Defects and degeneracies in supersymmetry protected phases

Fokkema, T.B.; Schoutens, C.J.M.

Published in:
Europhysics Letters

DOI:
10.1209/0295-5075/111/30007

Citation for published version (APA):
Defects and degeneracies in supersymmetry protected phases

Thessa Fokkema\textsuperscript{1,*} and Kareljan Schoutens\textsuperscript{1,2}\textsuperscript{*}

\textsuperscript{1} Institute for Theoretical Physics, University of Amsterdam Science Park 904, 1098 XH Amsterdam
\textsuperscript{2} Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP

(Dated: v3, October 8, 2015)

We analyse a class of 1D lattice models, known as $M_k$ models, which are characterised by an order-$k$ clustering of spin-less fermions and by $\mathcal{N} = 2$ lattice supersymmetry. Our main result is the identification of a class of (bulk or edge) defects, that are in one-to-one correspondence with so-called spin fields in a corresponding $\mathbb{Z}_k$ parafermion CFT. In the gapped regime, injecting such defects leads to ground state degeneracies that are protected by the supersymmetry. The defects, which are closely analogous to quasi-holes over the fermionic Read-Rezayi quantum Hall states, display characteristic fusion rules, which are of Ising type for $k = 2$ and of Fibonacci type for $k = 3$. Possible choices for the $\lambda_{a,b,j}$ have been studied in \cite{10,11}. Here we choose for the $M_2$ model

$$\lambda_{[1,1],j} = \sqrt{2} \lambda_j, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = \lambda_j,$$

where $\lambda_j$ is a fermionic creation operator which creates a particle at lattice site $j$ in such a way that a string of $a$ particles is formed, with the newly created particle at position $b$. Choosing the $\lambda_{a,b,j}$ such that $(Q^+)^2 = 0$, we define a $\mathcal{N} = 2$ supersymmetric hamiltonian through

$$H = \{Q^+, Q^-\},$$

with $Q^- = (Q^+)\dagger$. This hamiltonian combines hopping terms with local potential and interaction terms. By construction, $[H, Q^+] = [H, Q^-] = 0$. All states in the spectrum are doublets with $[f, f+1]$ particles, with the exception of the supersymmetric groundstates at $E = 0$, which are annihilated by both $Q^+$ and $Q^-$. Possible choices for the $\lambda_{a,b,j}$ have been studied in \cite{10,11}. Here we choose for the $M_2$ model

$$\lambda_{[1,1],j} = \sqrt{2} \lambda_j, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = \lambda_j,$$

where $\lambda_j$ is a fermionic creation operator which creates a particle at lattice site $j$ in such a way that a string of $a$ particles is formed, with the newly created particle at position $b$. Choosing the $\lambda_{a,b,j}$ such that $(Q^+)^2 = 0$, we define a $\mathcal{N} = 2$ supersymmetric hamiltonian through

$$H = \{Q^+, Q^-\},$$

with $Q^- = (Q^+)\dagger$. This hamiltonian combines hopping terms with local potential and interaction terms. By construction, $[H, Q^+] = [H, Q^-] = 0$. All states in the spectrum are doublets with $[f, f+1]$ particles, with the exception of the supersymmetric groundstates at $E = 0$, which are annihilated by both $Q^+$ and $Q^-$. Possible choices for the $\lambda_{a,b,j}$ have been studied in \cite{10,11}. Here we choose for the $M_2$ model

$$\lambda_{[1,1],j} = \sqrt{2} \lambda_j, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = \lambda_j,$$
with the \( \lambda_j \) staggered as \( \ldots 1\lambda\lambda\lambda\ldots \). The factor \( \sqrt{2} \) guarantees that the model is integrable [11] and, if \( \lambda = 1 \), critical [10].

For the general \( M_k \) model we choose parameters describing a critical point perturbed by a specific, integrable, staggering [12]. The staggering, with lattice periodicity \( k + 2 \), connects the critical regime with one of ‘extreme staggering’ \( \lambda \ll 1 \), where the \( \lambda_{a,b,j} \) follow a simple pattern.

For \( k = 3 \), to lowest order in \( \lambda \),

\[
\begin{align*}
\lambda_{[1,1],j} : & \quad 1 \sqrt{2} \sqrt{2}\lambda \sqrt{2} 1 \ldots \\
\lambda_{[2,1],j} : & \quad 1 1 \lambda \sqrt{2} \lambda \ldots \\
\lambda_{[2,2],j} : & \quad \lambda \sqrt{2} \lambda 1 1 \lambda \ldots \\
\lambda_{[3,1],j} : & \quad 1 \lambda \lambda 1 \lambda/\sqrt{2} \ldots \\
\lambda_{[2,3],j} : & \quad \lambda/\sqrt{2} 1 \lambda^2/\sqrt{2} 1 \lambda/\sqrt{2} \ldots \\
\lambda_{[3,3],j} : & \quad \lambda/\sqrt{2} 1 \lambda \lambda 1 \ldots 
\end{align*}
\]

(4)

with the dots indicating repetition modulo 5. We denote this as \( \ldots \star \lambda \star \ldots \), with the ‘\( \lambda \)’ indicating the central position in the staggering pattern.

The Witten index for the \( M_k \) model with periodic boundary conditions (PBC) and with \( L = l(k + 2) \) sites is \( W_k = k + 1 \); indeed, for \( \lambda > 0 \) the models have precisely this number of supersymmetric groundstates, all at \( E = 0 \) and filling \( \nu = k/(k + 2) \) [10, 13]. They are protected against perturbations that commute with supersymmetry and do not affect the \( k \)-clustering constraints. For open BC there are either zero or a single supersymmetric groundstate with \( E = 0 \), the latter for \( L \equiv 0 \pmod{(k + 2)} \). We find, however, that in the presence of suitable boundary or bulk defects, the open systems have states with energies that are exponentially suppressed, \( E \propto e^{-\alpha L_i} \), with \( L_i \) characteristic distances among defects and boundaries. The exponential degeneracies are protected by supersymmetry.

**M₂ model.** - We now zoom in on the \( M_2 \) model on an open chain. At criticality (\( \lambda = 1 \)), the finite size spectra can be matched with those of the 2nd minimal model of \( N = 2 \) superconformal field theory (CFT), of central charge \( c = \frac{3}{2} \). The match can be made with the help of numerical spectra (we analysed open chain spectra up to length \( L = 25 \), fig. 1) and are similar to the results of [14] for the \( M_1 \) model. The relevant CFT modules are \( V_m, \psi V_m \) with \( m \in \mathbb{Z} + \frac{1}{2} \) and \( \sigma V_m \) with \( m \in \mathbb{Z} \). Here the \( V_m \) are charge \( m \) vertex operators for a \( c = 1 \) scalar field and the \( \psi, \sigma \) arise from the \( c = \frac{3}{2} \) Ising CFT factor. States in \( V_m \) have an even number of \( \psi \)-modes while those in \( \psi V_m \) contain an odd number. The supercharges are the (Ramond sector) zero modes of the supercurrents \( \psi V_{\pm 2}(z) \).

Up to an overall \( 1/L \) scaling, the lattice model energies correspond to \( E_{\text{CFT}} = L_0 - \frac{1}{16} \). The lowest energies are \( (4m^2 - 1)/16 \) for \( V_m \), \( (4m^2 + 7)/16 \) for \( \psi V_m \) and \( m^2/4 \) for \( \sigma V_m \). The lowest-energy states in \( V_{\pm \frac{1}{2}} \) and \( \sigma V_0 \) are supersymmetry singlets with \( E_{\text{CFT}} = 0 \), all other states have \( E_{\text{CFT}} > 0 \) and pair up into doubles. The critical \( M_2 \) spectrum with open BC is easily described: with \( m = 2f - L - \frac{1}{2} \), one finds the CFT modules \( V_m \) for \( f \) even and \( \psi V_m \) for \( f \) odd (fig. 1).

Reducing \( \lambda \) below 1 sends the theory off criticality, with RG flow leading to the supersymmetric sine-Gordon (ssG) theory at coupling \( \beta^2 = 8\pi \) [11]. The \( M_2 \) model off-critical finite size spectra can be analysed in terms of ssG bulk \( S \)-matrices and boundary reflection matrices [15]. The ssG theory holds important clues for the topological aspects of the \( M_2 \) model degeneracies [15–17].

Rather than following the RG flow, we will here consider the limit \( \lambda \ll 1 \) (‘extreme staggering’), where the \( M_2 \) eigenstates approach a simple factorized form. This is analogous to a special tuning in the Kitaev chain, which leads to perfectly decoupled Majorana edge states [9]. This simple setting enables us to demonstrate how different BC result in exponential ground state degeneracy beyond this idealised limit. The \( \lambda \ll 1 \) limit is also similar to the thin-torus limit of the MR and RRk quantum Hall states [18–20]. Indeed, the systematics of the fusion channel degeneracies is highly analogous between the two settings.

For \( \lambda = 0 \) and PBC, the \( M_2 \) groundstates are

\[
|\ldots - \rangle = \ldots 0(\cdot 1)0(\cdot 1)\ldots , \quad |\ldots + \rangle = \ldots 1(\cdot 1)0(\cdot 1)\ldots ,
\]

\[
|0\rangle = \ldots 1010101\ldots ,
\]

where \( (\cdot 1) = 110 + 011 \) and \( |\ldots - \rangle \) and \( |\ldots + \rangle \) are related by a shift over two lattice sites. For open BC, there is at the most a single \( E = 0 \) groundstate for given particle number \( f \). For \( L = 4l - 1 \), staggering \( 1\lambda \ldots \lambda 1, f = 2l \),

\[
|\ldots + \rangle_{0,0} = [(-1)0(-1)\ldots (-1)] ,
\]

where \( |\ldots + \rangle_{0,0} \) refers to open/open BC. For \( \lambda > 0 \) this state remains at \( E = 0 \), where it is protected by the Witten index, \( W = 1 \), and it is separated from all other states by a gap that remains finite as long as \( \lambda < 1 \).

**Boundary defects.** - To steer into a case with exponentially degenerate groundstates at given particle number \( f \), we need to enforce a defect at both boundaries that allows all three \( \lambda = 0 \) PBC groundstates to connect to the edge at zero energy cost. For this we impose the constraint that the two sites adjacent to a boundary cannot both be occupied by a particle [21]. With this BC (which we call ‘\( \sigma \)-type’ and denote by a bracket \( \ldots \)\( \sigma \)), all three \( \lambda = 0 \) PBC vacua can connect to the boundary at zero energy cost. This gives, for \( L = 4l + 1, A_1 \ldots 1A \),

\[
|\ldots - \rangle_{\sigma,\sigma} = \sigma[0(\cdot 1)\ldots 0]\sigma , \quad |\ldots + \rangle_{\sigma,\sigma} = \sigma[100(\cdot 1)\ldots 001]\sigma ,
\]

\[
|0\rangle_{\sigma,\sigma} = \sigma[01010\ldots 01010]\sigma .
\]

(7)

For \( \lambda > 0 \) the state \( |\ldots - \rangle_{\sigma,\sigma} \) at \( f = 2l \) remains at \( E = 0 \) while \( |\ldots + \rangle_{\sigma,\sigma} \) at \( f = 2l+1 \) pair into a doublet of energy \( \delta E(\lambda) > 0 \). The energy difference
between $|\pm\rangle_{\sigma,\sigma}$ originates from the boundary, which implies that it involves a power of $\lambda$ that scales with the length of the system. We checked numerically that

$$\delta E(\lambda) \propto \lambda^{(L-1)/2},$$

which gives exponential splitting $\delta E \propto e^{-\alpha L}$ as long as $\lambda < 1$. We checked that this behaviour is robust against perturbations obtained by deforming some of the parameters $\lambda_{a,b,i,j}$ in eq. (1), provided we do not break the $\mathcal{N} = 2$ supersymmetry. We thus identify supersymmetry as the agent protecting the exponential degeneracy of our ‘qubit’ $|\pm\rangle_{\sigma,\sigma}$. We remark that many ‘natural’ perturbations of the $M_2$ model do break supersymmetry.

An example is the local fermion density operator $\rho_i$ at site $i = 4p + 1$. At $\lambda = 0$ the expected value is $\langle \rho_i \rangle = 0$ in the state $|\rangle_{\sigma,\sigma}$ while $\langle \rho_i \rangle = 1$ in the state $|+\rangle_{\sigma,\sigma}$.

Following the ‘qubit’ states all the way to the CFT point, $\lambda = 1$, we find (fig. 2)

$$|{-}\rangle_{\sigma,\sigma} \leftrightarrow |V_{-\frac{1}{2}}\rangle, \quad |{+}\rangle_{\sigma,\sigma} \leftrightarrow |\psi_{V_{-\frac{1}{2}}}\rangle,$$

where $|\rangle$ denotes the lowest weight state of the corresponding CFT module. At the CFT point the boundary Majorana modes have delocalised and the energy splitting is of order $1/L$.

The degeneracy of the ‘qubit’ $|\pm\rangle_{\sigma,\sigma}$ can be traced to the fusion channel degeneracy $|\sigma\sigma = 1 + \psi$ of the Ising spin-field $\sigma$ in the underlying CFT. Through the qH-CFT connection $[3]$, this choice of fusion channel carries over to the fusion product of two quasi-holes over the MR state. We consider the MR state in spherical geometry, which we view as an open ‘tube’ capped by specific boundary conditions at the two poles. For $N = 2l$ particles, the MR groundstate has the following thin-torus form

$$\text{MR, } N = 2l : \quad |11001100 \ldots 110011\rangle.$$

The analogous groundstate of the $M_2$ model is precisely the state $|+\rangle_{o,o}$ in eq. (6). The simplest case with two-fold fusion channel degeneracy is that of the MR states with $n = 4$ quasi-holes. The general counting formula for $n$ quasi-holes reads $[22]$

$$\sum_{F \equiv N \text{ mod } 2} \left( \frac{N-E}{2} + n \right) \left( \frac{n/2}{F} \right).$$

Here the first binomial counts orbital degeneracies of the $n$ quasi-holes, while the second, together with the sum over $F$, pertains to the fusion channel degeneracy. We fix the orbital degeneracy by selecting the states with two quasi-holes at both the north and the south poles,

$$F = 0 : \quad [01100 \ldots 110], \quad F = 2 : \quad [10011 \ldots 001].$$

These ‘MR qubit’ states correspond to the ‘$M_2$ qubit’ states $|\pm\rangle_{\sigma,\sigma}$ of eq. (7). To understand this we have to compare the open $M_2$ chains with the MR states not on the sphere but on the cylinder. On a cylinder with vacua ‘1100’ at far left and right, we can extend the $F = 0$ state

FIG. 1. Numerical $M_2$ spectra with $L = 25$, $f = 11, 12, 13, 14$ up to $E = 4$. The labels specify the corresponding CFT modules.

FIG. 2. Flow between ‘extreme staggering’ (left, $\lambda = 0$) and critical (right, $\lambda = 1$) limits of the $M_2$ model with $L = 17$ sites, $f = 8$ particles, staggering $\lambda_1 \ldots \lambda_7$ and $\sigma/\sigma$ BC. The lower two states constitute the ‘qubit’ eq. (9).
\[ \cdots 1100_\sigma [01100 \ldots 110]_{\sigma_0} 0011 \ldots, \]  
where \( \sigma \sigma \) denote the two quasi-holes at the boundaries. We can move one of the quasi-holes out from each of the boundaries to get
\[ \cdots 1100_\sigma 1010 \ldots 10_\sigma [01100 \ldots 110]_{\sigma_0} 0101 \ldots 01_\sigma 0011 \ldots. \]  
This corresponds to the situation that we have in the \( M_3 \) model, where \( \sigma \)-type BC arise from the presence of a single \( \sigma \) quantum at a boundary. This interpretation is confirmed by the CFT content of the open chain finite size spectra at \( \lambda = 1 \) (fig. 1). For open/\( \sigma \) BC, we find all sectors \( \sigma V_m \) with \( m \) shifted to \( m = 2f - L \). For \( \sigma / \sigma \) BC, with non-Abelian defects at both ends, we find, at \( m = 2f - L + \frac{1}{2} \), both the \( V_m \) and \( \psi V_m \) modules, in accordance with the fusion rule \( \sigma \sigma = 1 + \psi \). The states in eq. (9) are a particular example.

**Bulk defects and quantum register.** - At extreme staggering, kinks connecting any two of the \( \lambda = 0 \) vacua come at finite energy cost. For example, kinks/anti-kinks connecting \( 0 \) and \( + \), written as
\[ \cdots 10101_0 0(\cdot \cdot \cdot) \ldots, \cdots 10101_1 0(\cdot \cdot \cdot) \ldots, \]  
have \( E = 1 \); the same holds true for all kink types \( (a, b) = (0, \pm), (\pm, 0) \). Note that kinks/anti-kinks are connected by \( \mathcal{N} = 2 \) supersymmetry,
\[ Q^+: K_{a, b} \rightarrow \overline{K}_{a, b}. \]  
Multi-kink/anti-kink states can be counted through formulas similar to those for the MR state, see eq. (11), for all choices of open chain BC. Followed through to the CFT limit, these counting formulas provide novel expressions for characters of the CFT [15].

We now define \( \sigma \)-type bulk defects. These will allow some of the bulk kink states to exist at zero energy (for \( \lambda = 0 \)). In the example of eq. (15) this can be done by excluding the configuration ‘11’ at the kink location: this eliminates the anti-kink and turns the kink into an \( E = 0 \) state! To treat the \( \pm \) vacua on equal footing, we repeat the same constraint two steps to the right, and define
\[ \sigma : \cdots \lambda_1 \lambda_1 \lambda_1 \ldots, \sigma' : \cdots \lambda_1 \lambda_1 \lambda_1 \ldots \]  
This definition holds for general \( \lambda \) and represents a constraint in the Hilbert space of states. At \( \lambda = 0 \), a defect \( \sigma \) can connect a vacuum \( 0 \) coming in from the left to \( |+ \rangle, |0 \rangle \) or \(|- \rangle \) extending to the right, and opposite for \( \sigma' \), all at zero energy.

We can now conceive a ‘quantum register’ by taking an open chain, length \( L = 4f + 1 \), staggering type \( \lambda 1 \ldots \lambda \), \( \sigma \)-type BC at both ends, and injecting a sequence of \( 2n \) well-separated defects in the order \( \sigma' \sigma \ldots \sigma' \sigma \). This leads to \( 2^{n+1} \) degenerate groundstates at \( \lambda = 0 \), with \( 2^{n+1} \) of them having the minimal particle number \( f = 2l - n \). These \( 2^{n+1} \) states form an Ising anyon ‘quantum register’. At \( \lambda > 0 \), a unique \( E = 0 \) groundstate at \( f = 2l - n \) remains, while all other states pair up into doublets at energies of order \( e^{-\alpha L} \).

**M₃ model.** - Turning to \( k = 3 \) we identify the following four PBC groundstates at extreme staggering
\[ |1 \rangle = \cdots 11100 \ldots, |\frac{1}{2} \rangle = \cdots (\cdot \cdot \cdot) \ldots, \]  
\[ |0 \rangle = \cdots 01011 \ldots, |\frac{3}{2} \rangle = \cdots 0(\cdot \cdot \cdot) \ldots, \]  
with \( \wedge \) indicating the position of ‘\( \lambda \)’ in the staggering pattern eq. (4), \( (\cdot \cdot \cdot ) = 01101 - 01110 - 01101 - 11010 \) and \( (\cdot \cdot \cdot ) = 1110 - 0111 \). The energies of the kinks \( K_{a, b} \) are \( m_{a, b} = 2|a - b| \). This is in agreement with the kink masses in the \( \mathcal{N} = 2 \) symmetric massive integrable QFT with Chebyshev superpotential \( W(X) = \frac{1}{2} X^3 - \beta X^2 + \beta X \) [23–26], which we expect to result from the RG flow set by the staggering perturbation.

On open chains, we define \( \sigma_1 \) type BC by excluding ‘111’ near a given edge, and \( \sigma_2 \) type by excluding ‘11’. At the level of the open-chain CFT spectra, a \( \sigma_1 \) type BC precisely corresponds to the \( \mathbb{Z}_3 \) parafermion spin field \( \sigma_1 \); upon changing BC, the various CFT sectors shift according to the fusion products with these two spin fields. The \( k = 3 \) CFT contains, besides a free boson, \( \mathbb{Z}_3 \) parafermions \( \psi_{1, 2} \) and parafermion spin fields \( \sigma_1, \varepsilon \). The supercharge \( Q^+ \) is the zero-mode of the super current \( \psi_1 V_{\frac{3}{2}}(z) \). For open/open BC, the \( M_3 \) model realises the sectors, with \( m = \frac{5}{2} f - \frac{3}{2} L - \frac{3}{4} \),

\[ \{ V_m, \psi_1 V_m, \psi_2 V_m \}, \]  
for \( k = 0, 1, 2 \) with \( k \equiv f \mod 3 \). For open/\( \sigma_1 \) BC this becomes, with \( m = \frac{5}{2} f - \frac{3}{2} L - \frac{1}{4} \),

\[ \{ \sigma_1 V_m, \varepsilon V_m, \sigma_2 V_m \}, \]  
in accordance with the parafermion fusion rules \( \sigma_1 \psi_1 = \varepsilon \) and \( \sigma_1 \psi_2 = \sigma_2 \). For open/\( \sigma_2 \) BC, with \( m = \frac{5}{2} f - \frac{3}{2} L + \frac{1}{4} \),

\[ \{ \sigma_2 V_m, \sigma_1 V_m, \sigma_2 V_m \}, \]  
in agreement with \( \sigma_2 \psi_1 = \sigma_1 \) and \( \sigma_2 \psi_2 = \varepsilon \). The supersymmetric groundstates are \( |\sigma_{1, 2} V_{\frac{3}{2}} \rangle \) and \( |V_{\pm \frac{3}{2}} \rangle \).

Putting \( \sigma_i \) type BC on both ends, the open-chain CFT sectors follow the fusion rules \( \sigma_1 \sigma_1 = \psi_1 + \sigma_2, \sigma_1 \sigma_2 = 1 + \varepsilon \) and \( \sigma_1 \sigma_2 = \psi_2 + \sigma_1 \). As for \( k = 2 \), these fusion rules lead to exponential groundstate degeneracies in the extreme staggering limit. A characteristic case would be \( L = 15 \) sites, \( \sigma_2 \)-type BC on both ends, and staggering positioned as \( \cdots \cdot \cdot \cdot \lambda \ldots \). Here the lowest CFT states are two supersymmetry doublets (at \( f = 8, 9 \)) at CFT energies.
in the $M_k$ models correspond to operators that are local in the dual variables of the topological phase, and are thus unable to split the exponential degeneracies. Many extensions of the ideas presented here are feasible. In the $M_2$ model, alternative bulk defects, based on a ‘no 0’ rather than a ‘no 11’ condition, lead to Fibonacci number degeneracies. The $M_1$ model on a square ladder is in many ways similar to the $M_2$ model [27] and it is natural to introduce $\sigma$-type defects in the 2D $M_1$ model on the octagon-square and square lattices [28, 29]. In all these cases, we expect non-trivial fusion relations to emerge.

We thank Erez Berg, Tarun Grover, Yingfei Gu, Christian Hagendorf, Liza Huijse, Steve Simon, and, in particular, Paul Fendley and Ville Laihtinen, for discussions, and the INFN for hospitality in Firenze, where part of this work was done. T.F. is supported by the Netherlands Organisation for Scientific Research (NWO).

**Conclusions.** We have demonstrated that (boundary and bulk) defects in the $M_k$ models off-criticality lead to quantum registers that are protected by supersymmetry. It will be interesting to explore ways to manipulate these registers, so as to act on the quantum information that can be stored in the supersymmetry protected ground states. One idea is that of a 1D braid protocol (as in [9]); one expects that this will result in the braid matrices as they are known for the corresponding 2D (RR$_k$) topological phases. It will also be interesting to see if a dual formulation of the $M_k$ models, with 1D topological order rather than order of CDW type, can be obtained. One expects that operators that preserve supersymmetry.

$$E_{\text{CFT}} = 1/5, E_{\text{CFT}} = 4/5,$$

while the $\lambda = 0$ $M_3$ model has four $E = 0$ vacua. The states connect according to

$$|\frac{1}{2}\sigma_2, \sigma_2\rangle \leftrightarrow |\psi_1 V_{-5/4}\rangle, |0\rangle_{\sigma_2, \sigma_2}\leftrightarrow |\psi_2 V_{+5/4}\rangle,$$

$$|\sigma_2 V_{-5/4}, 1\rangle_{\sigma_2, \sigma_2}\leftrightarrow |\sigma_1 V_{+5/4}\rangle. \quad (22)$$

We define bulk defects as

$$
\begin{align*}
\sigma_1 : \ldots \lambda \ast \ast \ldots , & \quad \sigma_2 : \ldots \lambda \ast \ast \ldots , \\
\sigma_1' : \ldots \ast \ast \lambda \ldots , & \quad \sigma_2' : \ldots \ast \ast \lambda \ldots . 
\end{align*}
$$

(23)

Fig. 3 depicts the corresponding zero-energy fusion rules. They determine the size and structure of the quantum register opened up by an alternating sequence of defects $\sigma_i, \sigma'_i$. In all cases the states at maximal particle number are made up entirely of vacua $|0\rangle$ and $|1\rangle$. Restricting the fusion rules to $|0\rangle, |1\rangle$ (drawn lines) gives Fibonacci number degeneracies. The $N = 2$ supersymmetry acts within the register, with $Q^-$ mapping the $|0\rangle, |1\rangle$ into linear combinations of $|\frac{1}{2}\rangle, |\frac{1}{2}\rangle$. In the example of $L = 5l$ sites, $\sigma_2/\sigma_2'$ BC, staggering type $\ast \ast \lambda \ast \ast \ldots$ and $n \sigma_2/\sigma_2'$-type defects, the degeneracy at $f = 3l$ is Fibonacci, with Fibonex$+3$, with Fibo$_0 = 1, 1, 2, 3, \ldots$ . Eq. (22) is the case $n = 0$. For $n = 1$ one finds 3, 4, 1 states at $f = 3l, \ldots, 3l - 2, n = 2$ gives 5, 9, 5, 1 states at $f = 3l, \ldots, 3l - 3$, etc.

For the general $M_k$ model, defects eliminating $k + 1 - j$ consecutive ‘1’s will correspond to the $\mathbb{Z}_k$ parafermion spin fields $\sigma_j, j = 1, \ldots, k - 1$ [15].

---

\[ \text{t.b.fokkema@uva.nl} \]
\[ \text{c.j.m.schoutens@uva.nl} \]


[12] C. Hagendorf and L. Huijse, manuscript in preparation. We thank the authors for sharing their results prior to publication.


[21] Similar BC were introduced independently in [13].


