambda^s}),
\]
\[
x_u = \mu + B_2 x_u^{-\lambda^s} + O(x_{ss} x_u^{-\lambda^s}) + O(x_u^{-2\lambda^s}).
\] (1.2.7)

We rewrite the first equation in Equation (1.2.7) as

\[
f(x_{ss}, x_u, \mu) = x_{ss} - C_1(\mu) + B_1 x_u^{-\lambda^s} + O(x_{ss} x_u^{-\lambda^s}) + O(x_u^{-2\lambda^s}) = 0. \] (1.2.8)

Note that \( f \) is \( C^1 \) in the variable \( x_{ss} \). The heteroclinic orbit corresponds to the solution \((x_{ss}, x_u, \mu) = (C_1(\mu), 0, 0)\). We compute that

\[ D_{x_{ss}} f(C_1(\mu), 0, 0) = 1 \neq 0. \]
We apply the implicit function theorem, see Berger [1977]. Note that the implicit function theorem as stated in Berger [1977] does not need differentiability of the function \( f \) for all variables, but only for those that we want to solve for. This gives a solution \( x_{ss}(x_u, \mu) \) as a continuous function of \( x_u \) and \( \mu \) so that \( x_{ss}(0, 0) = C_1(\mu) \):

\[
x_{ss} = (C_1(\mu) + B_1 x_u^{-\lambda^u} + O(x_u^{-2\lambda^u}))(1 + O(x_u^{-\lambda^u})))
= C_1(\mu) + B_1 x_u^{-\lambda^u} + O(x_u^{-2\lambda^u}) + O(x_u^{-\lambda^u}).
\]

Insertion into the second equation of Equation (1.2.7) yields the reduced bifurcation equation for the fixed point problem:

\[
x_u = \mu + B_2 x_u^{-\lambda^u} + O(x_u^{-\lambda^u}) + O(x_u^{-2\lambda^u}). \tag{1.2.9}
\]

The existence of periodic orbits for \( \mu < 0 \) or \( \mu > 0 \) depends on the sign of \( B_2(0) \) and the magnitude of \(-\lambda^s(0)/\lambda^u(0)\). For \( B_2(0) \neq 0 \) and \(-\lambda^s(0)/\lambda^u(0) \neq 1 \) there is a small nonzero solution \( x_u^*(\mu) \) of Equation (1.2.9) for either \( \mu < 0 \) or \( \mu > 0 \). This corresponds to a fixed point of the first return map. This fixed point is close to the unstable manifold of the equilibrium \( p \). The solution \( x_u^*(\mu) \) tends to zero if we let \( \mu \) go to zero. It corresponds to a periodic orbit close to the homoclinic orbit (for \( \mu = 0 \)) with a minimal period denoted by \( T(\mu) \). It holds that \( T(\mu) \to \infty \) for \( \mu \to 0 \). If \( \mu \) goes through zero the periodic orbit disappears. Note that everything in the fixed point equation depends on \( \mu \), but for \( \mu \) small the constants and the expression \( \lambda^s + \lambda^u \) do not change sign because of the assumptions.

We now discuss the case that \( B_2(0) > 0 \) and \(-\lambda^s/\lambda^u > 1 \) in more detail. The functions \( x_u \) and

\[
g(x_u) = \mu + B_2 x_u^{-\lambda^u} + O(x_u^{-\lambda^u}) + O(x_u^{-2\lambda^u})
\]

intersect for small \( \mu > 0 \) which follows because the derivative of \( g(x_u) \) goes to zero for \( x_u \downarrow 0 \), see figure 1.2.4. The fixed point we found corresponds to a periodic orbit, see the schematic picture in Figure 1.2.5. The period of this orbit tends to infinity for \( \mu \downarrow 0 \). Moreover, the periodic orbit is attracting. This is because the magnitude of the eigenvalues of the linearization of the first return map at the fixed point are smaller than one. This is done a few times in more detail in chapter 2. For \( B_2(0) > 0 \) we have thus found an attracting periodic orbit for \( \lambda^s(0) + \lambda^u(0) < 0 \) and \( \mu > 0 \). For \( \mu < 0 \), there are no periodic orbits. For \( \lambda^s(0) + \lambda^u(0) > 0 \) there are periodic orbits for \( \mu < 0 \). When \( B_2(0) < 0 \) there are periodic orbits if \( \lambda^s(0) + \lambda^u(0) \) and \( \mu \) are both positive or both negative. Only when both are negative, the periodic orbit is stable, which is what we expect, because then the leading stable eigenvalue is stronger than the leading unstable eigenvalue.

More complicated dynamics can be found when studying saddle-focus homoclinic orbits. That is when the two stable eigenvalues of \( Df(p) \) is a complex pair of
Figure 1.2.4: For $\mu$ small, there is a fixed point $x_u^*$ of Equation (1.2.9). The curve depicting $g(x_u)$ for small values of $x_u$ intersects with the function $x_u$. Eigenvalues and the unstable manifold of $p$ intersects with the stable manifold of $p$. A systematic study of these homoclinic orbits was pioneered by Shil’nikov. Under the assumption that the unstable eigenvalue dominates the pair of complex conjugate stable eigenvalues, infinitely many periodic orbits of saddle type were shown to occur in each neighborhood of the homoclinic orbit, see Shil’nikov [1965] and Tresser [1984]. These periodic orbits are organized in suspended horseshoes accumulating onto the homoclinic orbit, see Shil’nikov [1970]. In fact, the periodic orbits near two coexisting saddle-focus homoclinic orbits under the eigenvalue condition are known to span every possible knot and link type, we refer to Ghrist et al. [1997]. Dynamical features beyond hyperbolic suspended horseshoes, including the existence of periodic and strange attractors accumulating onto the homoclinic orbit, were described in later papers by Gonchenko et al. [1997], Homburg [2002], Ovsyannikov & Shilnikov [1987].

1.2.2 Two parameter vector fields

In this section we consider homoclinic orbits in generic two parameter vector fields, i.e. we consider (1.2.4) for $m = 2$. We write $\mu = (\mu_1, \mu_2)$. Since homoclinic orbits are a codimension one phenomenon, we expect them to occur along one-dimensional curves $\mu = \mu(\tau)$ in two-parameter space. In this way we can track a homoclinic orbit rather than hitting it out of the blue. Suppose there is a homoclinic orbit attached to the hyperbolic fixed point $p$, by following the curve $\mu(\tau)$. Also assume that the eigenvalues of the linearization at $p$ are at resonance for some $\mu = \mu_0$ on this curve, i.e. $\lambda^s(\mu_0) + \lambda^u(\mu_0) = 0$. Furthermore we assume Hypothesis 1.2.2(ii), Hypothesis 1.2.1 and Hypothesis 1.2.3 (expressing a nondegenerate homoclinic orbit.
Figure 1.2.5: A schematic picture of the blue sky catastrophe. In this case $B_2(0) > 0$ and $\lambda^s(0) + \lambda^u(0) < 0$.

and absence of inclination flips or orbit flips). After a suitable reparametrization of the parameter $\mu$ we may assume that there is a homoclinic orbit at $\{\mu_2 = 0\}$ and that the resonance occurs at $\mu = 0$, i.e. $\mu_0 = 0$. It will follow that at $\mu = 0$ there is a codimension two bifurcation. More precise, there are two different bifurcation diagrams possible, depending on the homoclinic orbit being twisted or non-twisted, described below. To distinguish between twisted and non-twisted homoclinic orbits we first define the unit vectors

$$e^\pm := \pm \lim_{t \to \mp \infty} \frac{\dot{h}(t)}{|\dot{h}(t)|}.$$ 

Also define the hyperplanes $T_{h(t)} = T_{h(t)} W^s + T_{h(t)} W^u$. These are codimension one subspaces by Hypothesis 1.2.1(i). For simple real principal eigenvalues, it is a consequence of the strong $\lambda$-lemma, see Chow et al. [1990], Deng [1989], that generically

$$\lim_{t \to -\infty} T_{h(t)} = T_0 W^{ss} \oplus T_0 W^u,$$

$$\lim_{t \to +\infty} T_{h(t)} = T_0 W^s \oplus T_0 W^{uu}. \quad (1.2.10)$$

This implies that for $t = t_+$ large enough and for $t = t_-$ negative enough, we have

$$e^+ \not\in T_{h(t_+)}; \quad e^- \not\in T_{h(t_-)}. \quad (1.2.11)$$

Because $\text{codim} T_{h(t_+)} = 1$ the hyperplane $T_{h(t_+)}$ divides the space $\mathbb{R}^n$ into two half-spaces. The hyperplane $T_{h(t_-)}$ divides $\mathbb{R}^n$ also in two half-spaces. Note that $T_{h(t)}$ for $t^- \leq t \leq t^+$ defines a homotopy from $T_{h(t_-)}$ to $T_{h(t_+)}$. Let $H_1(t_-)$ and $H_2(t_-)$
be the two half-spaces that are divided by $T_{h(t_-)}$. The hyperplane is transported along the homoclinic orbit by the homotopy which induces the transportation of the two half-spaces continuously along the homoclinic orbit. This gives rise to defining $H_1(t_+)$ and $H_2(t_+)$ as the two half-spaces that are divided by $T_{h(t_+)}$. We do this in such a way that $H_i(t_-)$ is transported to $H_i(t_+)$ for $i = 1, 2$. This allows us to define a twisted homoclinic orbit.

**Definition 1.2.5.** Assume 1.2.1, 1.2.2(ii) and 1.2.3 hold. We call the homoclinic orbit $h(t)$ twisted, if

$$e^- \in H_i(t_-) \text{ and } e^+ \in H_j(t_+), \text{ for } i \neq j, \ i, j \in \{1, 2\}.$$  

We call the homoclinic orbit $h(t)$ non-twisted if

$$e^- \in H_i(t_-) \text{ and } e^+ \in H_i(t_+), \ i \in \{1, 2\}.$$  

Note that $B_2(0) > 0$ in Equation (1.2.5) corresponds to a non-twisted homoclinic orbit and $B_2(0) < 0$ corresponds to a twisted homoclinic orbit for $\mu$ small. For twisted and non-twisted homoclinic orbits at resonance there are two different codimension two bifurcations. We first define a period doubling bifurcation. This is a codimension one bifurcation. We therefore consider a curve $\mu(\alpha)$ for $\alpha \in \mathbb{R}$ close to zero with $\mu(0) = 0$.

**Definition (Period doubling bifurcation).** Suppose there is a periodic orbit $p_\alpha(t)$ for $\alpha$ small. There is a period doubling bifurcation at $\alpha = 0$ if for $\alpha$ small enough the following holds:

there is a tubular neighborhood $T$ of $p_0$ so that for a cross section $V$ transverse to $p_0$, $p_\alpha(t)$ intersects $V \cap T$ once for $\alpha \leq 0$ and twice for $\alpha > 0$.

One can also define a homoclinic doubling bifurcation.

**Definition (Homoclinic doubling bifurcation).** Suppose there is a homoclinic orbit $h_\alpha(t)$ for $\alpha$ small. There is a homoclinic doubling bifurcation at $\alpha = 0$ if for $\alpha$ small enough the following holds:

there is a tubular neighborhood $T$ of the closure of $h_0$ so that for a cross section $V$ transverse to $h_0$, $h_\alpha(t)$ intersects $V \cap T$ once for $\alpha \leq 0$ and twice for $\alpha > 0$.

As the following theorem shows, in the case of a twisted resonant homoclinic orbit there is a curve through $\mu = 0$ on which there is a homoclinic doubling bifurcation. The part of this curve that corresponds to the double homoclinic orbit (i.e. after the doubling) is a branch of homoclinic bifurcations in full parameter space. In this thesis we will refer to this branch as a branch of double homoclinic bifurcations. Moreover, there also is a branch of periodic doubling bifurcations at $\mu = 0$. In the case of a non-twisted homoclinic orbit there is an exponentially flat branch of
saddle-node bifurcations of periodic orbits. We only discuss the resonant homoclinic doubling case, where the constant $B_2(0)$ from the global return map is smaller than zero. The following theorem is due to Chow et al. [1990].

**Theorem 1.2.6 (Resonant homoclinic doubling).** Let $f = f(x, \mu)$ smooth enough, possess a generic resonant homoclinic orbit at $\mu_1 = \mu_2 = 0$. Assume this homoclinic orbit is twisted, i.e. $B_2(0) < 0$, then there is a branch of period doubling bifurcations at $\mu = 0$ and a branch of double homoclinic bifurcations at $\mu = 0$. These branches are exponentially flat to the branch of homoclinic bifurcations. More precisely, after a suitable reparametrization the bifurcation diagram looks like in figure 1.2.6.

![Bifurcation diagram](image)

**Figure 1.2.6:** Bifurcation diagram for $B_2 < 0$, i.e. resonant homoclinic doubling. At the line (a) there is a homoclinic bifurcation. There is a period doubling bifurcation at the curve (b). And at the curve (c) there is a double homoclinic bifurcation. Both bifurcations at (a) and (c) are blue sky catastrophes. In the striped areas there is precisely one periodic orbit close to $h(t)$. In the double striped area there are precisely two periodic orbits close to $h(t)$, see Shil’nikov et al. [2001].

To gain more insight in this theorem we look at the first return map as discussed in the previous section. Again we assume that the system is linear close to the equilibria to simplify the computations. We assume that after a reparametrization there is a homoclinic orbit at $\mu_2 = 0$ and that the resonance $\lambda^s + \lambda^u = 0$ occurs at $\mu_1 = 0$. Periodic orbits that get close to the equilibrium twice can be found by
solving the first return map for period two points $x_0$ and $x_1$:

$$x_1 = \mu_2 + B_2 x_0^{1-\mu_1} + \text{h.o.t.},$$

$$x_0 = \mu_2 + B_2 x_1^{1-\mu_1} + \text{h.o.t.},$$

where $x_0$ and $x_1$ must be small. We will assume that $-1 < B_2(0) < 0$ but similar computations can be done for $B_2(0) < -1$.

To find the branch of double homoclinic bifurcations we set $x_0 = 0$, from which it follows that we have to solve:

$$x_1 = \mu_2 + \text{h.o.t.},$$

$$0 = \mu_2 + B_2 x_1^{1-\mu_1} + \text{h.o.t.}$$

For simplicity, we neglect higher order terms and assume that $B_2(\mu)$ is constant. Note that $\mu_2 = x_1 = 0$ solves this equation, which will give the homoclinic bifurcation. The other solution is:

$$\mu_1 = \frac{\ln(-B_2(0))}{\ln \mu_2}.$$ 

This gives a flat branch of double homoclinic bifurcations. Of course we neglected some terms, but considering the graphs of $g_1(x) = x$ and $g_2(x) = \mu_2 + B_2 x^{1-\mu_1} + \text{h.o.t.}$ the existence of a flat branch of double homoclinic bifurcations can also be concluded, see Figure 1.2.7. For fixed small $\mu_1$ and for $\mu_2$ small, the graphs of $g_1$ and $g_2$ are as in the left frame of Figure 1.2.7. From this we conclude that for $\mu_2$ small enough there are values of $\mu_1$ for which the graphs look like in the left frame of Figure 1.2.7 and there are values of $\mu_1$ for which they look like in the right frame of Figure 1.2.7. Then for each $\mu_2 > 0$ there is also a value for $\mu_1$ so that there is a periodic point of length 2 as in the middle frame of Figure 1.2.7. Suppose we have a parameter value $\mu^* = (\mu_1^*, \mu_2^*)$ for which we found this periodic point of order two. We can then look at lower values of $\mu_2$. Simple computations and drawings show that at first there is a periodic point of order two and that finally these two points come together into one fixed point that disappears when $\mu_2$ goes through zero, see Figure 1.2.8. In this way we have found the branch of period doubling bifurcations and again the homoclinic bifurcation.

### 1.3 Equivariant systems

In a general vector field, homoclinic orbits break under small perturbations. In a vector field that is equivariant under the action of a symmetry, global bifurcations can differ substantially from those in general flows. Restricting perturbations to equivariant ones can make the codimension of a bifurcation much smaller and also
1.3. Equivariant systems

Figure 1.2.7: For fixed values of $\mu_2$ there is a value $\mu_1^-$ so that for $\mu = \mu_1^-$ the graphs are as in the left frame. There is also a value $\mu_1^+$ so that for $\mu = \mu_1^+$ they are as in the right frame. Then there must be a value $\mu_1^- < \mu_1^* < \mu_1^+$ so that the graphs are as in the middle frame. The dashed line is the graph of $\mu_2 - B(2)$ and the other two solid lines are $x$ and $\mu_2 - x$. Note that the intersection of $g_2(x)$ with the function $x$ yields a periodic orbit that originated from the homoclinic bifurcation.

limits the possible dynamics one sees in an unfolding. Most important difference (for this thesis) is the existence of robust heteroclinic cycles in equivariant systems. In this thesis we restrict ourselves to systems that are equivariant under the linear action of a finite group. We will address topics like existence, geometry, stability properties and bifurcations of homoclinic networks. To avoid confusion, the term orbit will be used only as the orbit of a group and will not be used for a solution. Instead we will sometimes say trajectory for a solution.

1.3.1 Symmetry

Let $G$ be a finite group with a linear action $x \mapsto gx$ on $\mathbb{R}^n$, $g \in G$. We may assume that $G \subset O(n)$, leaving the standard inner product invariant, and that it has a faithful action, see e.g. Golubitsky et al. [1988]. We consider a family of ordinary differential equations

$$\frac{dx}{dt} = f(x, \mu), \quad (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (1.3.12)$$

This system is called $G$-equivariant, if for all $g \in G$ it holds that $x(t)$ is a solution to (1.3.12) if and only if $gx(t)$ is a solution to (1.3.12). Equivalently, if

$$gf(x) = f(gx), \quad \forall g \in G.$$
Figure 1.2.8: We see that there is a periodic point of order 2 closely after the double homoclinic bifurcation. Eventually this point disappears into the periodic orbit.

For a subgroup $H \subset G$, write

$$\text{Fix}(H) = \{ x \in \mathbb{R}^n \mid gx = x, g \in H \}$$

for the fixed point space of $H$. For $x \in \text{Fix}(H)$ we have $f(x) = f(g(x)) = gf(x)$ for $g \in H$, so the vector $f(x)$ is inside $\text{Fix}(H)$. Thus such a fixed point space is a flow invariant linear space. We can also look at all the symmetries that fix a certain point or solution. Write

$$G_x = \{ g \in G \mid gx = x \}$$

for the isotropy group of $x \in \mathbb{R}^n$.

### 1.3.2 Robust cycles

Equivariant flows can posses heteroclinic cycles connecting equilibria in a robust way. This can occur as symmetry forces subspaces to be invariant, and a heteroclinic solution within an invariant subspace may connect an equilibrium to a second equilibrium that is attracting within this invariant subspace. Small perturbations that respect this symmetry do not break the heteroclinic connection. Note that in general (i.e. not equivariant) systems a single heteroclinic connection can be robust, for example if one of the equilibria is a sink, nevertheless a heteroclinic cycle is not robust in general ODE’s. The first examples of robust heteroclinic cycles were obtained in dos Reis [1984] and Guckenheimer & Holmes [1988]. Note that for non-equivariant systems for which we only assume the existence of invariant spaces, there can also exist robust heteroclinic cycles. In figure 1.3.9 is a system that has multiple robust heteroclinic cycles and it has three flow-invariant (linear) spaces. Recall that
for a $G$-equivariant differential equation $\dot{x} = f(x)$ we defined a heteroclinic cycle to be a collection of heteroclinic trajectories $h_j(t)$ that connects the equilibrium $p_j$ with the equilibrium $p_{j+1}$, $1 \leq j \leq l$ where indices are taken modulo $l$. The isotropy subgroups $G_{h_j(\tilde{t})}$ for $\tilde{t} \in \mathbb{R}$ do not depend on $\tilde{t}$ so that one can speak of the isotropy subgroup of $h_j$. We write $\Sigma_j = G_{h_j}$. Define the fixed point space $P_j = \text{Fix}(\Sigma_j)$. If restricted to $P_j$, $W^u(p_j) \cap P_j$ and $W^s(p_{j+1}) \cap P_j$ intersect transversally, then the connecting trajectories are robust. In this way there can even exist a manifold of heteroclinic connections from $p_j$ to $p_{j+1}$. We will use a stronger assumption that implies robustness of isolated connections:

**Hypothesis 1.3.1 (Robustness properties).** Assume that the following properties are satisfied:

(i) $\dim W^u(p_j) = 1$ and $p_{j+1}$ is a sink for the vector field restricted to $P_j$.

(ii) Each fixed point space $P_j$ is two dimensional.

In the rest of this thesis we will only consider simple heteroclinic cycles.

**Definition (Simple heteroclinic cycle).** A heteroclinic cycle that satisfies Hypothesis 1.3.1 is called a simple heteroclinic cycle.
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In the literature there are some different definitions for networks. These are the definitions we use:

**Definition (Heteroclinic network).** A connected component in the image under $G$ of a heteroclinic cycle is called a heteroclinic network.

More in general a connected invariant set that can be written as a finite union of heteroclinic cycles is called a polycycle.

**Definition (Homoclinic cycle).** A homoclinic cycle is a polycycle that is equal to a group orbit $\langle \gamma \rangle h$ for a heteroclinic trajectory $h$ connecting equilibria $p$ to $\gamma p$ for some $\gamma \in G$.

The element $\gamma \in G$ is called the twist for the homoclinic cycle. Note that the twist is well defined modulo the isotropy group of $p$.

**Definition (Homoclinic network).** A homoclinic network is a connected component of the group orbit $GH$ of a homoclinic cycle $H$.

We will see that for different twists, there are different type of homoclinic cycles inside the same homoclinic network. These different homoclinic cycles become important in higher dimensional ($> 3$) homoclinic networks. For four dimensions this is discussed in section 1.3.4.

### 1.3.3 Classification and explicit examples in $\mathbb{R}^3$

In this section we discuss the classification of simple homoclinic cycles in $\mathbb{R}^3$. Together with $\mathbb{R}^4$ these are the only dimensions in which a complete classification exists in the literature. In section 1.3.4 we discuss the classification for $\mathbb{R}^4$. The classification is given in Chapter 2. There are two different homoclinic networks for ODE’s in $\mathbb{R}^3$, one with two and one with six equilibria. Identifying twists that generate cycles that can be mapped onto each other, there are only two different twists and thus two types of cycles for each network. The classification of simple homoclinic cycles in $\mathbb{R}^3$ is in Table 1.1 listing the number of heteroclinic trajectories $l$ and the symmetry group $G$ of the network that contains the cycle. Note that the length of a homoclinic cycle times the number of homoclinic cycles of that length in the homoclinic network is equal to two times the number of heteroclinic connections in the homoclinic network. For example if $G = \mathbb{Z}_2 \times \mathbb{Z}_2^2$, then there are four cycles of length 2 and two cycles of length 4. Figure 1.3.10 contains pictures of the four different homoclinic cycles in $\mathbb{R}^3$. The symmetry group for the cycles in the two top frames equals $G = \mathbb{Z}_2 \times \mathbb{Z}_2^2$ where $\mathbb{Z}_2$ is generated by $(x, y, z) \mapsto (-x, z, y)$ and $\mathbb{Z}_2^2$ is generated by $(x, y, z) \mapsto (x, \pm y, \pm z)$. The two equilibria are on the $x$-axis. Depending on the twist $\gamma$, we find four homoclinic cycles with two heteroclinic connections (if
Table 1.1: Classification of simple homoclinic cycles in $\mathbb{R}^3$.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2^3$</td>
</tr>
<tr>
<td>4</td>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2^3$</td>
</tr>
<tr>
<td>3</td>
<td>$\mathbb{Z}_3 \times \mathbb{Z}_2^3$</td>
</tr>
<tr>
<td>6</td>
<td>$\mathbb{Z}_3 \times \mathbb{Z}_2^3$</td>
</tr>
</tbody>
</table>

$\gamma(x, y, z) = (-x, z, y)$ or two heteroclinic cycles with four heteroclinic connections (if $\gamma(x, y, z) = (-x, z, -y)$). Note that the flow can only follow the cycles of length 2 and not those with length 4. An explicit family of ODEs which contains such a homoclinic network is given by

$$
\begin{align*}
\dot{x} &= \nu x + z^2 - y^2 - x^3 + \beta x(y^2 + z^2), \\
\dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2) + yx, \\
\dot{z} &= z(\lambda + az^2 + by^2 + cx^2) - zx,
\end{align*}
$$

for $\nu > 0$ small, $\lambda \in (\lambda_H(\nu), \sqrt{\nu} + c\nu)$ for some $\lambda_H(\nu) = -\frac{1}{2}\nu + O(\nu)$, see Sandstede & Scheel [1995]. The homoclinic cycle connects the equilibria $(\pm \sqrt{\nu}, 0, 0)$. This homoclinic network is asymptotically stable if $c < -\lambda/\nu$, $\mu < 0$ and $|d\sqrt{\nu}| < |\nu|$. This follows from stability results stated in section 1.4. We choose the constants so that they satisfy the conditions for asymptotic stability and numerically solve the equations in Mathematica. The solutions as given by Mathematica are given in Figure 1.3.11. There is fast convergence to the homoclinic network. Because some of the coordinates quickly become smaller than the computer precision, the solution as solved by mathematica follows the entire network. Of course this is not a correct picture, but it shows the fast convergence and it also shows the network.

The symmetry group for the other two pictures in Figure 1.3.10 is $G = \mathbb{Z}_3 \times \mathbb{Z}_2^3$ where $\mathbb{Z}_3$ is generated by $(x, y, z) \mapsto (y, z, x)$ and $\mathbb{Z}_2^3$ is generated by $(x, y, z) \mapsto (\pm x, \pm y, \pm z)$. Using the two different twists we find that there are eight heteroclinic cycles with three heteroclinic connections and four heteroclinic cycles with six heteroclinic connections. An example of a $\mathbb{Z}_3 \times \mathbb{Z}_2^3$-equivariant family of ODEs containing such a network is given by

$$
\begin{align*}
\dot{x} &= x(\lambda + ax^2 + by^2 + cz^2), \\
\dot{y} &= y(\lambda + ay^2 + bz^2 + cx^2), \\
\dot{z} &= z(\lambda + az^2 + bx^2 + cy^2),
\end{align*}
$$

with $a < 0$ and $\lambda > 0$. This ODE has a robust homoclinic cycle if and only if $b < a < c$ or $c < a < b$, see Guckenheimer & Holmes [1988]. This homoclinic
network is asymptotically stable if $2a > b + c$, which follows from stability results in section 1.4.

### 1.3.4 Classification in $\mathbb{R}^4$

Simple cycles in $\mathbb{R}^4$ are distinguished in three types. Of these three types the type B homoclinic cycles are those homoclinic cycles contained in a three dimensional fixed point space. Following Chossat et al. [1997] we adopt the following classification for heteroclinic connections in a simple heteroclinic cycle in $\mathbb{R}^n$.

**Hypothesis 1.3.2 (Type A, B, C homoclinic cycles).** The following configurations for simple heteroclinic trajectories in $\mathbb{R}^n$ are distinguished:

(i) **Type A**: $P_j + P_{j+1}$ is not a fixed point space.

(ii) **Type B**: $P_j + P_{j+1}$ is a fixed point space containing the homoclinic cycle.
Figure 1.3.11: Numerical results for $\nu = 0.1, \beta = 1, \lambda = 0.1, a = 1, b = 1, c = -1.2, \mu = -0.2, d = 0.5$. Note that the entire network is revealed because of the (lacking) computer precision.

(iii) Type C: $P_j + P_{j+1}$ is a fixed point space not containing the homoclinic cycle

It turns out that for simple heteroclinic cycles in $\mathbb{R}^4$ all equilibria are of the same type. We therefore have three different types of simple heteroclinic cycles in $\mathbb{R}^4$ which are also denoted by the letters A, B and C. For $n > 4$ there can be equilibria of different types within a simple heteroclinic network. This makes determining all types of cycles, including their main properties, more elaborate for higher dimensions ($n > 4$). A description of different simple homoclinic cycles of type B and C in $\mathbb{R}^4$ is contained in Krupa & Melbourne [2004]. Following Krupa & Melbourne [2004], Sottocornola [2002, 2003, 2005] we give a classification of simple homoclinic cycles in $\mathbb{R}^4$ in table 2.1 in section 2.1. We remark that for all different networks there will be two different twists which gives rise to two different groups of cycles. In $\mathbb{R}^3$ only the shortest cycles can be followed by the flow. For homoclinic (and heteroclinic) networks in $\mathbb{R}^4$ the flow can follow both types of cycles. For each ODE only one type of cycle is followed. The distinction between the different twists is thus more important for $n > 3$. 
1.4 Asymptotic stability

For the networks that we consider, the unstable separatrices of equilibria in the network form heteroclinic connections in the network. Thus points close to the network that flow along the unstable manifold of an equilibrium can stay close to the network. While in general systems a heteroclinic cycle will not attract all nearby points this is possible for a heteroclinic network. The stability of the network depends only on the eigenvalues of the linearization at the equilibria. In homoclinic networks all equilibria are symmetry related so that at each equilibrium we have the same eigenvalues.

Consider a robust (simple) homoclinic network in $\mathbb{R}^4$, i.e. we assume hypothesis 1.3.1 is met. The geometry given by the invariant subspaces $P_i$ allows us to divide the spectrum of $Df(p_i)$ into four classes. We set $p = p_j$ and $P = P_j$ for some $j$.

1. Radial eigenvalue $\lambda_r$: the negative eigenvalue restricted to $V(r) = h^{-1}P \cap P$.

2. Contracting eigenvalue $\lambda_c$: the negative eigenvalue restricted to $V(c) = h^{-1}P \oplus V(r)$.

3. Expanding eigenvalue $\lambda_e$: the positive eigenvalue restricted to $V(e) = P \ominus V(r)$.

4. Transverse eigenvalue $\lambda_t$: the (negative) eigenvalue restricted to $V(t) = (h^{-1}P + P)^\perp$.

More general for systems in $\mathbb{R}^n$ with $n > 4$ one can define radial, contracting, expanding and transverse eigenvalues as in paragraph 2.1. We assume that the transverse eigenvalue is negative. For a positive transverse eigenvalue the homoclinic cycle is automatically unstable due to two-dimensional unstable manifolds at each equilibrium. Krupa & Melbourne [1995, 2004] showed under which eigenvalue conditions simple heteroclinic cycles are asymptotically stable. For homoclinic cycles in $\mathbb{R}^4$ this theorem reduces to:

**Theorem 1.4.1.** Let $\dot{x} = f(x)$ be a $G$-equivariant ODE on $\mathbb{R}^4$, for a finite group $G$ acting linearly. Suppose it admits a simple homoclinic network $G\Gamma$. Assume there are $C^1$ linearizing coordinates near the equilibria in $G\Gamma$.

Then $G\Gamma$ is asymptotically stable precisely if $-\lambda_c > \lambda_e$ in case $G\Gamma$ is of type $A$ or $B$ and $-\lambda_c - \lambda_t > 1$ in case $G\Gamma$ is of type $C$.

In Chapter 2 a resonance bifurcation is considered, where $-\lambda_c - \lambda_t$ goes through one and the homoclinic cycle changes stability. If we assume that we are close to the resonance $-\lambda_c - \lambda_t = 1$, then we can drop the assumption of $C^1$ linearizing coordinates near the equilibria in Theorem 1.4.1. This is proven in Chapter 2.
1.5 Bifurcations and essential asymptotic stability

In this section we introduce bifurcations, studied in detail in the following chapters, where an asymptotically stable homoclinic network loses its stability. We consider a family of $G$-equivariant ODE’s

\[ \dot{x} = f(x, \mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R} \]  

(1.5.13)

that has a robust simple homoclinic network for all $\mu$ near 0. There are two ways for the simple homoclinic network to become unstable. Assume that for $\mu < 0$ the homoclinic network is asymptotically stable, i.e. $-\lambda_c > \lambda_e$ and $\lambda_t < 0$. If for $\mu > 0$ it holds that $-\lambda_c < \lambda_e$ and $\lambda_t < 0$ we speak of a resonance bifurcation at $\mu = 0$. Note that at $\mu = 0$ there is a resonance, i.e. $-\lambda_c = \lambda_e$. The other way of having a bifurcation from an asymptotically stable homoclinic network to an unstable one is when the transverse eigenvalue goes through zero. This is called a transverse bifurcation. After the bifurcation, the homoclinic network may still have some strong attracting property called essential asymptotic stability. Furthermore, all points in a small neighborhood, except for a codimension one manifold, are attracted to the homoclinic network. The resonance bifurcation is discussed in chapter 2 and the transverse bifurcation in chapter 3.

1.5.1 Resonance bifurcation

In Chapter 2 we discuss the resonance bifurcation. We show the existence of periodic orbits close to the network. In this section, as an illustration, we treat the resonance bifurcation for homoclinic networks of type B. We will follow the general (nonsymmetric) case, where we used the first return map to derive a reduced bifurcation equation. For full details, see Chapter 2.

We choose local coordinates $(u, v, w, z)$ as respectively the radial, contracting, expanding and transverse direction. We reduce to the three dimensional invariant subspace \{z = 0\} that contains the network, i.e. we look at homoclinic networks in $\mathbb{R}^3$. Asymptotic stability in this subspace implies asymptotic stability in $\mathbb{R}^4$ if $\lambda_t < 0$.

An important difference in the first return map for an equivariant system with the first return map of a general system is that symmetry forces coefficients in the expansion to be zero. Recall the first return map at lowest order, see equation (1.2.6). In the general (nonsymmetric) case we assumed that the heteroclinic connections were tangent to the weak unstable manifold. We drop this assumption and so there is no restriction on the relative sizes of $\lambda_c$ and $\lambda_r$. It becomes necessary to take a larger cross section $\Sigma^{in}$:

\[ \Sigma^{in} = \{(u, v) | < \delta, |w| < \delta\}. \]
We write \((u^*, v^*, 0)\) for the point where the heteroclinic orbit intersects \(\Sigma^{in}\).

We will write the local system in a suitable normal form, noting that smooth linearizations are in general not possible due to the resonance conditions. Then we derive asymptotic expansions for the first return map, which we use to find a periodic trajectory close to the homoclinic network. The differential equations are locally smoothly equivalent to differential equations of the form

\[
\begin{align*}
\dot{u} &= \lambda_r u + P(u, v, w), \\
\dot{v} &= \lambda_c v + vwQ(u, v, w), \\
\dot{w} &= \lambda_e w,
\end{align*}
\]

where \(P(u, v, w) = O(||(u, v)||/(u, v, w)||)\), \(Q(u, v, w) = O(||(u, v)||)\). These equations are \(G\)-equivariant and smoothly dependent on the parameter \(\mu\). The first return map \(\Pi : \Sigma^{in} \to \Sigma^{in}\) has asymptotic expansions

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} \xrightarrow{\Pi} \begin{bmatrix}
u^* + O(w^\Theta) \\
v^* + O(w^\Theta) \\
\gamma vw^\eta + O(w^{\eta+\Theta})
\end{bmatrix},
\]

for \(\eta(\mu) = -\lambda_c/\lambda_e\) and some \(\Theta > 0\). There are similar expansions for the derivatives.

Assume the unfolding condition \(\frac{\partial}{\partial \mu} \eta(\mu)|_{\mu=0} < 0\) holds. This enables a reparameterization \(\eta = 1 - \mu\) for \(\mu\) close to zero. The first return map can now be written as

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} \xrightarrow{\Pi_*} \begin{bmatrix}
u^*(\mu) + O(w^\Theta) \\
v^*(\mu) + O(w^\Theta) \\
\Psi(v, \mu)w^{1-\mu} + O(w^{1-\mu+\Theta})
\end{bmatrix}.
\]

To find periodic orbits close to the homoclinic cycle we want to solve the fixed point problem \((u, v, w) - \Pi_* (u, v, w) = 0\). The homoclinic orbit is given by the solution \((u^*(\mu), v^*(\mu), 0)\). We find another solution by applying the implicit function theorem, see Chapter 2. This leads to the reduced bifurcation equation for the fixed point map.

\[
w = \Psi^*(\mu)w^{1-\mu} + \text{h.o.t.},
\]

where \(\Psi^*(\mu) = \Psi(v^*(\mu), \mu)\). Depending on the size of \(\Psi^*(0)\) this gives a fixed point that is either attracting or repelling. We summarize this in a theorem. For a homoclinic cycle or network \(\Gamma\), write \(|\Gamma|\) for the number of connecting trajectories in \(\Gamma\). Note that \(|G\Gamma|\) is two times the number of equilibria in \(G\Gamma\), as each equilibrium counts two outgoing connecting trajectories (from the two unstable separatrices).

**Theorem 1.5.1.** Let \(\dot{x} = f(x, \mu)\) be a one parameter family of \(G\)-equivariant smooth differential equations in \(\mathbb{R}^3\), possessing for \(\mu = 0\) a simple homoclinic network \(G\Gamma\). Choose \(\Gamma\) such that \(|\Gamma|\) is minimal. Suppose that \(G\Gamma\) is asymptotically stable for
1.5.3 Essential asymptotic stability

When a (simple) heteroclinic network has a two-dimensional unstable manifold at an equilibrium the network is automatically unstable. Melbourne [1991] discusses
an example of a robust heteroclinic network that is unstable because of this two-dimensional unstable manifold at one of the two equilibria in that network. He shows that this network still has some strong attracting property called essential asymptotic stability.

**Definition** (Brannath [1994], Melbourne [1991]). A flow-invariant compact set $H$ is *essentially asymptotically stable* if for any open neighborhood $U$ of $H$ there is a set $C$ so that for any given number $a \in (0, 1)$ there is an open $\varepsilon$-neighborhood $V \subseteq U$ of $H$ such that all trajectories starting in $V - C$ remain in $U$ and are asymptotic to $H$ and $\mu(V - C)/\mu(V) > a$, where $\mu$ is the Lebesgue measure.

The heteroclinic network in the example of Melbourne exists on a three dimensional invariant sphere in a system with broken spherical symmetry, see Lauterbach & Roberts [1992]. In figure 1.5.13 part of the heteroclinic cycle is shown. Also the eigenvalues at the equilibria are depicted, where $-c_i < 0$ are the contracting eigenvalues and $e_i > 0$ are the expanding eigenvalues. Note that there is one positive transverse eigenvalue $t_e$ and one negative transverse eigenvalue $-t_c$. We will only use this notation for this example. In this example the essential asymptotic behav-

Figure 1.5.13: The heteroclinic network discussed by Melbourne.
Theorem 1.5.2. Suppose that these conditions hold:
\[ c_1c_2 > e_1e_2, \]
\[ (e_1 - tc) c_2 > e_1e_2, \]
\[ t_c > c_2, \]
then generically, there exists a cuspoidal region \( C \) abutting the heteroclinic connections joining \( p_2 \) to \( p_1 \) with the following property: given any neighbourhood \( U \) of the heteroclinic cycle, there is an open neighbourhood \( V \) such that trajectories starting in \( V - C \) remain in \( U - C \) in forward time and are asymptotic to the heteroclinic cycle.

Note that in this example the set \( C \) from the definition of essential asymptotically stability is independent of \( U \).

In chapter 3 we will show that there can also be essentially asymptotically stable homoclinic networks. Because of the symmetry between equilibria all unstable manifolds have the same dimension. It is thus obvious that we need more than one transverse direction. We construct this 5-dimensional example of an essentially asymptotically stable homoclinic network by starting with the three dimensional example from section 1.3.3 with \( G = \mathbb{Z}_2 \rtimes \mathbb{Z}_2^2 \). We add two dimensions (\( u \) and \( v \) coordinates), where the symmetry group is extended nontrivially, so that the original three dimensional space is flow invariant. Let \( G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4 \) be generated by
\[
\begin{align*}
g_1 : (x, y, z, u, v) &\to (x, y, -z, u, v), \\
g_2 : (x, y, z, u, v) &\to (x, -y, z, u, v), \\
g_3 : (x, y, z, u, v) &\to (x, y, z, u, -v), \\
h : (x, y, z, u, v) &\to (-x, z, y, -v, u).
\end{align*}
\]
When one of the transverse eigenvalues is positive (say \( \lambda_u \)) and the other is negative the cycle can be essentially asymptotically stable. The exact construction is done in chapter 3.

Theorem 1.5.3. Generically, this example of a homoclinic network is essentially asymptotically stable if \( 0 < \lambda_u/\lambda_y < \min\{\lambda_u/\lambda_z, 1\} \), \( \lambda_u < 0 \) and \( -\lambda_z/\lambda_y > 1 \).

Explicit equations for such an ODE are obtained by taking the explicit equations from section 1.3.3 and adding the following equations
\[
\begin{align*}
\dot{u} &= \mu u + dux, \\
\dot{v} &= \mu v - dvx.
\end{align*}
\]
The homoclinic network in this example is essentially asymptotically stable for \( \mu < 0 \), \( c < -\lambda/\nu \), \( d < \mu/\sqrt{\nu} \) and \( \mu - d\sqrt{\nu} < \lambda + c\nu + \sqrt{\nu} \).
1.5.3 Transverse bifurcation and attractor

Consider the example as constructed in the previous section. We consider a transverse bifurcation, i.e. the transverse eigenvalue goes through zero. The homoclinic network loses its asymptotic stability and becomes essentially asymptotically stable. The bifurcation is a pitchfork bifurcation from which two new equilibria are created at each equilibrium. There are connections from the old to the new equilibria but there are also connections from the new to the old equilibria which is due to flow invariant spaces that are forced by the symmetry. Schematically half of the network is drawn in Figure 1.5.14.

We let the system depend on a parameter $\varepsilon$

$$\dot{x} = f(x, \varepsilon)$$

and assume there is a transverse bifurcation at $\varepsilon = 0$, in which the transverse eigenvalue $\lambda_v$ goes through zero. We assume the unfolding condition $\frac{\partial}{\partial \varepsilon} \lambda_v(\varepsilon)|_{\varepsilon=0} > 0$ and also that the transverse bifurcation results from a supercritical pitchfork bifurcation. The equilibria created at $p_1$ together with their unstable manifolds are inside the flow invariant space

$$Q = \{z = u = 0\}.$$ 

These unstable manifolds form a heteroclinic connection to $p_2$. In chapter 3 we prove the following theorem

**Theorem 1.5.4.** Generically, for $\varepsilon > 0$ small enough the heteroclinic network $A = G(W^u(p_1') \cup W^u(p_1))$ is an attractor. Furthermore there is an open invariant neighborhood $U$ of $G\Gamma$, so that for $\varepsilon > 0$ small enough, trajectories starting at $x \in U - GW^s(p_1')$ converge for positive time to $G\Gamma$. 

![Figure 1.5.14: The heteroclinic network after bifurcation.](image-url)