Chapter 2

Multiple Steady States, Limit Cycles and Chaotic Attractors in Logit Dynamics

2.1 Introduction

2.1.1 Motivation

A large part of the research on evolutionary game dynamics focused on identifying conditions that ensure uniqueness of and/or convergence to point-attractors such as Nash Equilibrium and Evolutionary Stable Strategy (ESS). Roughly speaking, within this ‘convergence’ literature one can further distinguish between literature focusing on classes of games (e.g. Milgrom and Roberts (1991), Nachbar (1990), Hofbauer and Sandholm (2002)) and literature on the classes of evolutionary dynamics (e.g. Cressman (1997), Hofbauer (2000), Hofbauer and Weibull (1996), Sandholm (2005), Samuelson and Zhang (1992)). Nevertheless, there are some examples of periodic and chaotic behaviour in the literature, mostly under a particular kind of evolutionary dynamics, the Replicator Dynamics. Motivated by the idea of adding an explicit
dynamical process to the static concept of Evolutionary Stable Strategy, Taylor and Jonker (1978) introduced the Replicator Dynamics. It soon found applications in biological, genetic or chemical systems, those domains where organisms, genes or molecules evolve over time via replication. The common feature of these systems is that they can be well approximated by an infinite population game with random pairwise matching, giving rise to a replicator-like evolutionary dynamics given by a low-dimensional dynamical system.

In the realm of the non-convergence literature two issues are particularly important: the existence of stable periodic and complicated solutions and their robustness (to slight perturbations in the payoffs matrix). Hofbauer et al. (1980) and Zeeman (1981) investigate the phase portraits resulting from three-strategy games under the replicator dynamics and conclude that only ‘simple’ behaviour - sinks, sources, centers, saddles - can occur. In general, an evolutionary dynamics together with a \( n \)-strategy game define a proper \( n-1 \) dynamical system on the \( n-1 \) simplex. An important result the proof of Zeeman (1980) that there are no generic Hopf bifurcations on the 2-simplex: "When \( n = 3 \) all Hopf bifurcations are degenerate"\(^1\). Bomze (1983), Bomze (1995) provide a thorough classification of the planar phase portraits for all 3-strategy games according to the number, location (interior or boundary of the simplex) and stability properties of the Replicator Dynamics fixed points and identify 49 different phase portraits: again, only non-robust cycles are created usually via a degenerate Hopf bifurcation.

Hofbauer (1981) proves that in a 4-strategy game stable limit cycles are possible under Replicator Dynamics; the proof consists in finding a suitable Lyapunov function whose time derivative vanishes on the \( \omega \)-limit set of a periodic orbit. Stable limit cycles are also reported in Akin (1982) in a genetic model where gene ‘replicates’ via the two allele-two locus selection; this is not surprising as the dynamical system modeling gametic frequencies is three-dimensional, the dynamics is of Replicator type

\(^1\)Zeeman (1980), pp. 493
and the Hofbauer (1981) proof applies here, as well. Furthermore, Maynard Smith and Hofbauer (1987) prove the existence of a stable limit cycle for an asymmetric, Battle of Sexes-type genetic model where the allelic frequencies evolution defines again a 3-D system. Their proof hinges on normal form reduction together with averaging and elliptic integrals techniques for computing the phase and angular velocity of the periodic orbit. Stadler and Schuster (1990) perform an impressive systematic search for both generic(fixed points exchanging stability) and degenerate(stable and unstable fixed points colliding into a one or two-dimensional manifold at the critical parameter value) transitions between phase portraits of the replicator equation on \(3 \times 3\) normal form game.

Chaotic behaviour is found by Schuster et al. (1991) in Replicator Dynamics for a 4-strategy game matrix derived from an autocatalytic reaction network. They report the standard Feigenbaum route to chaos: a cascade of period-doubling bifurcations intermingled with several interior crisis and collapses to a chaotic attractor. Numerical evidence for strange attractors is provided by Skyrms (1992), Skyrms (2000) for a Replicator Dynamics flow on two examples of a four-strategy game.

Although periodic and chaotic behaviour is substantially documented in the literature for the Replicator Dynamics, there is much less evidence for such complicated behaviour in classes of evolutionary dynamics that are more appropriate for humans interaction(fictitious play, best response dynamics, adaptive dynamics, etc.). While Shapley (1964) constructs an example of a non-zero sum game with a limit cycle under fictitious play and Berger and Hofbauer (2006) find stable periodic behaviour - two limit cycles bounding an asymptotically stable annulus - for a different dynamic - the Brown-von Newmann Nash (BNN) - a systematic characterization of (non) generic bifurcations of phase portraits is still missing for the Best Response dynamics.

In this Chapter we take a first step in this direction and use a smoothed version of the Best Response dynamics - the Logit Dynamics - to study evolutionary dynamics
in simple three and four strategies games from the existing literature. The qualitative
behaviour of the resulting ‘evolutionary’ games is investigated with respect to changes
in the payoff and behavioural parameters, using analytical and numerical tools from
non-linear dynamical system theory.

2.1.2 ‘Replicative’ vs. ‘rationalistic’ dynamics

Most of the earlier discussed evolutionary examples are inspired from biology and not
from social sciences. They concerned animal contests (Zeeman (1980), Bomze (1983),
Bomze (1995)), genetics (Maynard Smith and Hofbauer (1987)) or chemical catalytic
networks (Schuster et al. (1991), Stadler and Schuster (1990)). From the perspective
of strategic interaction the main criticism of the ‘biological’ game-theoretic models
is targeted at the intensive use of preprogrammed, simple imitative play with no role
for optimization and innovation. Specifically, in the transition from animal contests
and biology to humans interactions and economics the Replicator Dynamics seems
no longer adequate to model the rationalistic and ‘competent’ forms of behaviour
(Sandholm (2008)). Best Response Dynamics would be more applicable to human
interaction as it assumes that agents are able to optimally compute and play a (my-
opic) ‘best response’ to the current state of the population. But, while the Replicator
Dynamics appeared to impose an unnecessarily loose rationality assumption the Best
Response dynamics moves to the other extreme: it is too stringent in terms of ra-
tionality. Another drawback is that, technically, the best reply is not necessarily
unique and this leads to a differential inclusions instead of an ordinary differential
equation. One way of solving these problems was to stochastically perturb the matrix
payoffs and derive, via the discrete choice theory, a ‘noisy’ Best Response Dynamics,
called the Logit Dynamics. Mathematically it is a ‘smoothed’, well-behaved dynam-
ics while from the strategic interaction point of view it models a boundedly rational
player/agent. Moreover, from a nonlinear dynamical systems perspective the Replic-
ator Dynamic is non-generic in dimension two and only degenerate Hopf bifurcations
can arise on the 2-simplex. In sum, apart from its conjectured generic properties, the Logit is recommended by the need for modelling players with different degrees of rationality and for smoothing the Best Response correspondence.

Thus, the main part of this Chapter investigates the qualitative behaviour of simple evolutionary games under the Logit dynamics when the level of noise and/or underlying normal form game payoffs matrix is varied with the goal of finding attractors (periodic or perhaps more complicated) which are not fixed points or Nash Equilibria.

This Chapter is organized as follows: Section 2.2 introduces the Logit Dynamics, while Section 2.3 gives a brief overview of the Hopf bifurcation theory. In Sections 2.4 the Logit Dynamics is implemented on various versions of Rock-Scissors-Paper and Coordination games while Section 2.5 discusses an example of chaotic dynamics under a frequency-weighted version of the Logit Dynamics. Section 2.6 contains concluding remarks.

2.2 The Logit Dynamics

2.2.1 Evolutionary dynamics

Evolutionary game theory deals with games played within a (large) population over a long time horizon (evolution scale). Its main ingredients are the underlying normal form game - with payoff matrix \( A[n \times n] \) - and the evolutionary dynamic class which defines a dynamical system on the state of the population. In a symmetric framework, the strategic interaction takes the form of random matching with each of the two players choosing from a finite set of available strategies \( E = \{E_1, E_2, \ldots E_n\} \). For every time \( t \), \( x(t) \) denotes the \( n \)-dimensional vector of frequencies for each strategy/type \( E_i \) and belongs to the \( n - 1 \) dimensional simplex \( \Delta^{n-1} = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1\} \). Under the assumption of random interactions strategy \( E_i \) fitness would be simply determined by averaging the payoffs from each strategic interaction with weights.
given by the state of the population $\mathbf{x}$. Denoting with $f(\mathbf{x})$ the payoff vector, its components - individual payoff or fitness of strategy $i$ in biological terms - are:

$$ f_i(\mathbf{x}) = (\mathbf{Ax})_i $$

(2.1)

Sandholm (2006) rigorously defines an evolution dynamics as a map assigning to each population game a differential equation $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ on the simplex $\Delta^{n-1}$. In order to derive such an 'aggregate' level vector field from individual choices he introduces a revision protocol $\rho_{ij}(f(\mathbf{x}), \mathbf{x})$ indicating, for each pair $(i, j)$, the rate of switching($\rho_{ij}$) from the currently played strategy $i$ to strategy $j$. The mean vector field is obtained as:

$$ \dot{x}_i = \mathbf{V}_i(\mathbf{x}) = \text{inflow into strategy } i \text{ - outflow from strategy } i $$

$$ = \sum_{j=1}^{n} x_j \rho_{ji}(f(\mathbf{x}), \mathbf{x}) - x_i \sum_{j=1}^{n} \rho_{ij}(f(\mathbf{x}), \mathbf{x}). $$

(2.2)

Based on the computational requirements/quality of the revision protocol $\rho$ the set of evolutionary dynamics splits into two large classes: imitative dynamics and pairwise comparison ('competent' play). The first class is represented by the most famous dynamics, the Replicator Dynamic (Taylor and Jonker (1978)) which can be easily derived by substituting into (2.2) the pairwise proportional revision protocol: $\rho_{ij}(f(\mathbf{x}), \mathbf{x}) = x_j[f_j(\mathbf{x}) - f_i(\mathbf{x})]_+$ (player $i$ switches to strategy $j$ at a rate proportional with the probability of meeting an $j$-strategist($x_j$) and with the excess payoff of opponent $j$-$[f_j(\mathbf{x}) - f_i(\mathbf{x})]$- iff positive):

$$ \dot{x}_i = x_i[f_i(\mathbf{x}) - \bar{f}(\mathbf{x})] = x_i[(\mathbf{Ax})_i - \mathbf{xAx}] $$

(2.3)

where $\bar{f}(\mathbf{x}) = \mathbf{xAx}$ is the average population payoff.

Although widely applicable to biological/chemical models, the Replicator Dynamics lacks the proper individual choice, micro-foundations which would make it
attractive for modeling humans interactions. The alternative - Best Response dynamic - already introduced by Gilboa and Matsui (1991) requires extra computational abilities from agents, beyond merely sampling randomly a player and observing the difference in payoff: specifically being able to compute a best reply strategy to the current population state:

\[ \dot{x}_i = BR(x) - x_i \]  

where,

\[ BR(x) = \arg \max_y y f(x) \]

### 2.2.2 Discrete choice models-the Logit choice rule

Apart from the highly unrealistic assumptions regarding agents capacity to compute a perfect best reply to a given population state there is also the drawback that (2.4) defines a differential inclusion, i.e. a set-valued function. The best responses may not be unique and multiple trajectory paths can emerge from the same initial conditions. A ‘smoothed’ approximation of the Best Reply dynamics - the Logit dynamics - was introduced by Fudenberg and Levine (1998); it was obtained by stochastically perturbing the payoff vector \( f(x) \) and deriving the Logit revision protocol:

\[ \rho_{ij}(f(x), x) = \frac{\exp[\eta^{-1}f_j(x)]}{\sum_k \exp[\eta^{-1}f_k(x)]} = \frac{\exp[\eta^{-1}Ax]_i}{\sum_k \exp[\eta^{-1}Ax]_k}, \]

where \( \eta > 0 \) is the noise level. Here \( \rho_{ij} \) represents the probability of player \( i \) switching to strategy \( j \) when provided with a revision opportunity. For high levels of noise the choice is fully random (no optimization) while for \( \eta \) close to zero the switching probability is almost one. This revision protocol can be explicitly derived from a random utility model or discrete choice theory (McFadden (1981), Anderson et al. (1992)) by adding to the payoff vector \( f(x) \) a noise vector \( \varepsilon \) with a particular distribution, i.e. \( \varepsilon_i \) are i.i.d following the extreme value distribution \( G(\varepsilon_i) = \exp(-\exp(-\eta^{-1}\varepsilon_i - \gamma)) \), \( \gamma = 0.5772 \) (the Euler constant). The density of
this Weibull type distribution is
\[ g(\varepsilon_i) = G'(\varepsilon_i) = \eta^{-1} \exp(-\eta^{-1}\varepsilon_i - \gamma) \exp(-\exp(-\eta^{-1}\varepsilon_i - \gamma)). \]

With noisy payoffs, the probability that strategy \( E_i \) is a best response can be computed as follows:
\[
P(i = \arg \max_j [(Ax)_j + \varepsilon_i] = P[(Ax)_i + \varepsilon_i > (Ax)_j + \varepsilon_j], \forall j \neq i
= \int_{-\infty}^{\infty} g(\varepsilon_i) \prod_{j \neq i} G((Ax)_i + \varepsilon_i - (Ax)_j) d\varepsilon_i
= \int_{-\infty}^{\infty} \eta^{-1} \exp(-\eta^{-1}\varepsilon_i - \gamma) \exp(-\exp(-\eta^{-1}\varepsilon_i - \gamma)) \prod_{j \neq i} \exp(-\exp(-\eta^{-1}|(Ax)_i + \varepsilon_i - (Ax)_j| - \gamma)) d\varepsilon_i
\]
which simplifies to our logit probability of revision:
\[
\rho_{ij} = \frac{\exp[\eta^{-1}(Ax)_i]}{\sum_k \exp[\eta^{-1}(Ax)_k]}
\]
An alternative way to obtain (2.5) is to deterministically perturb the set-valued best reply correspondence (2.4) with a strictly concave function \( V(y) \) (Hofbauer (2000)):
\[
BR_\eta(x) = \arg \max_{y \in \Delta^{n-1}} [y \cdot (Ax) + V_\eta(y)]
\]
For a particular choice of the perturbation function \( V_\eta(y) = \eta \sum_{i=1,n} y_i \log y_i; y \in \Delta^{n-1} \) the resulting objective function is single-valued and smooth; the first order condition yields the unique logit choice rule:
\[
BR_\eta(x)_i = \frac{\exp[\eta^{-1}(Ax)_i]}{\sum_k \exp[\eta^{-1}(Ax)_k]}
\]
Plugging the Logit revision protocol (2.5) back into the general form of the mean field dynamic (2.2) and making the substitution \( \beta = \eta^{-1} \) we obtain a well-behaved
system of o.d.e.’s, the Logit dynamics as a function of the intensity of choice (Brock and Hommes (1997)) parameter $\beta$:

$$\dot{x}_i = \frac{\exp[\beta A x_i]}{\sum_k \exp[\beta A x_k]} - x_i$$

(2.6)

When $\beta \to \infty$ the probability of switching to the discrete ‘best response’ $j$ is close to one while for a very low intensity of choice ($\beta \to 0$) the switching rate is independent of the actual performance of the alternative strategies (equal probability mass is put on each of them). The Logit dynamics displays some properties characteristic to the logistic growth function, namely high growth rates($\dot{x}_i$) for small values of $x_i$ and growth ‘levelling off’ when close to the upper bound. This means that a specific frequency $x_i$ grows faster when it is already large in the Replicator Dynamics relative to the Logit dynamics.

### 2.3 Hopf and degenerate Hopf bifurcations

As the main focus of the thesis is the detection of stable cyclic behaviour this section will shortly review the main bifurcation route towards periodicity, the Hopf bifurcation. In a one-parameter family of continuous-time systems, the only generic bifurcation through which a limit cycle is created or disappears is the non-degenerate Hopf bifurcation. The planar case will be discussed first and then, briefly, the methods to reduce higher-dimensional systems to the two-dimensional one. The main mathematical result (see, for example, Kuznetsov (1995)) is:

Assume we are given a parameter-dependent, two dimensional system:

$$x = f(x, \alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}, \quad f \text{ smooth}$$

(2.7)

with the Jacobian matrix evaluated at the fixed point $x^* = 0$ having a pair of purely
imaginary, complex conjugates eigenvalues:

\[ \lambda_{1,2} = \mu(\alpha) \pm i\omega(\alpha), \mu(\alpha) < 0, \alpha < 0, \mu(0) = 0 \]

and \( \mu(\alpha) > 0, \alpha > 0 \).

If, in addition, the following genericity\(^2\) conditions are satisfied:

\begin{enumerate}
  \item [(i)] \[ \left[ \frac{\partial \mu(\alpha)}{\partial \alpha} \right]_{\alpha = 0} \neq 0 - \text{transversality condition} \]
  \item [(ii)] \( l_1(0) \neq 0 \), where \( l_1(0) \) is the first Lyapunov coefficient\(^3\) - nondegeneracy condition,
\end{enumerate}

then the system (2.7) undergoes a Hopf bifurcation at \( \alpha = 0 \). As \( \alpha \) increases the steady state changes stability from a stable focus into an unstable focus.

There are two types of Hopf bifurcation, depending on the sign of the first Lyapunov coefficient \( l_1(0) \):

\begin{enumerate}
  \item [(a)] If \( l_1(0) < 0 \) then the Hopf bifurcation is supercritical: the stable focus \( x \) becomes unstable for \( \alpha > 0 \) and is surrounded by an isolated, stable closed orbit (limit cycle).
  \item [(b)] If \( l_1(0) > 0 \) then the Hopf bifurcation is subcritical: for \( \alpha < 0 \) the basin of attraction of the stable focus \( x^* \) is surrounded by an unstable cycle which shrinks and disappears as \( \alpha \) crosses the critical value \( \alpha = 0 \) while the system diverges quickly from the neighbourhood of \( x^* \).
\end{enumerate}

In the first case the stable cycle is created immediately after \( \alpha \) reaches the critical value and thus the Hopf bifurcation is called supercritical, while in the latter the unstable cycle already exists before the critical value, i.e. a subcritical Hopf bifurcation (Kuznetsov (1995)). The supercritical Hopf is also known as a soft or

\(^2\)Genericity usually refer to transversality and non-degeneracy conditions. Roughly speaking, the transversality condition means that eigenvalues cross the real line at non-zero speed. The nondegeneracy condition implies non-zero higher-order coefficients in equation (2.10) below. It ensures that the singularity \( x^* \) is typical (i.e. ‘nondegenerate’) for a class of singularities satisfying certain bifurcation conditions (see Kuznetsov (1995)).

\(^3\)This is the coefficient of the third order term in the normal form of the Hopf bifurcation (see equation (2.10) below).
non-catastrophic bifurcation because, even when the system becomes unstable, it
still lingers within a small neighbourhood of the equilibrium bounded by the limit
cycle, while the subcritical case is a sharp/catastrophic one as the system now moves
quickly far away from the unstable equilibrium.

If \( l_1(0) = 0 \) then there is a degeneracy in the third order terms of normal form
and, if other, higher order nondegeneracy conditions hold (i.e. non-vanishing second
Lyapunov coefficient) then the bifurcation is called Bautin or generalized Hopf bi-
furcation. This happens when the first Lyapunov coefficient vanishes at the given
equilibrium \( x^* \) but the following higher-order genericity\(^4\) conditions hold:

(i) \( l_2(0) \neq 0 \), where \( l_2(0) \) is the second Lyapunov coefficient - nondegeneracy
condition

(ii) the map \( \alpha \rightarrow (\mu(\alpha), l_1(\alpha)) \) is regular (i.e. the Jacobian matrix is nonsingular)
at the critical value \( \alpha = 0 \) - transversality condition.

Depending on the sign of \( l_2(0) \), at the Bautin point the system may display a
limit cycle bifurcating into two or more cycles, coexistence of stable and unstable
cycles which collide and disappear, together with cycle blow-up.

**Computation of the first Lyapunov coefficient**

For the planar case, \( l_1(0) \) can be computed without explicitly deriving the normal
form, from the Taylor coefficients of a transformed version of the original vector
field. The computation of \( l_1(0) \) for higher dimensional systems makes use of the Center
Manifold Theorem by which the orbit structure of the original system near \( (x^*, \delta_0) \)
is fully determined by its restriction to the two-dimensional center manifold\(^5\). On

\(^4\)Technically, these ‘higher-order’ generivity conditions ensure that there are smooth invertible
coordinate transformations, depending smoothly on parameters, together with parameter changes
and (possibly) time re-parametrizations such that (2.7) can be reduced to a ‘simplest’ form, the
normal form. See Kuznetsov (1995) pp. 309 for more details on the Bautin (generalized Hopf)
bifurcation and for an expression for the second Lyapunov coefficient \( l_2(0) \).

\(^5\)The center manifold is the manifold spanned by the eigenvectors corresponding to the eigen-
values with zero real part.
the center manifold (2.7) takes the form (Wiggins (2003)):

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \Re \lambda(\delta) & - \Im \lambda(\delta) \\ \Im \lambda(\delta) & \Re \lambda(\delta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, \delta) \\ f^2(x, y, \delta) \end{pmatrix}$$

(2.8)

where $\lambda(\delta)$ is an eigenvalue of the linearized vector field around the steady state and $f_1(x, y, \delta), f_2(x, y, \delta)$ are nonlinear functions of order $O(|x|^2)$ to be obtained from the original vector field. Wiggins (2003) also provides a procedure for transforming (2.7) into (2.8). Specifically, for any vector field $\dot{x} = F(x), x \in \mathbb{R}^2$ let $DF(x^*)$ denote the Jacobian evaluated at the fixed point $x^*$. Then $\dot{x} = F(x)$ is equivalent to:

$$\dot{x} = Jx + T^{-1} \tilde{F}(Tx)$$

(2.9)

with $J$ stands for the real Jordan canonical form of $DF(x^*), T$ is the matrix transforming $DF(x^*)$ into the Jordan form, and $\tilde{F}(x) = F(x) - DF(x^*)x$. At the Hopf bifurcation point $\delta_0, \lambda_{1,2} = \pm i \omega$ and the first Lyapunov coefficient is (Wiggins (2003)):

$$l_1(\delta) = \frac{1}{16} [f^1_{xxx} + f^1_{xyy} + f^2_{xx} + f^2_{yy}] +$$

$$+ \frac{1}{16\omega} [f^1_{xy}(f^1_{xx} + f^1_{yy}) - f^2_{xy}(f^2_{xx} + f^2_{yy}) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}]$$

(2.10)

(2.11)

### 2.4 Three strategy games

We consider two well-known three-strategy games: the generalized Rock-Scissors-Paper and the Coordination Game. In subsection (2.4.1) we first perform a local bifurcation analysis for a classical example of three-strategy games, the Rock-Scissors-Paper. Two types of evolutionary dynamics - Replicator and Logit Dynamics - are considered while the qualitative change in their orbital structure is studied with respect to the behavioural parameter $(\beta)$ and the payoff $(\varepsilon, \delta)$ parameters. In subsection (2.4.2) we provide numerical evidence for a sequence of fold bifurcations in
the Coordination game under the Logit Dynamics and depict the fold curves in the parameter space.

2.4.1 Rock-Scissors-Paper Games

The Rock-Paper-Scissors class of games (or games of cyclical dominance) formalize strategic interactions where each strategy $E_i$ is an unique best response to strategy $E_{i+1}$ for $i = 1, 2$ and $E_3$ is a best rponse to $E_1$:

$$A = \begin{pmatrix} \gamma_1 & \delta_2 & \varepsilon_3 \\ \varepsilon_1 & \gamma_2 & \delta_3 \\ \delta_1 & \varepsilon_2 & \gamma_3 \end{pmatrix} ; \delta_i \geq \gamma_i \geq \varepsilon_i \quad (2.12)$$

Due to the invariance under positive linear transformations of the payoff matrix (2.12) (Zeeman (1980), Weissing (1991)) the main diagonal element can be set to zero, by, for instance, substracting the diagonal entry from each column entries:

$$A = \begin{pmatrix} 0 & \delta_2 & -\varepsilon_3 \\ -\varepsilon_1 & 0 & \delta_3 \\ \delta_1 & -\varepsilon_2 & 0 \end{pmatrix} ; \delta_i, \varepsilon_i \geq 0 \quad (2.13)$$

If matrix (2.13) is circulant (i.e. $\delta_i = \delta, \varepsilon_i = \varepsilon, i = 1, 2, 3$) then the RSP game is called circulant, while for a non-circulant matrix (2.13) we have a generalized RSP game. The behavior of Replicator and Logit Dynamics on the class of circulant RSP games will be investigated, both analytically and numerically, in the first and second part of this subsection, respectively. Numerical results about the generalized RSP class of games under Logit Dynamics are reported in the third part.

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6 see Weissing (1991) for a thorough characterization of the discrete-time Replicator Dynamics behavior on this class of games.

7 We stick to the same notations as in (2.12) although it is apparent that the new $\delta'$s and $\varepsilon'$s are different from the old ones as they are derived via the above mentioned linear transformation.
Circulant RSP Game and Replicator Dynamics

The circulant RSP game is a first generalization of the classical, zero-sum form of RSP game as discussed in, for instance, Hofbauer and Sigmund (2003):

\[
A = \begin{pmatrix}
0 & \delta & -\varepsilon \\
-\varepsilon & 0 & \delta \\
\delta & -\varepsilon & 0
\end{pmatrix}, \delta, \varepsilon > 0
\] (2.14)

Letting \(x(t) = (x(t), y(t), z(t))\) denote the population state at time instance \(t\) define a point from the 2-dimensional simplex, the payoff vector \([Ax]\) is obtained via (2.1):

\[
[Ax] = \begin{pmatrix}
y\delta - z\varepsilon \\
-x\varepsilon + z\delta \\
x\delta - y\varepsilon
\end{pmatrix}
\]

Average fitness of the population is:

\[
x Ax = x(y\delta - z\varepsilon) + y(-x\varepsilon + z\delta) + z(x\delta - y\varepsilon)
\]

The replicator equation (2.3) with the game matrix (2.14) induce on the 2-simplex the following vector field:

\[
\begin{bmatrix}
\dot{x} = x[y\delta - z\varepsilon - (x(y\delta - z\varepsilon) + y(-x\varepsilon + z\delta) + z(x\delta - y\varepsilon))] \\
\dot{y} = y[-x\varepsilon + z\delta - (x(y\delta - z\varepsilon) + y(-x\varepsilon + z\delta) + z(x\delta - y\varepsilon))] \\
\dot{z} = z[x\delta - y\varepsilon - (x(y\delta - z\varepsilon) + y(-x\varepsilon + z\delta) + z(x\delta - y\varepsilon))]
\end{bmatrix}
\] (2.15)

While Hofbauer and Sigmund (2003) use the Poincare-Bendixson theorem together with the Dulac criterion to prove that limit cycles cannot occur in games with three strategies under the replicator we will derive this negative result using tools from dynamical systems, in particular the Hopf bifurcation and ‘normal form’ theory.
The same toolkit will be applied next to the Logit Dynamic and a positive result - stable limit cycles do occur - will be derived.

As we are interested in limit cycles within the simplex we consider only interior fixed points of this system (any replicator dynamic has the simplex vertices as steady states, too). For the parameter range $\varepsilon, \delta > 0$ the barycentrum $x^* = [x = 1/3, y = 1/3, z = 1/3]$ is always an interior fixed point of (2.15). In order to analyze its stability properties we obtain first, by substituting $z = 1 - x - y$ into (2.15), a proper 2 dimensional dynamical system of the form:

$$
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = 
\begin{bmatrix}
-x\varepsilon + 2xy\varepsilon - x^2\delta + 2x^2\varepsilon + x^3\delta - x^3\varepsilon + xy^2\delta + x^2y\delta - xy^2\varepsilon - x^2y\varepsilon \\
y\delta - 2xy\delta - 2y^2\delta + y^3\varepsilon + y^3\delta - y^3\varepsilon + xy^2\delta + x^2y\delta - xy^2\varepsilon - x^2y\varepsilon
\end{bmatrix}
\tag{2.16}
$$

We can detect the Hopf bifurcation threshold at the point where the trace of the Jacobian matrix of (2.16) is equal to zero. The Jacobian evaluated at the barycentrical steady state $x^*$ is

$$
\begin{bmatrix}
\varepsilon & \delta + \varepsilon \\
-\delta - \varepsilon & -\delta
\end{bmatrix},
$$

with eigenvalues: $\lambda_{1,2}(\varepsilon, \delta) = -\frac{1}{2}(\varepsilon - \delta) \pm i \frac{\sqrt{3}}{2} \sqrt{(\varepsilon + \delta)^2}$ and trace $\varepsilon - \delta$. For $\delta < \varepsilon$, $x^*$ is an unstable focus, at $\delta = \varepsilon$ a pair of imaginary eigenvalues crosses the imaginary axis($\lambda_{1,2} = \pm i \sqrt{3}\delta$), while for $\delta > \varepsilon$ $x^*$ becomes a stable focus (see Fig. (2.1) below). This is consistent with Theorem 6 in Zeeman (1980) which states that the determinant of the matrix $A$ determines the stability properties of the interior fixed point. In example (2.14) $DetA = \varepsilon^3 - \delta^3$ vanishes for $\varepsilon = \delta$. By the same Theorem 6 in Zeeman (1980) the vector field (2.15) has a center in the 2-simplex and a continuum of cycles if $DetA = 0$.

Alternatively, our local bifurcation analysis suggests that a Hopf bifurcation occurs when $\varepsilon = \delta$ and in order to ascertain its features - sub/supercritical or degenerate - we have to further investigate the nonlinear vector field near the $(x^*, \varepsilon = \delta)$ point. At the Hopf bifurcation $\varepsilon = \delta$ necessary condition $\text{Re} \lambda_{1,2}(\varepsilon, \delta) = 0$, vector field (2.16)
takes a simpler form:

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} =
\begin{bmatrix}
-x\delta + 2xy\delta + x^2\delta \\
y\delta - 2xy\delta - y^2\delta
\end{bmatrix}
\]

Using formula (2.8) we can back out the nonlinear functions \(f^1, f^2\) needed for the computation of the first Lyapunov coefficient:

\[
\begin{align*}
    f_1(x, y) &= y\sqrt{3}\delta - x\delta + 2xy\delta + x^2\delta \\
    f_2(x, y) &= -x\sqrt{3}\delta + y\delta - 2xy\delta - y^2\delta
\end{align*}
\]

We can now state the following result:

**Lemma 1** All Hopf bifurcations are degenerate in the circulant Rock-Scissor-Paper game under Replicator Dynamics.

**Proof.** From the nonlinear functions \(f_1(x, y), f_2(x, y)\) derived above we can easily compute the first Lyapunov coefficient (2.10) as \(l_1(\varepsilon^{Hopf} = \delta^{Hopf}) = 0\) which means that there is a first degeneracy in the third order terms from the Taylor expansion of the normal form. The detected bifurcation is a *Bautin* or *degenerate Hopf* bifurcation (assuming away other higher order degeneracies: technically, the second Lyapunov coefficient \(l_1(\varepsilon^{Hopf} = \delta^{Hopf})\) should not vanish). ■

Although, in general, the orbital structure at a degenerate Hopf bifurcation may be extremely complicated (see Section (2.3)), for our particular vector field induced by the Replicator Dynamics it can be shown by Lyapunov function techniques (Hofbauer and Sigmund (2003), Zeeman (1980)) that a continuum of cycles is born *exactly* at the critical parameter value \(8\).

The absence of a generic Hopf bifurcation does not suffice to conclude that the vector field admits no isolated periodic orbits and 'global' results are required (e.g. the Bendixson-Dulac method, positive divergence of the vector field on the simplex).
Figure 2.1: Rock-Scissors-Paper and Replicator Dynamics for fixed $\varepsilon = 1$ and varying $\delta$. Qualitative changes in the phase portraits - unstable focus (Panel (a)), continuum of cycles (Panel (b)) and stable focus (Panel (c)) - occur as we increase $\delta$ from below to above $\varepsilon$.

It would be worth pointing out the connection between our local (in)stability results and the static concept of Evolutionary Stable Strategy (ESS). As already noted in the literature (Zeeman (1980), Hofbauer (2000)) ESS implies (global) asymptotic stability under a wide class of evolutionary dynamics - Replicator Dynamics, Best Response and Smooth Best Response Dynamics, Brown-von Newmann-Nash, etc. The reverse implication does not hold in general, i.e. the local stability analysis does not suffice to qualify an attractor as an ESS. However, for $\delta < \varepsilon$ we have shown that $x^*$ is an unstable focus and we can conclude that, for this class of RSP games, the barycentrum is not an ESS. Indeed, the case $\delta < \varepsilon$ is the so called bad RSP game (Sandholm (2008)) which is known not to have an ESS. For $\delta > \varepsilon$ we are in the good RSP game and it does have an interior ESS which coincides with the asymptotically stable rest point $x^*$ of the Replicator Dynamics. Last, if $\delta = \varepsilon$ (i.e. standard RSP) then $x^*$ is a neutrally stable strategy/state and we proved that in this zero-sum game the Replicator undergoes a degenerate Hopf bifurcation. The Hopf bifurcation scenario is illustrated in Fig. 2.1 for a particular value of $\varepsilon$. 
Circulant RSP Game and Logit Dynamics

The logit evolutionary dynamics (2.6) applied to our normal form game (2.14) leads to the following vector field:

\[
\begin{bmatrix}
\dot{x} &= \frac{\exp(\beta(y\delta-z\varepsilon))}{\exp(\beta(y\delta-z\varepsilon)) + \exp(\beta(-x\varepsilon+z\delta)) + \exp(\beta(x\delta-y\varepsilon))} - x \\
\dot{y} &= \frac{\exp(\beta(-x\varepsilon+z\delta))}{\exp(\beta(y\delta-z\varepsilon)) + \exp(\beta(-x\varepsilon+z\delta)) + \exp(\beta(x\delta-y\varepsilon))} - y \\
\dot{z} &= \frac{\exp(\beta(x\delta-y\varepsilon))}{\exp(\beta(y\delta-z\varepsilon)) + \exp(\beta(-x\varepsilon+z\delta)) + \exp(\beta(x\delta-y\varepsilon))} - z
\end{bmatrix}
\]  

(2.17)

By substituting \( z = 1 - x - y \) into (2.17) we can reduce the initial system to a 2-dimensional system of equations and solve for its fixed points in the interior of the simplex, numerically, for different parameters values \((\delta, \varepsilon, \beta)\):

\[
\begin{bmatrix}
\frac{\exp(\beta(y\delta-z\varepsilon(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon)) + \exp(\beta(-x\varepsilon+z\delta(-x-y+1))) + \exp(\beta(y\delta-z\varepsilon(-x-y+1)))} - x = 0 \\
\frac{\exp(\beta(-x\varepsilon+z\delta(-x-y+1)))}{\exp(\beta(x\delta-y\varepsilon)) + \exp(\beta(-x\varepsilon+z\delta(-x-y+1))) + \exp(\beta(y\delta-z\varepsilon(-x-y+1)))} - y = 0
\end{bmatrix}
\]  

(2.18)

The 2-dim simplex barycentrum \([x = 1/3, y = 1/3, z = 1/3]\) remains a fixed point irrespective of the value of \( \beta \). The Jacobian of (2.18) evaluated at this steady state is:

\[
\begin{bmatrix}
\frac{1}{6} \beta \varepsilon - 1 & \frac{1}{3} \beta \delta + \frac{1}{3} \beta \varepsilon \\
-\frac{1}{3} \beta \delta - \frac{1}{3} \beta \varepsilon & -\frac{1}{3} \beta \delta - 1
\end{bmatrix}
\], with eigenvalues: \( \lambda_{1,2} = \frac{1}{6} \beta \varepsilon - \frac{1}{6} \beta \delta - 1 \pm i \frac{1}{3} \sqrt{1/3 (\beta \delta + \beta \varepsilon)} \). The Hopf bifurcation (necessary) condition \( \text{Re}(\lambda_{1,2}) = 0 \) leads to:

\[
\beta^{\text{Hopf}} = \frac{6}{\varepsilon - \delta}, \quad 0 < \delta < \varepsilon
\]  

(2.19)

We notice that for the zero-sum RSP game \((\varepsilon = \delta)\) - unlike Replicator Dynamics which exhibited a degenerate Hopf at \( \varepsilon = \delta \) - the barycentrum is always asymptotically stable (\( \text{Re} \lambda_{1,2} = -1 \)) under Logit Dynamics. We have the following:

**Lemma 2** The Logit Dynamics (2.17) on the circulant Rock-Scissors-Paper game exhibits a generic Hopf bifurcation and, therefore, has limit cycles. Moreover, all such Hopf bifurcations are supercritical, i.e. the limit cycles are born stable.
Proof. Condition (2.19) gives the necessary first-order condition for Hopf bifurcation to occur; in order to show that the Hopf bifurcation is non-degenerate we have to compute, according to (2.10) the first Lyapunov coefficient \( l_1(\beta^{\text{Hopf}}, \varepsilon, \delta) \) and check whether it is non-zero. The analytical form of this coefficient takes a complicated expression of exponential terms (see appendix (2.A)) which, after some tedious computations, boils down to:

\[
l_1(\beta^{\text{Hopf}}, \varepsilon, \delta) = -\frac{864(\delta \varepsilon + \delta^2 + \varepsilon^2)}{3(3\varepsilon - 3\delta)^2} < 0, \quad \varepsilon > \delta > 0.
\]

Computer simulations of this route to a stable cycle are shown in Fig. 2.2 below. We notice that as \( \beta \) moves up from 10 to 35 (i.e. the noise level is decreasing) the interior stable steady state loses stability via a supercritical Hopf bifurcation and a small, stable limit cycle emerges around the unstable steady state. Unlike Replicator Dynamics, stable cyclic behavior does occur under the Logit dynamics even for three-strategy games.

![Diagram](image-url)

(a) Stable focus, \( \beta = 10 \)  
(b) Generic Hopf, \( \beta = 30 \)  
(c) Limit cycle, \( \beta = 35 \)

Figure 2.2: Rock-Scissors-Paper and Logit Dynamics for fixed game \( \varepsilon = 1, \delta = 0.8 \) and free behavioral parameter \( \beta \). Qualitative changes in the phase portraits: a stable interior fixed point (Panel (a)) loses stability when the critical threshold \( \beta = 30 \) is hit, via a generic, supercritical Hopf bifurcation (Panel (b)) and, if \( \beta \) is pushed up even further, a stable limit cycle is born (Panel (c)).
Similar periodic behaviour can be detected in the payoff parameter space as in Fig. 2.3 below where the noise level is kept constant and the game is allowed to change.

Figure 2.3: Rock-Scissors-Paper and Logit Dynamics for fixed behavioral parameter $\beta = 10$ and free game parameter $\delta [\varepsilon = 1]$. Qualitative changes in the phase portraits: a stable interior fixed point (Panel (a)) loses stability when the critical threshold $\delta = 0.399$ is hit, via a generic, supercritical Hopf bifurcation (Panel (b)). Panel (c) display a stable limit cycle for $\delta = 0.1$ that attracts trajectories originating both outside and inside the cycle.

A more thorough search in the parameters space is performed with the help of a sophisticated continuation software\(^9\)(Dhooge et al. (2003)) which enable us to determine all combination of relevant parameters giving rise to a specific singularity. Figure 2.4 depicts curves of Hopf bifurcations in the $(\beta, \varepsilon)$ and $(\beta, \delta)$ parameter space (fixing $\delta(\varepsilon)$ to 1). As we cross these Hopf curves from below the stable interior fixed point loses stability and a stable periodic attractor surrounds it.

---

\(^9\)MatCont (see package documentation at http://sourceforge.net/projects/matcont/).

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**Generalized RSP Game and Logit Dynamics**

In this subsection we discuss the occurrence of limit cycles in the most general specification of a Rock-Scissors-Paper game (2.13) under Logit Dynamics. For circulant RSP game (i.e. circulant payoff matrix) Weissing (1991) proved that the discrete and continuous time Replicator Dynamics are ‘qualitatively’ equivalent. However, for a non-circulant payoff matrix (2.20), co-existing stable and unstable limit cycles are found under discrete-time Replicator Dynamics. This sharply contrasts with the behaviour of the continuous-time Replicator Dynamics which is known not to give rise to generic Hopf bifurcations. On the other hand, as shown in the previous subsection stable limit cycles do occur under Logit Dynamics so a natural question to pose is whether multiple, interior periodic attractors can be detected under the logistic dynamics in the generalized version of the circulant RSP game.

Weissing (1991) presents the following generalized RSP game as an example of co-existing limit cycles under discrete-time Replicator Dynamics:
\[ W = \begin{pmatrix} k & b_1 & d_1 \\ d_2 & k & b_2 \\ b_1 + 3\mu & d_1 - 3\mu & k \end{pmatrix}, \quad b_i > k \geq d_i, d_1 \geq 3\mu > 0 \quad (2.20) \]

We have already seen that this general specification can be turned into (2.13) by a positive linear payoff transformations as the continuous-time dynamics remain invariant under such transformation\(^\text{10}\).

In the sequel we will investigate, numerically, the dynamical system resulting from the generalized game payoff matrix (2.13) and the Logit evolutionary dynamics:

\[
\dot{x}_i = \frac{e^{\beta(Ax)_i}}{\sum_{i=1}^3 e^{\beta(Ax)_i}} \quad A x = \begin{bmatrix} \delta_2 x_2 - \varepsilon_3 x_3 \\ \delta_3 x_3 - \varepsilon_1 x_1 \\ \delta_1 x_1 - \varepsilon_2 x_2 \end{bmatrix}
\]

Figure 2.5 displays typical curves of fold and Hopf bifurcations produced by MatCont, along with their codimension II singularities in various sub-regions of the parameters space. The curves are initiated at certain bifurcation points\(^\text{11}\) detected along the system’s equilibria curves. They first generalize results obtained in previous subsection for a circulant RSP payoff matrix to the entire class of RSP games. Second, from the bifurcation diagrams we can detect, the co-existence of multiple interior steady-states (born via Limit Point bifurcation) and limit cycle (created when a Hopf curve is crossed). However, the fold curves seem to emerge beyond the generalized RSP class of games (i.e. the positive \(\mathbb{R}^2\) orthant in each 2-parameter subspace)

In order to address the question of co-existing limit cycles under continuous-time

\(^{10}\)For continous-time Replicator Dynamics the proof could be found in Weissing (1991)). Yet, the dynamics on (2.20 ) and (2.13) may be qualitatively different under discrete-time version of the Replicator Dynamics.

\(^{11}\)A supercritical (first Lyapunov coefficient \(l_1 = -3.222\)) Hopf point is detected at \(\delta_1 = 0.626\) and two Fold points at \(\delta_1 = -0.554\) and \(\delta_1 = -0.952\). Benchmark parameterization: \(\beta = 8.3, \delta_1 = 4, \delta_2 = 3, \delta_3 = 2, \varepsilon_1 = 2, \varepsilon_{21} = 3, \varepsilon_3 = 4\).
Logit Dynamics, one should run a systematic search in the parameter space for a fold bifurcation of limit cycles: detect the Hopf points, continue them with respect to each of the 6-parameter in the game form (2.13) and, then track any codimension II singularity (most interestingly, the limit point of cycles bifurcation). A representative subset of the resulting Hopf curves is plotted in Fig. 2.5. Except for the Bogdanov-Takens (BT) bifurcation, no further codimension II singularities are detected by the continuation package in the positive $\mathbb{R}^2$ orthant corresponding to the generalized class of RSP game. In particular, there is no limit point/fold bifurcation of cycles, i.e. multiple, co-existing (un)stable limit cycles are ruled out.
Figure 2.5: Generalized Rock-Scissors-Paper - curves of codimension I - fold (LP) and Hopf (H) - bifurcations along with the detected codimension II singularities - Cusp (CP), Bogdanov-Taken (BT), Cusp (CP) and Zero-Hopf (ZH) points - in various two-parameter subspaces. Unless free to float, game parameters are fixed to following values - $\delta_1 = 0.62, \delta_2 = 4, \delta_3 = 3, \varepsilon_1 = 2, \varepsilon_2 = 3, \varepsilon_3 = 4$ - while the intensity of choice is $\beta = 8$
2.4.2 Coordination Game

Using topological arguments, Zeeman (1980) shows that three-strategies games have at most one interior, isolated fixed point under Replicator Dynamics (see Theorem 3 in Zeeman (1980)). In particular, fold catastrophe (two isolated fixed points which collide and disappear when some parameter is varied) cannot occur within the simplex. In this section we provide - by means of the classical coordination game - numerical evidence for the occurrence of multiple, isolated interior steady-states under the Logit Dynamics and show that fold catastrophe is possible when we alter the intensity of choice $\beta$. The coordination game we consider is given by the following payoff matrix:

$$A = \begin{pmatrix} 1 - \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \varepsilon \end{pmatrix}, \varepsilon \in (0, 1)$$

Coordination Game and Replicator Dynamics

The vector field $\dot{x} = x[Ax - xA]p$ becomes:

$$\begin{align*}
\dot{x} &= x[x(-\varepsilon + 1) - (y^2 + z^2(\varepsilon + 1) + x^2(-\varepsilon + 1))] \\
\dot{y} &= y[y - (y^2 + z^2(\varepsilon + 1) + x^2(-\varepsilon + 1))] \\
\dot{z} &= z[z(\varepsilon + 1) - (y^2 + z^2(\varepsilon + 1) + x^2(-\varepsilon + 1))] 
\end{align*}$$

This systems has the following fixed points in simplex coordinates:

(i) simplex vertices (stable nodes): $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

(ii) interior (source): $O \left(\frac{\varepsilon + 1}{3-\varepsilon^2}, \frac{1-\varepsilon^2}{3-\varepsilon^2}, \frac{1-\varepsilon}{3-\varepsilon^2}\right)$

(iii) on the boundary (saddles):

$M \left(\frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon}, 0\right), N \left(\frac{1}{2-\varepsilon}, 0, \frac{1-\varepsilon}{2-\varepsilon}\right), P \left(0, \frac{1}{2-\varepsilon}, \frac{1-\varepsilon}{2-\varepsilon}\right)$

The eigenvalues of the corresponding Jacobian evaluated at each of the above fixed points are:

$A \left[ x = 0, y = 1, z = 0 \right]$; eigenvalues: $\lambda_{1,2,3} = -1$
\[ B \left[ x = 1, y = 0, z = 0 \right]; \text{eigenvalues: } \lambda_{1,2,3} = \varepsilon - 1 \]
\[ C \left[ x = 0, y = 0, z = 1 \right]; \text{eigenvalues: } \lambda_{1,2,3} = -\varepsilon - 1 \]
\[ O \left[ x = \frac{\varepsilon + 1}{3 - \varepsilon}, y = \frac{1 - \varepsilon^2}{3 - \varepsilon^2}, z = \frac{1 - \varepsilon}{3 - \varepsilon^2} \right]; \text{eigenvalues: } \lambda_{1,2} = \frac{\varepsilon^2 - 1}{\varepsilon^2 - 3} > 0, \text{ for } \varepsilon \in (0,1) \]
\[ M \left[ x = \frac{1}{2 - \varepsilon}, y = \frac{1 - \varepsilon}{2 - \varepsilon}, z = 0 \right]; \text{eigenvalues: } \lambda_{1,2} = \pm \frac{1}{\varepsilon - 2} \]
\[ N \left[ x = \frac{1}{2 - \varepsilon}, y = 0, z = \frac{1 - \varepsilon}{2 - \varepsilon} \right]; \text{eigenvalues: } \lambda_{1,2} = \pm \frac{1}{\varepsilon + \varepsilon^2 + 4} \left( 2\varepsilon + \varepsilon^2 - \varepsilon^3 - 2 \right) \]
\[ P \left[ x = 0, y = \frac{1}{2 - \varepsilon}, z = \frac{1 - \varepsilon}{2 - \varepsilon} \right]; \text{eigenvalues: } \lambda_{1,2} = \pm \frac{1}{4\varepsilon + \varepsilon^2 + 4} \left( \varepsilon + \varepsilon^2 - \varepsilon^3 - 2 \right) \]

This fixed points structure is consistent with Zeeman (1980) result that no fold catastrophes (i.e. multiple, isolated and interior steady states) can occur under the Replicator Dynamics. The three saddles together with the interior source define the basins of attractions for the simplex vertices. The boundaries of the simplex are invariant under Replicator Dynamics\textsuperscript{12} and it suffices to show that the segment lines \([OM],[ON],\text{ and } [OP]\) (see Fig. 2.6 for a plot of these segments for \(\varepsilon = 0.1\)) are also invariant under this dynamics.

---

\textsuperscript{12}see proof in Hofbauer and Sigmund (2003) pp. 67-68.
Lemma 3  The segment lines $[OM],[ON]$, and $[OP]$ are invariant under Replicator Dynamics and they form, along with the simplex edges, the boundaries of the basins of attraction for the three stable steady states.

Proof. Using the standard substitution $z = 1 - x - y$ system (2.21) becomes:

$$
\begin{bmatrix}
\dot{x} = x^2 - x^3 - xy^2 - x^2\varepsilon + x^3\varepsilon - x(-x - y + 1)^2 - x\varepsilon(-x - y + 1)^2 \\
\dot{y} = y^2 - y^3 - x^2y + x^2y\varepsilon - y(-x - y + 1)^2 - y\varepsilon(-x - y + 1)^2
\end{bmatrix}
$$

(2.22)

Segment $[MO]$ is defined, in simplex coordinates, by $y = x(1 - \varepsilon)$. Along this line we have:

$$
\begin{bmatrix}
\dot{y} \\
\dot{x}
\end{bmatrix}_{[MO]:y=x(1-\varepsilon)} = \frac{x(-\varepsilon + 1)(x(-\varepsilon + 1) - x^2(-\varepsilon + 1) - x^2(-\varepsilon + 1)^2 - (\varepsilon + 1)(-x - y + 1)^2)}{x(x(-\varepsilon + 1) - x^2(-\varepsilon + 1) - x^2(-\varepsilon + 1)^2 - (\varepsilon + 1)(-x - y + 1)^2)} = 1 - \varepsilon,
$$

which is exactly the slope of $[MO]$. Similarly, invariance results are obtained for segments $[ON]$ and $[OP]$ defined by $x(1 - \varepsilon) = z(1 + \varepsilon)$ and $y = z(1 + \varepsilon)$.

Analytically, the basins of attraction are determined by the areas of the polygons delineated by the invariant manifolds $[OM],[ON]$, and $[OP]$ and the 2-dim simplex boundaries, as follows:

$$
A(0,1,0) = A[AMON] = \left[\begin{array}{c}
\frac{1}{8}\sqrt[3]{\frac{\varepsilon+1}{4-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-2\varepsilon}{6-2\varepsilon}} - \frac{1}{8}\sqrt[3]{\frac{\varepsilon+1}{3-2\varepsilon}} + \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{6-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{6-2\varepsilon}} \\
+ \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{6-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{\varepsilon+1}{4-2\varepsilon}}(3-\varepsilon) - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{4-2\varepsilon}}(3-\varepsilon)
\end{array}\right]
$$

$$
B(0,1,0) = A[BMOP] = \left[\begin{array}{c}
\frac{1}{8}\sqrt[3]{\frac{\varepsilon+1}{4-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-2\varepsilon}{6-2\varepsilon}} + \frac{1}{4}\sqrt[3]{\frac{\varepsilon+1}{3-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{6-2\varepsilon}}(3-\varepsilon) - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{4-2\varepsilon}}(3-\varepsilon)
\end{array}\right]
$$

$$
C(0,0,1) = A[CNOP] = \left[\begin{array}{c}
\frac{1}{8}\sqrt[3]{\frac{\varepsilon+1}{4-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-2\varepsilon}{6-2\varepsilon}} + \frac{1}{4}\sqrt[3]{\frac{\varepsilon+1}{3-2\varepsilon}} - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{6-2\varepsilon}}(3-\varepsilon) - \frac{1}{4}\sqrt[3]{\frac{3-\varepsilon}{4-2\varepsilon}}(3-\varepsilon)
\end{array}\right]
$$

The basins’ sizes vary with the payoff perturbation parameter $\varepsilon$ as in Table 2.1.

When the payoff asymmetries are small, the simplex of initial conditions is divided equally among the three fixed points/equilibria (Fig. 2.7a). As the payoff discrepancies increase most of the initial conditions are attracted by the welfare-maximizing equilibrium $(0,0,1)$ (see Fig. 2.7b).
### Table 2.1: Coordination Game, Replicator Dynamics–Long-run average fitness for different payoff-perturbation parameter $\varepsilon$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$A(1,0,0)$</th>
<th>$B(0,1,0)$</th>
<th>$C(0,0,1)$</th>
<th>Long-Run Average Fitness/Welfare</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>33.33%</td>
<td>33.33%</td>
<td>33.33%</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>29.9%</td>
<td>33.2%</td>
<td>36.7%</td>
<td>1.0048</td>
</tr>
<tr>
<td>0.5</td>
<td>15.5%</td>
<td>30.3%</td>
<td>54.54%</td>
<td>1.1986</td>
</tr>
<tr>
<td>0.6</td>
<td>8.92%</td>
<td>12.18%</td>
<td>78.90%</td>
<td>1.4199</td>
</tr>
<tr>
<td>0.9</td>
<td>1.2%</td>
<td>12%</td>
<td>86.7%</td>
<td>1.512</td>
</tr>
</tbody>
</table>

Figure 2.7: Coordination Game and Replicator Dynamics-basins of attraction for different values of the payoff parameter $\varepsilon$

#### Coordination Game and Logit Dynamics

We choose a small payoff perturbation, $\varepsilon = 0.1$, such that system is ‘close’ to the symmetric basins of attraction $A, B$ and $C$ in the Replicator Dynamics. Logit Dynamics together with payoff matrix $A$ define the following vector field on the simplex of frequencies $(x, y, z)$ of strategies $E_1, E_2, E_3$, respectively:
\[
\begin{align*}
\dot{x} &= \frac{\exp(0.9\beta x)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - x \\
\dot{y} &= \frac{\exp(\beta y)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - y \\
\dot{z} &= \frac{\exp(1.1\beta z)}{\exp(0.9\beta x) + \exp(\beta y) + \exp(1.1\beta z)} - z
\end{align*}
\]

In order to ascertain the number of (asymptotically) stable fixed points on the two-dimensional simplex we first run simulations for increasing values of \( \beta \) and for different initial conditions within the simplex:

Case I. \( \beta \approx 0 \). One interior steady state (initial conditions, time series and attracting point in Fig. 2.8a). The barycentrum is globally attracting, i.e. irrespective of the initial proportions, the population will settle down to a state with equalized fractions.

Case II. \( \beta \geq 10 \). Three stable steady states asymptotically approaching the vertices of the simplex as \( \beta \) increases (Fig. 2.8b-d) plus other unstable steady states. The size of their basins of attraction is determined both by the strategy relative payoff advantage and by the value of the intensity of choice. Whichever of the three strategy happens to outnumber the other two in the initial population, will eventually win the evolutionary competition.
(a) $\beta = 0$, any initial mixture $(x, y, z)$ converges to $(1/3, 1/3, 1/3)$

(b) $\beta = 10$, initial mixture $(0.4, 0.3, 0.3) \to (1, 0, 0)$

(c) $\beta = 10$, initial mixture $(0.25, 0.4, 0.35) \to (0, 1, 0)$

(d) $\beta = 10$, initial fractions $(0.33, 0.33, 0.34) \to (0, 0, 1)$

Figure 2.8: Coordination Game[$\varepsilon = 0.1$] and Logit Dynamics - Unique stable steady state for $\beta$ low (Panel(a)) and three, co-existing stable steady states for high $\beta = 10$ (Panels (b)-(d)).

Multiple Equilibria. The most interesting situations occur in transition from high to low values of the behavioral parameter $\beta$: we will show that the Logit Dynamics undertakes a sequence of fold bifurcations by which each of the three stable
steady states collides with another unstable steady state from within the simplex and disappears. Fig. 2.9cd shows simulations of time series converging to a unique steady state for low values of $\beta$ ($\beta = 2.6$). Panels (a)-(b) show the co-existence of two steady-states for $\beta = 3$ as different initial population mixtures converge to different steady states.

Figure 2.9: Coordination Game $[\varepsilon = 0.1]$ and Logit Dynamics with moderate level of noise -Panels (a)-(b) display two interior, isolated stable steady states, while Panels (c)-(d) shows convergence to a unique, interior steady state.
**Bifurcations.** Unlike Replicator, the Logit Dynamics displays multiple, interior isolated steady states created via a fold bifurcation. In the particular case of a 3-strategy Coordination game, three interior stable steady states emerge through a sequence of two saddle-node bifurcations.

Fig. 2.10 depicts the fold bifurcations scenario by which the multiple, interior fixed points appear when the intensity of choice (Panel (a)) or the payoff parameter (Panel (b)) changes. For small values of $\beta$ the unique, interior stable steady state is the simplex barycentrum $(1/3, 1/3, 1/3)$. As $\beta$ increases this steady state moves towards the Pareto-superior equilibrium $(0, 0, 1)$. A first fold bifurcation occurs at $\beta = 2.77$ and two new fixed points are created, one stable and one unstable. If we increase $\beta$ even further ($\beta \approx 3.26$) a second fold bifurcation takes place and two additional equilibria emerge, one stable and one unstable. Last, two new fixed points arise at $\beta = 4.31$ via a saddle-source bifurcation\(^\text{13}\). There is co-existence of three stable steady states for large values of the intensity of choice $\beta$, the ‘logit equilibria’. A similar sequence of bifurcations is visible in the payoff parameter space (Fig. 2.10b) where a family of fold bifurcations is obtained for a particular value of the switching intensity $\beta$.

\(^\text{13}\)At a saddle-source bifurcation two additional *unstable* steady states emerge.
Figure 2.10: Coordination Game and Logit Dynamics. Curves of equilibria along with codimension I singularities, in this case fold (LP) points. Panel (a)-(b): the multiplicity of steady states arises through a sequence of three limit point (fold) bifurcations as one of the two parameters varies.

The continuation of the fold curves in the $(\beta, \varepsilon)$ parameter space allows the detection of codimension II bifurcations. The cusp points in Fig. 2.11 below organize the entire bifurcating scenario and capture the emergence of multiple, interior point-attractors in Coordination Game under the smoothed best response dynamics. Once such a curve of fold points is crossed from below two additional equilibria are created: one stable and one unstable after a saddle-node bifurcation and two unstable steady states after a saddle-source bifurcation. If choice is virtually random ($\beta = 0$) there is an unique steady state while for large $\beta$ there are 7 steady states (3 stable and 4 unstable).
Figure 2.11: Coordination Game and Logit Dynamics. Curves of fold points along with detected codimension II singularities - in this case, cusp (CP) and zero-Hopf (ZH) points - traced in the \((\varepsilon, \beta)\) parameter space.

**Basins of Attraction.** The numerical computation of the basins of attraction for different equilibria reveals interesting properties of the Logit dynamics from a social welfare perspective. We construct a measure of long-run aggregate welfare as the payoff at the stable steady state weighted by its corresponding basin of attraction size. While for extreme values of the intensity of choice the basins of attraction are similar in size with the Replicator Dynamics (Panels (a), (c), (d) in Fig. 2.12,2.13), for moderate levels of rationality the population manages to coordinate close to the Pareto optimal Nash equilibria (Panel (b) in Fig. 2.12,2.13). The aggregate welfare evolves non-monotonically with respect to the behavioural parameter \(\beta\) and, for a given payoff perturbation \(\varepsilon\), it is maximal just before the first bifurcation occurs \(\beta_{LP_1} = 2.77\).
(a) $\beta = 1$, one barycentrical steady state

(b) $\beta = 10/4$, one steady state 'close' to the Pareto superior equilibrium

(c) $\beta = 10/3$, two stable steady states

(d) $\beta = 15$, three stable steady states

Figure 2.12: Coordination Game $|\varepsilon = 0.1|$ and Logit Dynamics: Panels (a)-(d), basins of attraction for increasing values of the intensity of choice $\beta$. Fractions converging to each of the steady state are indicated in the box.
The long run average welfare evolves non-monotonically with respect to the behavioural parameter $\beta$ (Fig. 2.14ab): it increases as the fully mixed equilibrium slides slowly towards the Pareto optimal $(0, 0, 1)$ vertex, attains a maximum before the first fold bifurcation occurs at $\beta^{LP_1} = 2.77$ and then decreases, approaching the Replicator Dynamics average welfare, in the limit of $\beta \to \infty$. There are two effects driving the welfare peak before $\beta^{LP_1}$ is hit: first, the steady state payoff is higher the
closer the steady state is to the Pareto optimal equilibrium. Second there is a ‘basin of attraction’ effect: for $\beta < \beta^{LP}$ the entire simplex of initial conditions is attracted by the unique steady state lying close to the optimal equilibrium. Intuitively, the noisy choices in the low-beta regime help players escape the path-dependence built into Coordination games while for super-rational play are trapped into best, albeit payoff-inferior, responses.

In the limiting case $\beta \to \infty$, the fixed points of the Logit Dynamics (i.e. the logit equilibria) coincide with the Nash equilibria of the underlying game which, for this Coordination Game, are exactly the fixed points of the Replicator Dynamics. Thus the analysis (stable fixed points, basins’ of attraction sizes) of the ‘unbounded’ rationality case is identical to the one pertaining to the Replicator Dynamics in Coordination game (see subsection (2.4.2)).

Figure 2.14: Coordination Game and Logit Dynamics-Long-run average welfare plots as function of payoff perturbation parameter $\varepsilon$
2.5 Weighted Logit Dynamics (wLogit)

In this last section we run computer simulations for a three-strategy - Rock-Scissors-Paper - and a four-strategy - Schuster et al. (1991) example-game, from the perspective of a different type of evolutionary dynamics closely related to the Logit dynamic, namely the frequency-weighted Logit:

\[ \dot{x}_i = \frac{x_i \exp(\beta Ax)_i}{\sum_k x_k \exp(\beta Ax)_k} - x_i, \quad \beta = \eta^{-1} \] (2.23)

This evolutionary dynamic has the appealing property that when \( \beta \) approaches 0 it converges to the Replicator Dynamics (with adjustment speed scaled down by a factor \( \beta \)) and when the intensity of choice is very large it approaches the Best Response dynamic (Hofbauer and Weibull (1996)).

2.5.1 Rock-Scissors-Paper and wLogit Dynamics

Dynamic (2.23) together with game payoff matrix (2.14) give rise to the following dynamical system:

\[
\begin{align*}
\dot{x} &= \frac{x \exp[\beta x - y/3\delta]}{x \exp[\beta x - y/3\delta] + y \exp[x \beta - z/3\delta] + z \exp[y \beta - x/3\delta]} - x \\
\dot{y} &= \frac{y \exp[x \beta - z/3\delta]}{x \exp[\beta x - y/3\delta] + y \exp[x \beta - z/3\delta] + z \exp[y \beta - x/3\delta]} - y \\
\dot{z} &= \frac{z \exp[y \beta - x/3\delta]}{x \exp[\beta x - y/3\delta] + y \exp[x \beta - z/3\delta] + z \exp[y \beta - x/3\delta]} - z
\end{align*}
\]

To contrast Replicator and the Logit dynamics with the help of the wLogit we can start by fixing two parameters \( \beta = 5 \) and \( \varepsilon = 1 \), and let \( \delta \) free. We obtain that the barycentrum is an interior fixed point of the system, irrespective of the value of \( \delta \). When the critical threshold \( \delta^{Hopf} = 1 \) is hit a supercritical Hopf bifurcation arises (first Lyapunov coefficient = -1.249) and a stable limit cycle is born exactly as in the Logit dynamic (cf. Fig. 2.15ab). However, unlike Logit, fluctuations are not limited in amplitude by some Shapley polygon, but they can easily approach the boundary

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of the simplex (Fig. 2.15cd) a feature reminiscent of Replicator Dynamics\textsuperscript{14}. The limit cycles are stable because they are produced by a generic Hopf bifurcation. With Replicator Dynamics the Hopf bifurcations are degenerate and there is a continuum of periodic orbits. These features are illustrated in Fig. 2.15 where stable cycles are detected and continued in the game and behavioral parameters. For instance, one family of limit cycles parameterized by the payoff matrix parameter \(\varepsilon\) that gets closer to the simplex boundaries as \(\varepsilon\) decreases is shown in Panel (c).

\textsuperscript{14}Notice the difference with the Replicator Dynamics where system converges to a \textit{heteroclinic} cycle from each simplex vertex to the other.
(a) $\beta = 5, \varepsilon = 1$. Line of equilibria, free $\delta; \delta^{Hopf} = 1$.

(b) $\beta = 5, \varepsilon = 1$. Phase portrait, $\delta^{Hopf} = 1$

(c) Curve of limit cycles $(x, y, \varepsilon)$. $\beta = 5, \delta = \delta^{Hopf}$

(d) Curve of limit cycles $(x, y, \beta)$. $\varepsilon = 1, \delta = \delta^{Hopf}$

Figure 2.15: RSP Game and Weighted Logit Dynamics. Panels (a)-(d): long-run behavior inherits traits from both Logit (stable limit cycles, Panels(a)-(b)) and Replicator Dynamics (cycles of large amplitude, nearing the simplex boundaries, Panels (c)-(d)). Unless free, parameters set to $\varepsilon = 1, \beta = 5, \delta = 1$
2.5.2 Schuster et al. (1991) Game and wLogit Dynamics

Schuster et al. (1991) derive the following payoff matrix from models of biological interaction:

\[
A = \begin{pmatrix}
0 & 0.5 & -0.1 & 0.1 \\
1.1 & 0 & -0.6 & 0 \\
-0.5 & 1 & 0 & 0 \\
1.7 + \mu & -1 - \mu & -0.2 & 0 \\
\end{pmatrix}
\] (2.24)

The Weighted Logit (2.23) with the underlying game (2.24) generates the following vector field on the 4-simplex:

\[
\begin{align*}
\dot{x} &= \frac{x e^{[\beta(0.1t+0.5+y-0.1z)]}}{x e^{[\beta(0.1t+0.5+y-0.1z)]} + y e^{[\beta(1.1+0.6-0.2)]} + z e^{[\beta(-0.5+y)]} + t e^{[\beta(-0.2+y(-\mu-1)+x(\mu+1.7)]}}} - x \\
\dot{y} &= \frac{y e^{[\beta(0.1t+0.5+y-0.1z)]}}{x e^{[\beta(0.1t+0.5+y-0.1z)]} + y e^{[\beta(1.1+0.6-0.2)]} + z e^{[\beta(-0.5+y)]} + t e^{[\beta(-0.2+y(-\mu-1)+x(\mu+1.7)]}}} - y \\
\dot{z} &= \frac{z e^{[\beta(0.1t+0.5+y-0.1z)]}}{x e^{[\beta(0.1t+0.5+y-0.1z)]} + y e^{[\beta(1.1+0.6-0.2)]} + z e^{[\beta(-0.5+y)]} + t e^{[\beta(-0.2+y(-\mu-1)+x(\mu+1.7)]}}} - z \\
\dot{t} &= \frac{t e^{[\beta(0.1t+0.5+y-0.1z)]}}{x e^{[\beta(0.1t+0.5+y-0.1z)]} + y e^{[\beta(1.1+0.6-0.2)]} + z e^{[\beta(-0.5+y)]} + t e^{[\beta(-0.2+y(-\mu-1)+x(\mu+1.7)]}}} - t
\end{align*}
\]

Schuster et al. (1991) show that, within a specific payoff parameter region \([-0.2 < \mu < -0.105]\), a Feigenbaum sequence of period doubling bifurcations unfolds under the Replicator Dynamics and, eventually, chaos sets in. Here we consider the question whether the weighted version of the Logit Dynamics displays similar patterns, for a given payoff perturbation \(\mu\), when the intensity of choice \(\beta\) is varied. First we fix \(\mu\) to \(-0.2\) (the value for which the Replicator generates 'only' periodic behaviour in Schuster et al. (1991)) and run computer simulations for different \(\beta\). Cycles of increasing period multiplicity are reported in Fig. 2.16.
For $\beta = 2.5$ the wLogit Dynamics already enters the chaotic regime on a strange attractor (Fig. 2.17).
Figure 2.17: Schuster et. al. (1991) game and iLogit Dynamics-Strange Attractor. Panels (a)-(d) projections of the strange attractor onto various two-dimensional state subspaces. Parameters set to $\mu = -0.2, \beta = 2.5$. 
2.6 Conclusions

The main goal of this Chapter was to show that, even for ‘simple’ three-strategy games, periodic attractors do occur under a rationalistic way of modelling evolution in games, the Logit dynamics. The resulting dynamical systems were investigated with respect to changes in both the payoff and behavioral parameters. Identifying stable cyclic behaviour in such a system translates into proving that a generic, non-degenerate Hopf bifurcation occurs. By means of normal form computations, we showed first that a non-degenerate Hopf can not occur for Replicator Dynamics, when the number of strategies is three for games like Rock-Scissors-Paper. In these games, under the Replicator Dynamics, only a degenerate Hopf bifurcation can occur. However, in Logit dynamics, even for three strategy case, stable cycles are created, via a generic, non-degenerate, supercritical Hopf bifurcation. Another finding is that the periodic attractors can be generated either by varying the payoff parameters ($\varepsilon, \delta$) or the intensity of choice ($\beta$). Moreover, via computer simulations on a Coordination game, we showed that the Logit may display multiple, isolated, interior steady states together with the fold catastrophe, a bifurcation which is also known not to occur under the Replicator. Interestingly, a measure of aggregate welfare reaches a maximum only for intermediate values of $\beta$, when most of the population manages to coordinate close to the Pareto-superior equilibrium. Last, in a frequency-weighted version of Logit dynamics and for a $4 \times 4$ game, period-doubling route to chaos along with strange attractors emerged when the intensity of choice took moderate, ‘boundedly rational’ values.
2.A Rock-Scissor-Paper game with Logit Dynamics: Computation of the first Lyapunov coefficient

In order to discriminate between a degenerate and a non-degenerate bifurcations we need to compute the first Lyapunov coefficient. For this, we first use equations (2.8) to obtain the nonlinear functions:

\[ f_1(x, y) = y \sqrt{\frac{3\varepsilon + \delta}{\varepsilon - \delta}} - x + \frac{\exp\left(\frac{6(y\varepsilon - \varepsilon(x-y+1))}{-\delta + \varepsilon}\right)}{\exp\left(\frac{6(x\delta - y\varepsilon)}{-\delta + \varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon + \delta(x-y+1))}{-\delta + \varepsilon}\right) + \exp\left(\frac{6(y\varepsilon - \varepsilon(x-y+1))}{-\delta + \varepsilon}\right)} \]

\[ f_2(x, y) = -x \sqrt{\frac{3\varepsilon + \delta}{\varepsilon - \delta}} - y + \frac{\exp\left(\frac{6(x\delta - y\varepsilon)}{-\delta + \varepsilon}\right)}{\exp\left(\frac{6(x\delta - y\varepsilon)}{-\delta + \varepsilon}\right) + \exp\left(\frac{6(-x\varepsilon + \delta(x-y+1))}{-\delta + \varepsilon}\right) + \exp\left(\frac{6(y\varepsilon - \varepsilon(x-y+1))}{-\delta + \varepsilon}\right)} \]

Next, applying formula (2.10) for the computation of the first Lyapunov coefficient, we obtain after some further simplifications:

\[ l_1(\varepsilon, \delta) = \left[ \frac{1728\delta \varepsilon - 4320\delta^2 - 4320\varepsilon^2 - 4320\delta \varepsilon + 1728\delta^2 + 1728\varepsilon^2}{19\delta^2 - 38\delta \varepsilon + 19\varepsilon^2 - 16\delta \varepsilon + 8\delta^2 + 8\varepsilon^2} \right] = \frac{-2592\delta \varepsilon - 2592\varepsilon^2 - 2592\delta^2}{27\delta^2 - 54\delta \varepsilon + 27\varepsilon^2} \]

\[ = -2592\frac{\delta \varepsilon + \delta^2 + \varepsilon^2}{3(3\varepsilon - 3\delta)^2} \]