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Chapter 3

Heterogenous Learning Rules in Cournot Games

3.1 Introduction

Stability of the Cournot equilibrium in homogenous oligopoly has been constantly scrutinized: from the seminal work of Theocharis (1960) who showed that the Cournot-Nash equilibrium of the standard linear inverse demand-linear cost model, loses stability once we allow for more than two oligopolists, through the ‘exotic’ chaotic dynamics discovered by Rand (1978) in specific *non*-linear Cournot models and to the relatively recent work on ‘evolutionary’ Cournot models be it in a stochastic (Alos-Ferrer (2004)) or deterministic framework (Droste et al. (2002)). Sources of instability vary from non-monotonicity in the reaction curves (Kopel (1996)) to alternative specification of the quantity adjustment or expectations-formation processes (Szidarovszky et al. (2000)). While it may not appear too surprising to generate complicated behaviour when one allows for ‘strange’ reaction functions, the sensitivity of the (linear) Cournot game outcome to the learning dynamics or to the interplay of different learning ‘heuristics’ seems more intriguing given also the growing body

of experimental evidence on the heterogeneity of such learning rules¹.

As our goal is to explore the impact of such learning rules heterogeneity on the “limiting” outcome in Cournot oligopolies, we will review a few existing results concerning adjustment processes in oligopoly either in a static framework (two boundedly rational players endowed with Cournot expectations playing a best-reply to each other’s expectations) or in a population game (large group of oligopolists, randomly paired, with the resulting ecology of learning rules determined endogenously). Adaptive expectations are a straightforward generalization of Cournot expectations. They have been studied by Szidarovszky et al. (1994) who found that, in a multi-player oligopoly game, if the expectation adjustment speed is sufficiently low then the unique Cournot-Nash equilibrium is stabilized. Deschamp (1975) and Thorlund-Petersen (1990) investigate Fictitious Play duopolists, i.e. players choosing a quantity which is a best-reply to the empirical frequency distribution of the opponent past choices. Under certain restrictions on the cost structure (marginal costs not decreasing too rapidly) the process is shown to generate stable convergence to the Cournot solution. Milgrom and Roberts (1991) prove convergence for a wide range of adaptive processes provided that the oligopoly game is of so-called Type I - as coined by Cox and Walker (1998), and meaning that the reaction curves cross in such a way that the interior Cournot-Nash equilibrium is unique, i.e. there are no additional ‘boundary’ equilibria; this has to do, again, with the ‘marginal cost not decreasing too fast’ condition in Thorlund-Petersen (1990) .

Turning to the ‘evolutionary’ type of modelling oligopoly, Vega-Redondo (1997) proved that imitation with experimentation dynamics leads away from the Cournot Nash equilibrium in a finite population of Cournot oligopolists. The proof uses techniques from perturbed Markov chain theory introduced to game theory by Kandori et al. (1993) to show that the Walrasian equilibrium is the only stochastically stable

¹See Camerer (2003) for a broad overview of the experimental games literature, Cheung and Friedman (1997) for experiments on learning in specific normal form games and Huck et al. (2002) for experimental evidence on learning dynamics in Cournot games.

state when the noise parameter goes to zero. (Alos-Ferrer (2004)) shows, in a similar stochastic setting, that Vega-Redondo (1997) result is not robust to adding finite memory to the imitation rule: anything between Cournot and Walras is possible as ‘long-run’ outcomes with only one period memory. Last, Droste et al. (2002) consider, in a deterministic set-up, evolutionary competition between ‘Cournot’ players (endowed with freely available naive expectations) and rational or ‘Nash’ players (who can perfectly predict the choice of the opponent but incur a cost for information gathering). The resulting evolutionary dynamics is governed by the replicator dynamics with noise and leads to local and global bifurcations of equilibria and even strange attractors for some parameter constellations.

In this chapter, we construct an evolutionary oligopoly model where players are endowed with heterogenous learning procedures about the opponent’s expected behavior and where they update these routines according to a logistic evolutionary dynamic. In contrast with Droste et al. (2002), our focus is the interplay between adaptive and ‘fictitious play’ expectations, but other ecologies of rules (e.g. weighted fictitious play vs. rational or Nash expectations) are explored, as well. The choice of the Logit updating mechanism is motivated by the fact that this perturbed version of the Best-Reply dynamics, allows for an imperfect switching to the myopic best reply to the existing strategies distribution. The main finding is that in a population of fictitious (i.e best-responders to the empirical distribution of past history of the opponent’s choices) and adaptive players (i.e. those who place large weight on the last period choice and heavily discard more remote past observations) the Cournot-Nash equilibrium can be destabilized and complicated dynamics arise. The chapter is organized as follows: Sections 3.2 and 3.3 briefly revisit the original Cournot analysis and the set of learning rules agents are endowed with. Section 3.4 introduces the model of evolutionary Cournot duopoly and evaluates alternative heuristics ecologies. Some concluding remarks are presented in section 3.5.

3.2 Standard Static Cournot Analysis

We consider a linear Cournot duopoly model much in line with the specification used in, for example, Cox and Walker (1998) or Droste et al. (2002). Assume a linear inverse demand function:

$$P = a - b(q_1 + q_2), \quad a, b \geq 0, q_1 + q_2 \leq a/b, \quad (3.1)$$

and quadratic *concave* costs (i.e. decreasing marginal cost),

$$C_i(q_i) = cq_i - \frac{d}{2}q_i^2, \quad c, d \geq 0. \quad (3.2)$$

In a standard maximization problem firm I chooses q_1 such that it maximizes instantaneous profits taking as given the quantity choice of its competitor q_2 :

$$\begin{aligned} q_1^* &= \arg \max_{q_1} (Pq_1 - C_1) \\ &= \arg \max_{q_1} [(a - b(q_1 + q_2))q_1 - (cq_1 - \frac{d}{2}q_1^2)] \end{aligned} \quad (3.3)$$

and similarly for firm II ,

$$q_2^* = \arg \max_{q_2} [(a - b(q_1 + q_2))q_2 - (cq_2 - \frac{d}{2}q_2^2)]. \quad (3.4)$$

This gives firm I and II 's reaction curves as:

$$q_1 = R_1(q_2) = \frac{a - c - bq_2}{2b - d}; \quad q_2 = R_2(q_1) = \frac{a - c - bq_1}{2b - d} \quad (3.5)$$

and their intersection yields the interior Cournot-Nash equilibrium of the game:

$$q_1^* = q_2^* = \frac{a - c}{3b - d} \quad (3.6)$$

The question of how equilibrium is reached and, in particular, how could one of the duopolists know the strategic choice of the opponent at the moment when she is making the quantity choice decision generated much debate in the literature. Cournot (1838) was already pointing to an equilibration process by which firms best-respond to the opponent's choice in the previous period (in current terminology 'naive' expectations). However, there is a vast body of literature on whether and under what restrictions - for instance, the number of players and demand and cost function specifications - the Cournot process converges.

3.3 Heterogenous Learning Rules

3.3.1 Adaptive Expectations

In a boundedly rational environment, one possible reading of the reaction curves (3.5) is that each player best-responds to *expectations* about the other player's strategic choice.

$$\begin{aligned} q_1(t+1) &= R_1(q_2^e(t+1)) \\ q_2(t+1) &= R_2(q_1^e(t+1)). \end{aligned} \tag{3.7}$$

A number of 'expectation formation' or 'learning' rules are encountered in the literature. Among them, particularly appealing are adaptive expectations where *current* expectations about a variable adapt to past *realizations* of that variable. Formally, expectations $q_2^e(t+1)$ about the current period opponent quantity $-q_2(t+1)-$ are determined as a weighted average of last period's expectations and last period actual choice.

$$\begin{aligned} q_1^e(t+1) &= \alpha_1 q_1^e(t) + (1 - \alpha_1) R_1(q_2^e(t)) \\ &= \alpha_1 q_1^e(t) + (1 - \alpha_1) q_1(t) \end{aligned} \tag{3.8}$$

$$\begin{aligned}
q_2^e(t+1) &= \alpha_2 q_2^e(t) + (1 - \alpha_2) R_2(q_1^e(t)) \\
&= \alpha_2 q_2^e(t) + (1 - \alpha_2) q_2(t)
\end{aligned} \tag{3.9}$$

3.3.2 Fictitious Play

Fictitious play, introduced originally as an algorithm for computing equilibrium in games (Brown (1951)), asserts that each player best-responds to the *empirical distribution* of the opponent past record of play. In the context of Cournot ‘quantity-setting’ games this boils down to the average of the opponent past ‘played’ quantities where equal weight is attached to each past observation:

$$q_2^e(t+1) = \frac{1}{t} \sum_{k=1}^t q_2(k)$$

or FP given recursively:

$$q_2^e(t+1) = \frac{t-1}{t} q_2^e(t) + \frac{1}{t} q_2(t), t \geq 1 \tag{3.10}$$

3.3.3 Weighted Fictitious Play

Standard fictitious play could be generalized to allow for discarding the remote past observations at a higher rate. Cheung and Friedman (1997) use an exponentially-weighted scheme:

$$q_2^e(t+1) = \frac{q_2(t) + \sum_{u=1}^{t-1} \gamma^u q_2(t-u)}{1 + \sum_{u=1}^{t-1} \gamma^u}$$

or, recursively:

$$q_2^e(t+1) = \left\{ \begin{array}{l} \frac{\gamma - \gamma^t}{1 - \gamma^t} q_2^e(t) + \frac{1 - \gamma}{1 - \gamma^t} q_2(t), \gamma \in [0, 1) \\ \gamma \frac{t-1}{t} q_2^e(t) + \frac{1}{t} q_2(t), \gamma = 1 \end{array} \right\} \tag{3.11}$$

It can be easily verified that Weighted Fictitious Play expectations nest, as special cases, the following types of expectations:

- (i) $\gamma = 0$: Cournot or 'naive' expectations/short memory;
- (ii) $0 < \gamma < 1$: adaptive expectations with *time-varying* weights/intermediate memory;
- (iii) $\gamma = 1$: Fictitious Play/long memory.

3.4 Evolutionary Cournot Games

In this section we will consider an evolutionary version of the "static" model outlined in Section 3.2. Basically, at each time instance, two players are drawn randomly from an infinite population and matched to play a standard duopoly quantity-setting game. In order to form expectations about the other player's quantity choice a set of learning rules is available. Players switch between different rules based on some performance measure. Rules that perform relatively better are more likely to spread in the population and thus, fractions using each rule are expected to evolve over time according to a specified updating mechanism. A more sophisticated expectation-formation rule will come at a cost compared to the 'simple' rules which are assumed costless. Furthermore, this heterogeneity of the learning process adds uncertainty over the players expectation formation process. Since opponents may be of different types, expectations are formed about an *average* opponent strategy or quantity choice. A variety of such pairwise interactions and resulting evolution of the learning rules is discussed below, analytically (when possible) and via simulations.

3.4.1 Adaptive Expectations vs. Rational/Nash play

We start with a generalization of Droste et al. (2002) behaviors' ecology by considering a mixture of adaptive (nesting a special case of naive expectations as in Droste et al. (2002)) and rational or 'Nash' players. The adaptive players behave

according to (3.8). Rational firms are able to correctly compute the quantity picked by an adaptive firm as well as the updated fractions of Adaptive and Rational players: their reaction function is playing a best-reply to the mixture of rules (equations (3.12), (3.13)):

$$q_1(t) = R(q_1^e(t)) = \frac{a - c - bq_1^e(t)}{2b - d}$$

$$q_2^e(t) = (1 - n(t))q_1(t) + n(t)q_2(t) \tag{3.12}$$

$$q_2(t) = R(q_2^e(t)) = \frac{a - c - bq_2^e(t)}{2b - d} \tag{3.13}$$

$$\Rightarrow q_2(t) = \frac{a - c - b(1 - n(t))R(q_1^e(t))}{2b - d + bn(t)} \tag{3.14}$$

where, $n(t)$ is the fraction of type II, rational players and $q_1^e(t), q_2^e(t)$ are the expectations about a *randomly* encountered opponent, of the adaptive and rational types, respectively.

The fraction of rational players $n(t)$ updates according to the logistic mechanism with asynchronous updating (Hommes et al. (2005)). The asynchronous logistic updating has two components: an inertia component, parameterized by δ , showing the fraction of players who do not (are not given the opportunity to) update and the logistic probability of updating for the fraction $1 - \delta$ of players for which a revision opportunity arises:

$$n(t + 1) = \delta n(t) + \frac{(1 - \delta) \exp(\beta(U_2(q_2(t)) - k))}{\exp(\beta(U_1(q_1(t))) + \exp(\beta(U_2(q_2(t)) - k))} \tag{3.15}$$

with U_1, U_2 representing the average profits of an adaptive and rational player at time t , while β denotes the inverse of the noise/or ‘intensity of choice’ parameter and k stands for the extra-costs incurred by the agents employing the more sophisticated (here rational expectations) heuristic. Each strategy performance measure $U_1(U_2)$ is computed as a fractions-weighted average of profits accrued in encounters with own

and other expectation-formation types:

$$U_1 = (a - c)q_1 + \left(\frac{1}{2}d - b\right)q_1^2 - b(nq_1 + (1 - n)q_2)q_1 \quad (3.16)$$

$$U_2 = (a - c)q_2 + \left(\frac{1}{2}d - b\right)q_2^2 - b(nq_1 + (1 - n)q_2)q_2 \quad (3.17)$$

Expectations - $q_1^e(t), q_2^e(t)$ for the adaptive and rational type, respectively - about the quantity of an *average* opponent quantity, evolve according to:

$$q_1^e(t + 1) = \alpha q_1^e(t) + (1 - \alpha)((1 - n(t))q_1(t) + n(t)q_2(t)) \quad (3.18)$$

$$q_2^e(t + 1) = (1 - n(t + 1))q_1(t + 1) + n(t + 1)q_2(t + 1) \quad (3.19)$$

Equations (3.15), (3.18) and (3.19) define a three-dimensional, discrete dynamical system $(q_1^e(t + 1), q_2^e(t + 1), n(t + 1)) = \Phi(q_1^e(t), q_2^e(t), n(t))$ describing both the evolution of the adaptive and rational types expectations and of the learning heuristics' ecology:

$$\begin{aligned} q_1^e(t + 1) &= \alpha q_1^e(t) + (1 - \alpha)(n(t)R(q_1^e(t)) + (1 - n(t))R(q_2^e(t))) \\ q_2^e(t + 1) &= \gamma q_1^e(t) + (1 - \gamma)(n(t)R(q_1^e(t)) + (1 - n(t))R(q_2^e(t))) \\ n(t + 1) &= \frac{1}{1 + \exp(\beta(U_2(R(q_1^e(t)), R(q_2^e(t)), n(t)) - U_1(R(q_1^e(t)), R(q_2^e(t)), n(t)) - k))} \end{aligned} \quad (3.20)$$

Using (3.12), (3.13) we can express firm/type *II* expectations as a function of firm *I* expectations about opponent quantity choice, effectively reduce (3.20) to a two-dimensional system:

$$\begin{aligned} q_1^e(t + 1) &= \alpha q_1^e(t) + (1 - \alpha)\left((1 - n(t))R(q_1^e(t)) + n(t)\frac{a - c - b(1 - n(t))R(q_1^e(t))}{2b - d + bn(t)}\right) \\ n(t + 1) &= \delta n(t) + (1 - \delta)\frac{e^{(\beta(U_2(R(q_1^e(t)), n(t)) - k))}}{e^{(\beta(U_1(R(q_1^e(t)), n(t)))} + e^{(\beta(U_2(R(q_1^e(t)), n(t)) - k))}} \end{aligned} \quad (3.21)$$

Lemma 4 *The Cournot-Nash equilibrium $q_1^* = q_2^* = \frac{a-c}{3b-d}$ in (3.6) together with*

$n^* = \frac{1}{1+\exp(\beta k)}$ is the unique, interior, fixed point of (3.21). For low values of the intensity of choice $\beta < \beta^* = \ln \frac{(d-3b)(\alpha+1)}{b-d+3b\alpha-da}$ the Cournot-Nash equilibrium is stable; at $\beta = \beta^*$ a period-doubling bifurcation occurs and a stable two-cycle is born. The 2-cycle loses stability via a secondary Neimark-Sacker bifurcation when β hits a second threshold $\beta^{**} = \beta^* + \frac{1+\delta}{(1-\delta)(b-d+\alpha(3b-d))(\alpha+1)(1-\frac{2}{b}\frac{k}{1-\alpha})}$ with a limit cycle surrounding each point of the 2-cycle.

Proof. At the fixed point $q_1^e(t+1) = q_1^e(t)$ the following should hold:

$\frac{a-c-b(1-n(t))R(q_1^e(t))}{2b-d+bn(t)} = R(q_1^e(t)) \Leftrightarrow R(q_1^e(t)) = \frac{a-c}{3b-d}$. But, $R(x) = \frac{a-c}{3b-d}$ implies $x = \frac{a-c}{3b-d}$ and, thus, $q_1^{e*}(t) = \frac{a-c}{3b-d}$ is the unique, interior² rest point of (3.21) with corresponding fraction given by $n^* = \frac{1}{1+\exp(\beta k)}$.

It can be shown that system (3.21) may be rewritten in terms of deviations $Q_t = q_1^e(t) - q_1^*$ from the Cournot-Nash equilibrium steady state as $(Q(t+1), n(t)) = F(Q(t), n(t))$ where:

$$\begin{aligned} Q(t+1) &= -Q(t) \frac{b-3b\alpha+d\alpha-bn(t)}{2b-d+bn(t)} \\ n(t+1) &= \delta n(t) + \frac{1-\delta}{1+\exp(\beta(k - \frac{(2b-d)(Q(t)b(3b-d)}{2b-d+bn(t)})^2))} \end{aligned} \quad (3.22)$$

This system has a fixed point at $(Q^* = 0, n^* = \frac{1}{1+\exp(\beta k)})$. In order to investigate its local stability we first derive the Jacobi matrix JF when evaluated at the steady state (Q^*, n^*) as:

$$\begin{bmatrix} -\frac{1}{2b-d+bn^*} (b-3b\alpha+d\alpha-bn^*) & 0 \\ 0 & \delta \end{bmatrix}. \quad (3.23)$$

²We note that, besides the *interior* Cournot-Nash equilibrium, which is main focus of this Chapter, the type II (Cox and Walker (1998)) Cournot game considered, has two additional *boundary* equilibria (two extra intersections of the reaction functions on the x and y axis), namely: $A = (q_1^A, q_2^A) = (\frac{a-c}{2b-d}, 0)$ and $B = (q_1^B, q_2^B) = (0, \frac{a-c}{2b-d})$. One can show that, in a non-evolutionary environment with homogenous Cournot (naive) expectations, the Cournot-Nash equilibrium is unstable and all initial conditions (except for the 45° line) converge to a 2-cycle formed by the two Nash equilibria (A, B) . However, these two additional boundary equilibria do *not* form a 2-cycle of our dynamical system.

Its eigenvalues are $\lambda_1 = -\frac{1}{2b-d+bn^*} (b - 3b\alpha + d\alpha - bn^*)$ and $\lambda_2 = \delta$. We see that the fixed point loses stability via a, primary, period-doubling, bifurcation when λ_1 hits the unit circle:

$\lambda_1 = -\frac{1}{2b-d+bn^*} (b - 3b\alpha + d\alpha - bn^*) = -1 \Leftrightarrow \beta = \beta^* = \frac{1}{k} \ln \frac{(d-3b)(\alpha+1)}{b-d+3b\alpha-d\alpha}$. At the resulting symmetric 2-cycle $\{(\bar{Q}, \bar{n}), (-\bar{Q}, \bar{n})\}$ the following equalities hold:

$$\begin{aligned} Q(t+1) &= -Q(t) \frac{b-3b\alpha+d\alpha-bn(t)}{2b-d+bn(t)} \\ Q(t) &= -Q(t+1) \frac{b-3b\alpha+d\alpha-bn(t)}{2b-d+bn(t)} \end{aligned}$$

Thus, $\frac{b-3b\alpha+d\alpha-bn(t)}{2b-d+bn(t)} = -1 \Leftrightarrow \bar{n} = \frac{d-b-\alpha(3b-d)}{2b}$. Next, \bar{Q} solves $\bar{n} = \delta\bar{n} + (1 - \delta) \frac{1}{1 + \exp(\beta(k - \frac{(2b-d)}{2} [\frac{Qb(3b-d)}{2b-d+b\bar{n}}]^2))}$, yielding

$$\begin{aligned} \bar{Q}_{\pm} &= \pm \frac{1}{2} \frac{\sqrt{2}}{b\beta} (1-\alpha)(2b-d) \sqrt{\frac{\beta}{2b-d} \left(\beta k - \ln(\alpha+1) \frac{-3b+d}{b-d+3b\alpha-d\alpha} \right)} \\ &= \pm \frac{1}{2} \frac{\sqrt{2}}{b\beta} (1-\alpha)(2b-d) \sqrt{\frac{\beta k}{2b-d} (\beta - \beta^*)} \end{aligned}$$

After further simplifications, the 2-cycle $\{(\bar{Q}, \bar{n}), (-\bar{Q}, \bar{n})\}$ reads:

$$\left[\begin{array}{l} \left\{ \left(\frac{(1-\alpha)}{b} \sqrt{\frac{k(\beta-\beta^*)}{2\beta(2b-d)}}, \frac{d-b-\alpha(3b-d)}{2b} \right), \right. \\ \left. \left(-\frac{(1-\alpha)}{b} \sqrt{\frac{k(\beta-\beta^*)}{2\beta(2b-d)}}, \frac{d-b-\alpha(3b-d)}{2b} \right) \right\} \end{array} \right] \quad (3.24)$$

To investigate the stability of $\{(\bar{Q}, \bar{n}), (-\bar{Q}, \bar{n})\}$ we have to look at the Jacobian of the compound map $G = F^2$ evaluated at the two-cycle.

$$JG(\bar{Q}, \bar{n}) = JF(\bar{Q}, \bar{n})JF(F((\bar{Q}, \bar{n}))) = JF(\bar{Q}, \bar{n})JF(-\bar{Q}, \bar{n}). \quad (3.25)$$

The Jacobian matrix of F is:

$$JF(Q, n) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

with entries,

$$\begin{aligned}
g_{11} &= -\frac{1}{2b-d+bn} (b - 3b\alpha + d\alpha - bn) \\
g_{12} &= Qb(\alpha - 1) \frac{d-3b}{(2b-d+bn)^2} \\
g_{21} &= \frac{2Qb^2\beta(1-\delta)(d-3b)^2(b-\frac{1}{2}d) \exp\left(\beta\left(k-Q^2b^2(d-3b)^2\frac{b-\frac{1}{2}d}{(2b-d+bn)^2}\right)\right)}{\left(\exp\left(\beta\left(k-Q^2b^2(d-3b)^2\frac{b-\frac{1}{2}d}{(2b-d+bn)^2}\right)\right)+1\right)^2 (2b-d+bn)^2} \\
g_{22} &= \delta + \frac{2Q^2b^3\beta(\delta-1)(d-3b)^2(b-\frac{1}{2}d) \exp\left(\beta\left(k-Q^2b^2(d-3b)^2\frac{b-\frac{1}{2}d}{(2b-d+bn)^2}\right)\right)}{\left(\exp\left(\beta\left(k-Q^2b^2(d-3b)^2\frac{b-\frac{1}{2}d}{(2b-d+bn)^2}\right)\right)+1\right)^2 (2b-d+bn)^3}
\end{aligned}$$

The stability analysis can be simplified by exploiting the symmetry of the system (3.22):

$$G = F^2(\bar{Q}, \bar{n}) = F(-\bar{Q}, \bar{n}) = F(T(\bar{Q}, \bar{n}))$$

with the transformation matrix T given by: $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

Each point of the 2-cycle is a fixed point of the map $G = FT$ and so, the 2-cycle of F is stable iff the fixed points of G are stable. The stability of the 2-cycle is governed by the Jacobian of FT :

$$JG(\bar{Q}, \bar{n}) = JF(-\bar{Q}, \bar{n})T$$

It can be shown that, after further algebraic manipulations, the Jacobian takes the following form:

$$JG(\bar{Q}, \bar{n}) = \begin{bmatrix} -1 & -\frac{4}{d-3b} \sqrt{-\frac{1}{2\beta} k \frac{\beta-\beta^*}{d-2b}} \\ j_{21} & j_{22} \end{bmatrix}$$

with $j_{21} = \frac{1}{b} \frac{\beta}{1-\alpha} (\alpha + 1) (1 - \delta) (3b - d) (2b - d) (b - d + 3b\alpha - d\alpha) \sqrt{k \frac{\beta-\beta^*}{2\beta(2b-d)}}$ and $j_{22} = \delta - (\alpha + 1) (1 - \delta) (\beta - \beta^*) (b - d + 3b\alpha - d\alpha)$

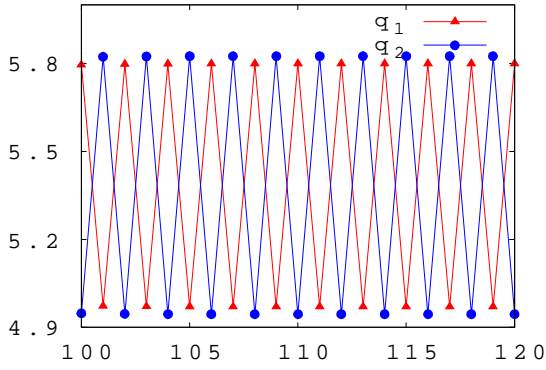
The matrix has a pair of complex conjugates eigenvalues as lengthy expressions of all the model parameters. Still, the determinant takes a more tractable form: $\det JG = (\alpha + 1) (1 - \delta) (\beta - \beta^*) (b - d + 3b\alpha - d\alpha) - \delta + \frac{2}{b} \frac{k(\alpha+1)(1-\delta)(\beta-\beta^*)(b-d+3b\alpha-d\alpha)}{\alpha-1}$

The Neimark-Sacker bifurcation condition ($\det JG = 1$) yields the intensity of choice threshold β^{**} at which a second bifurcation arises, namely:

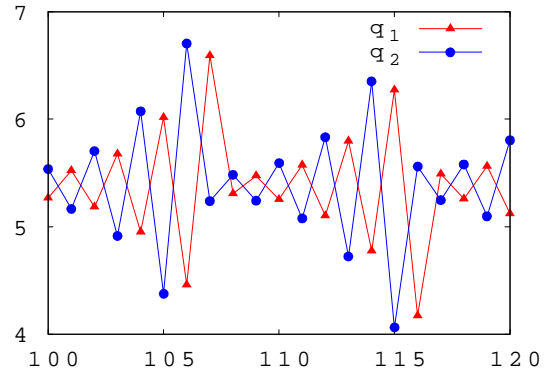
$$\beta^{**} = \beta^* + \frac{1 + \delta}{(1 - \delta)(b - d + \alpha(3b - d))(\alpha + 1)\left(1 - \frac{2}{b} \frac{k}{1 - \alpha}\right)}$$

This completes the proof of Lemma 4. ■

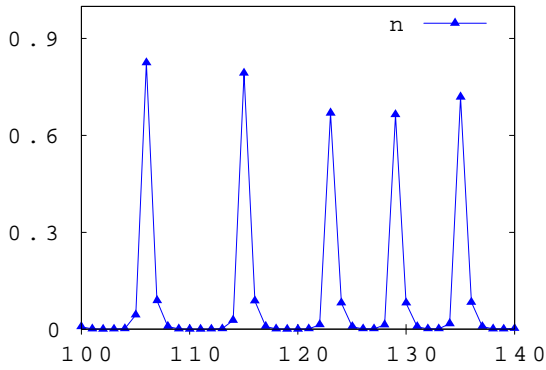
Summing up, as the intensity of choice varies from low to high values the system transits from a unique steady state, through a 2-cycle to two limit cycles at $\beta = \beta^{**}$ when it undergoes a Neimark-Sacker bifurcation and each of the 2-cycle branches becomes surrounded by a closed orbit as in Fig. 3.1d. Panels (a)-(b) illustrate the limiting behavior of quantities picked by the two players as they become more responsive to payoff differences, while Panel (c) displays the time evolution of the proportion of the costly rational (or Nash) expectation-formation heuristic. The intuition is similar to Brock and Hommes (1997): when the system is close to the Cournot-Nash equilibrium the simple, costless adaptive rule performs well enough and, due to the evolutionary switching mechanism, a larger number of players makes use of this cheap heuristic. However with a vast majority of agents using adaptive rule the CNE destabilizes and, as fluctuations of increasing amplitude arise, it pays off to switch to the more involved, though costly, behavior (i.e. Nash) pays-off. Once a large majority of agents switches to the rational rule fluctuations are stabilized and agents switch back to the costfree heuristic, re-initializing the cycle. Fig. 3.2a-b illustrates the primary (period-doubling) and secondary (Neimark-Sacker) bifurcations along with the "breaking of the invariant circle" route to chaos as the intensity of choice and degree of expectations adaptiveness are varied, respectively. Finally, Panels (c)-(d) show a strange attractor in the quantity-fraction of Nash players and quantity-quantity subspaces.



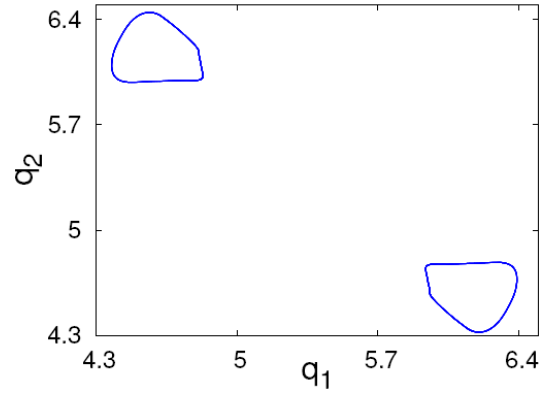
(a) $\beta = 2$. Time series



(b) $\beta = 9$. Time series

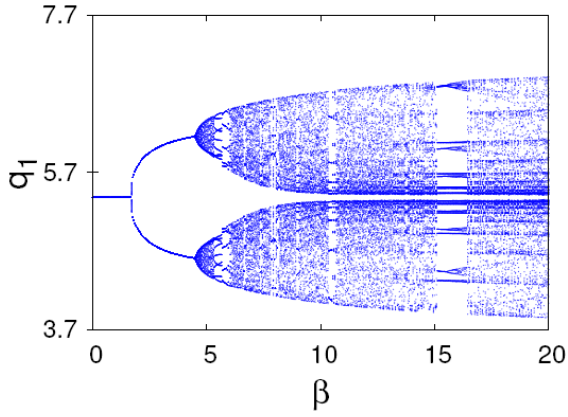


(c) $\beta = 9$. Evolution of fraction of RE

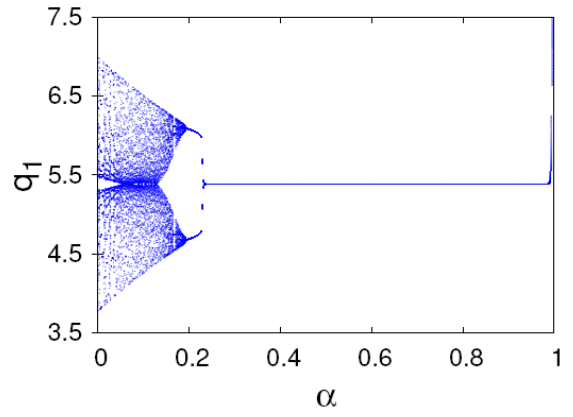


(d) $\beta = 5.2$. Qasiperiodic behavior

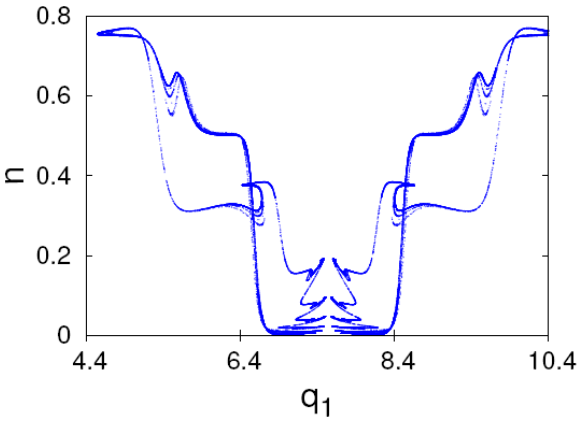
Figure 3.1: Cournot duopoly game with an ecology of Adaptive and Rational players. Panels (a)-(b): quantities evolution for different values of the intensity of choice (β). The evolution of Nash players fraction is shown in Panel (c). Panel (d) displays a limit cycle, consisting of two closed curves around the two points of the unstable 2-cycle, arising from the secondary Hopf bifurcation. Game parameters set to $a = 17, b = 0.8, c = 10, d = 1.1, k = 1, \alpha = 0.05, \delta = 0.1$.



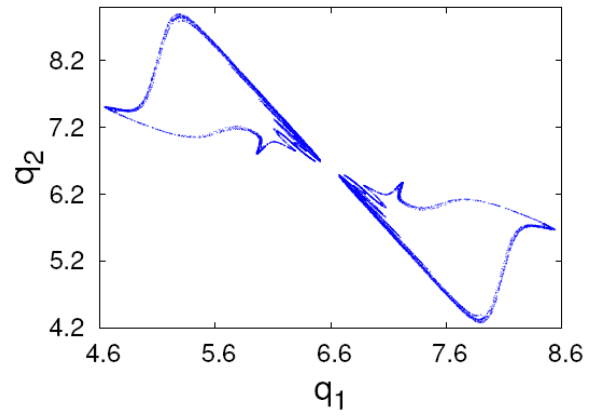
(a) Bifurcation diagram: $\beta - q_1, \alpha = 0.05$



(b) Bifurcation diagram: $\alpha - q_1, \beta = 15$



(c) Strange attractor, $q_1 - n$



(d) Strange attractor, $q_1 - q_2$

Figure 3.2: Cournot duopoly game with an ecology of Adaptive and Rational players. Bifurcation diagrams of the equilibrium quantity chosen by the Adaptive players (q_1) with respect to the intensity of choice (β) (Panel (a)) and degree of expectation adaptiveness (α) in Panel (b). Game parameters set to $a = 17, b = 0.8, c = 10, d = 1.1, \delta = 0.1$. Panels (c), (d) display projection of a strange attractor on the quantity-fraction of rational players and quantity-quantity subspaces, respectively. A strange attractor is obtained for the following parameterization: $a = 17.54, b = 0.754, c = 10, d = 1.1, k = 1.5, \alpha = 0.1, \beta = 7.8, \delta = 0.5$.

3.4.2 Adaptive vs. Exponentially Weighted Fictitious Play

As was already pointed out, players form expectations about the strategy of the *average* opponent given the existing heuristics ecology (frequencies of each learning type). In this subsection, the subset available is restricted to Adaptive Expectations and Weighted Fictitious Play as introduced in subsection 3.3. Thus, the rational rule is replaced by a long-memory rule that relies only on past observed quantity information with, possibly, different weighting schemes for each observation. Similar to the previous sections, the more sophisticated heuristic comes at a cost k . Considering that a WFP process gives rise to a non-autonomous dynamical system we will discuss analytically the limit as $t \rightarrow \infty$ when the system becomes autonomous. Let $q_{AE}^e(t+1)$ and $q_{WFP}^e(t+1)$ denote the expectations formed about an *average* encountered opponent by an AE-player and a WFP-player, respectively. The evolution of these two learning rules, in a Cournotian strategic environment is given by the following system of difference equations:

$$\begin{aligned} q_{AE}^e(t+1) &\equiv q_1^e(t+1) = \alpha q_1^e(t) + (1-\alpha)q_{opp}^{avg}(t) \\ q_{WFP}^e(t+1) &\equiv q_2^e(t+1) = \gamma q_1^e(t) + (1-\gamma)q_{opp}^{avg}(t) \end{aligned}$$

with $\gamma > \alpha$ ³. By letting $n(t)$ denote the fraction of agents employing the more computationally-involved heuristic II (WFP expectations in this case) this system

³In subsection 3.3 we have seen that α close to 0 renders adaptive expectations, while γ approaching 1 yields the standard fictitious play. These two parameters could also be interpreted as proxies for "memory": when they are close to the lower bound 0 the past is heavily discarded and only recent observations matter for constructing expectations about opponent choice, while as they are near the upper bound 1 past information becomes increasingly important for future choices. Thus naive expectations [$\alpha = 0$] is a *short-memory* rule, as the entire past, but last period, is discarded, while fictitious play [$\gamma = 1$] is a *long-memory* rule, as each past observation is equally important.

can be explicitly re-written as:

$$q_1^e(t+1) = \alpha q_1^e(t) + (1-\alpha)((1-n(t))q_1(t) + n(t)q_2(t)) \quad (3.26)$$

$$q_2^e(t+1) = \gamma q_1^e(t) + (1-\gamma)((1-n(t))q_1(t) + n(t)q_2(t)) \quad (3.27)$$

The actual strategic choices of an *AE*- and *WFP*-player are simply best-responses to their expectations:

$$q_1(t) = R(q_1^e(t)) = \frac{a-c-bq_1^e(t)}{2b-d} \quad (3.28)$$

$$q_2(t) = R(q_2^e(t)) = \frac{a-c-bq_2^e(t)}{2b-d} \quad (3.29)$$

The fraction of the more sophisticated *WFP* players evolves according to an asynchronous updating logistic mechanism (Hommes et al. (2005)):

$$n(t+1) = \delta n(t) + (1-\delta) \frac{\exp(\beta(U_2(t) - k))}{\exp(\beta U_1(t)) + \exp(\beta(U_2(t) - k))}, \quad (3.30)$$

where U_1, U_2 represent the average profits of an *AE* and *WFP* player at time t , and β denotes the inverse of the noise/or "intensity of choice" parameter and k stands for the extra-costs incurred by the agents employing the more sophisticated (i.e. with a higher weight placed on past observations about opponent choices) *WFP* heuristic. The performance measures U_1, U_2 are determined as in equations (3.16)-(3.17) from subsection 3.4.1. Similarly, we derive a three-dimensional, discrete dynamical system $(q_1^e(t+1), q_2^e(t+1), n(t+1)) = \Phi(q_1^e(t), q_2^e(t), n(t))$ describing both the evolution of the adaptive and fictitious play type expectations and of the learning heuristics' ecology:

$$\begin{aligned} q_1^e(t+1) &= \alpha q_1^e(t) + (1-\alpha)((1-n(t))R(q_1^e(t)) + n(t)R(q_2^e(t))) \\ q_2^e(t+1) &= \gamma q_2^e(t) + (1-\gamma)((1-n(t))R(q_1^e(t)) + n(t)R(q_2^e(t))) \\ n(t+1) &= \delta n(t) + \frac{1-\delta}{1 + e^{(\beta(U_1(R(q_1^e(t)), R(q_2^e(t)), n(t)) - U_2(R(q_1^e(t)), R(q_2^e(t)), n(t)) + k))}} \end{aligned} \quad (3.31)$$

Lemma 5 *The (static) Cournot-Nash equilibrium quantities (q_1^*, q_2^*) determined in (3.6) together with $n^* = \frac{1}{1+\exp(\beta k)}$ is an (interior) fixed point of system (3.31).*

Proof. At the Cournot-Nash Equilibrium(CNE) $q_1^* = q_2^* = \frac{a-c}{3b-d}$ we have, by definition, $R(q_1^e(t)) = q_1^e(t)$ and $R(q_2^e(t)) = q_2^e(t)$. Substituting into the first two equations of (3.31) gives the required fixed point equality. Then, $n^* = \frac{1}{1+\exp(\beta k)}$ is the solution of $n(t+1) = n(t)$ when this equation is evaluated at the CNE. ■

3.4.3 Local stability analysis

It can be shown that the Jacobian matrix of (3.31) evaluated at the equilibrium (q_1^*, q_2^*, n^*) takes the following form:

$$J_{\Phi/SS} = \begin{bmatrix} \alpha + (1-\alpha)(1-n^*)r & (1-\alpha)n^*r & 0 \\ (1-\gamma)(1-n^*)r & \gamma + (1-\gamma)n^*r & 0 \\ 0 & 0 & \delta \end{bmatrix} \quad (3.32)$$

where $r = \frac{\partial R(q_i^e)}{q_i^e} = -\frac{b}{2b-d}$, $r < -1$.

We know that under homogenous, naive (or adaptive with small α) expectations the CNE is unstable with a two- cycle emerging at the boundary. On the other hand, fictitious play alone (or weighted fictitious play with large weights put in remote past observations) stabilizes the CNE. Once the interplay between the cheap adaptive and costly WFP heuristics is introduced we can prove the following result:

Proposition 6 *A system with evolutionary switching between adaptive and weighted fictitious play expectations (i.e. $0 < \alpha \ll \gamma < 1$) is destabilized when the sensitivity to payoff difference between alternative heuristics reaches a threshold $\beta^{PD} = \frac{1}{k} \ln \frac{\gamma(r-1)-(1+\alpha)(r+1)}{(\gamma+1)(r+1)+\alpha(1-r)}$ and a stable two-cycle bifurcates from the interior CN steady state $(q_1^* = q_2^* = \frac{a-c}{3b-d}, n^* = \frac{1}{1+\exp(\beta k)})$.*

Proof. For an ecology of Adaptive and Weighted Fictitious Play rules - $\alpha \in$

$(0, 1)$, $\gamma \in (0, 1)$, $\gamma \gg \alpha$ = the eigenvalues of the Jacobian matrix (3.32) are : $\lambda_{1,2} = \frac{1}{2}r + \frac{1}{2}\alpha + \frac{1}{2}\gamma - \frac{1}{2}r\alpha + \frac{1}{2}r\alpha n^* - \frac{1}{2}r\gamma n^* \pm \frac{1}{2}\sqrt{S}$, $\lambda_3 = \delta$

$$\begin{aligned} \text{with } S = & r^2\alpha^2(n^*)^2 - 2r^2\alpha^2n^* + r^2\alpha^2 - 2r^2\alpha\gamma(n^*)^2 + 2r^2\alpha\gamma n^* + 2r^2\alpha n^* \\ & - 2r^2\alpha + r^2\gamma^2(n^*)^2 - 2r^2\gamma n^* + r^2 + 2r\alpha^2n^* - 2r\alpha^2 + 2r\alpha\gamma - 4r\alpha n^* \\ & + 2r\alpha - 2r\gamma^2n^* + 4r\gamma n^* - 2r\gamma + \alpha^2 - 2\alpha\gamma + \gamma^2 \end{aligned}$$

Next the period-doubling condition $\lambda_2 = -1$ yields:

$$\begin{aligned} n^*(\beta) &= \frac{(1 + \gamma)[(1 + r) + \alpha(1 - r)]}{2r(\gamma - \alpha)} \\ \beta^{PD} &= \frac{1}{k} \ln \frac{\gamma(r - 1) - (1 + \alpha)(r + 1)}{(\gamma + 1)(r + 1) + \alpha(1 - r)} \end{aligned} \quad (3.33)$$

We can also derive the intensity of choice threshold at which instability arises for an ecology of Naive vs. Weighted Fictitious Play - $\alpha = 0$, $\gamma \in (0, 1)$ - as a special case of (3.33):

$$\beta^{PD} = \beta^* = \frac{1}{k} \ln \frac{r(\gamma - 1) - (1 + \gamma)}{(\gamma + 1)(r + 1)}$$

For an ecology of Fictitious vs. Adaptive Players - $\gamma = 1$, $\alpha \in (0, 1)$ - system (3.31) is non-autonomous and we will investigate this case in a subsequent section, numerically. ■

We notice that if there are no costs associated with the sophisticated predictor *WFP* (i.e. $k = 0$) the system is stable for any value of the intensity of choice. This is so because what trigger switching into the destabilizing adaptive rule are the costs incurred when using *WFP*: thus, for $k = 0$ there is no incentive to switch away from *WFP* even in tranquil or stable periods.

The 2-cycle, in deviations from the Cournot -Nash steady state - $((x, y, n), (X, Y, n))$

with $X = -x, Y = -y$ - solves the following system of equations:

$$\begin{aligned}
X &= \alpha x + (1 - \alpha)\left((1 - n)\frac{a-c-bx}{2b-d} + n\frac{a-c-by}{2b-d}\right) - \frac{(a-c)(1-\alpha)}{2b-d} \\
Y &= \gamma y + (1 - \gamma)\left((1 - n)\frac{a-c-bx}{2b-d} + n\frac{a-c-by}{2b-d}\right) - \frac{(a-c)(1-\gamma)}{2b-d} \\
x &= \alpha X + (1 - \alpha)\left((1 - n)\frac{a-c-bX}{2b-d} + n\frac{a-c-bY}{2b-d}\right) - \frac{(a-c)(1-\alpha)}{2b-d} \\
y &= \gamma Y + (1 - \gamma)\left((1 - n)\frac{a-c-bX}{2b-d} + n\frac{a-c-bY}{2b-d}\right) - \frac{(a-c)(1-\gamma)}{2b-d} \\
n &= \delta n + \frac{1-\delta}{\exp\left(\frac{1}{2}\frac{b^2}{(d-2b)^2}\right) \exp(\beta((x-y)(2a-2c-4bx-2by+dx+dy+2bnx-2bny)+k))}
\end{aligned} \tag{3.34}$$

Given that the two-cycle above satisfies $X = -x, Y = -y$, we obtain, after some manipulations, that the ratio of the two expectation rules deviations from equilibrium prediction is given by:

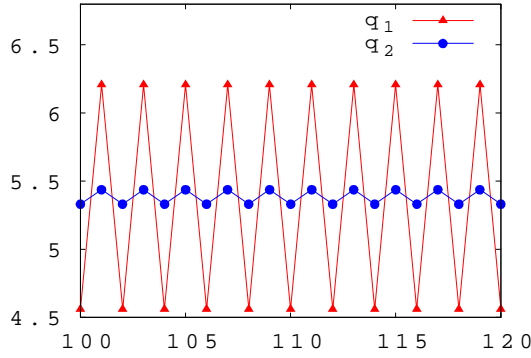
$$\frac{x}{y} = \frac{(1 - \alpha)/(1 + \alpha)}{(1 - \gamma)/(1 + \gamma)}$$

For $\alpha \ll \gamma$ (an adaptive vs. fictitious play ecology) we get $x > y$. Thus, the fluctuations of the adaptive, short-memory rule exceed in amplitude the oscillations incurred by the long memory heuristic (see Fig. 3.3ab).

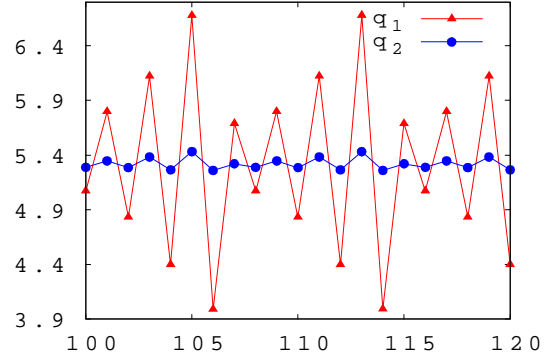
If we analyze the time evolution of the costly *WFP* fraction we observe a similar pattern as for the naive vs. Nash ecology discussed in the previous subsection. In tranquil times, when the dynamics is close to the Cournot-Nash equilibrium, the cheap adaptive rule outperforms the more involved and costly *WFP* rule. Thus, through the selection mechanism, almost the entire population becomes adaptive. However, a Cournot duopoly with adaptive(near naive) expectations is unstable and, as fluctuations emerge, conditions are created for the costly, more sophisticated *WFP* rule, to take over the population and re-stabilize the system.

We can also show, numerically, that the 2-cycle loses stability via a secondary Neimark-Sacker bifurcation with a quasi-periodic attractor consisting of two closed curves, arising around the unstable branches of the 2-cycle (Fig. 3.3d). Furthermore, the invariant curves break into a strange attractor if the intensity of choice β is increased even further. Bifurcations diagrams with respect to different behavioral

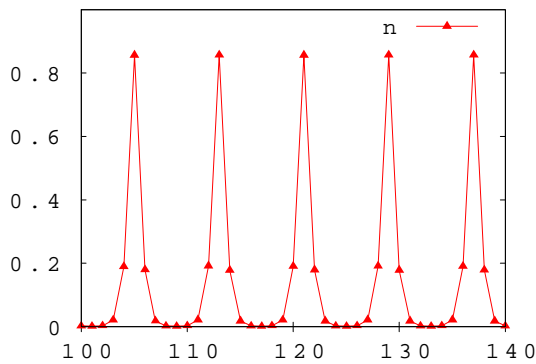
parameters together with the evolution of a strange attractor are reported in Figures 3.4 and 3.5.



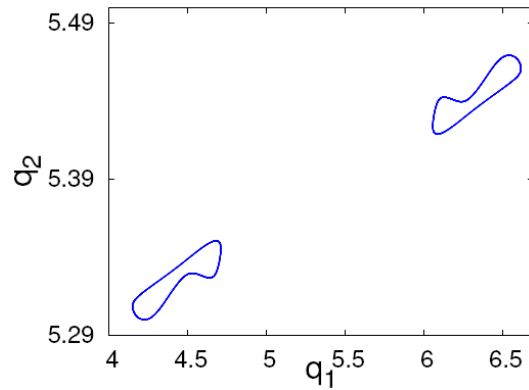
(a) $\beta = 2$. Time series



(a) $\beta = 7$. Time series

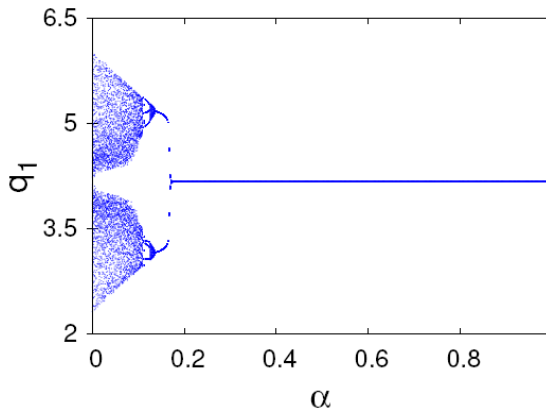


(c) $\beta = 7$. Evolution of WFP share

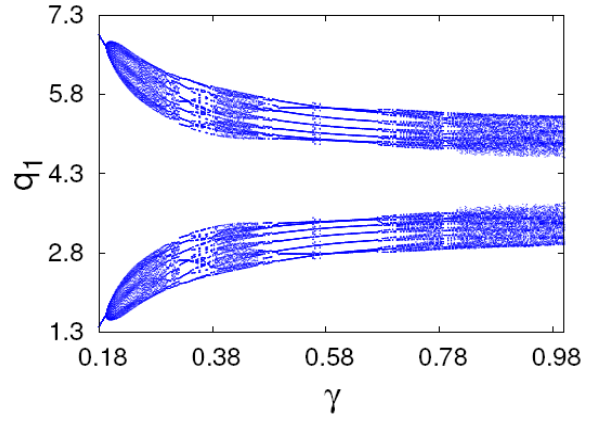


(d) $\beta = 4.25$. Phase plot

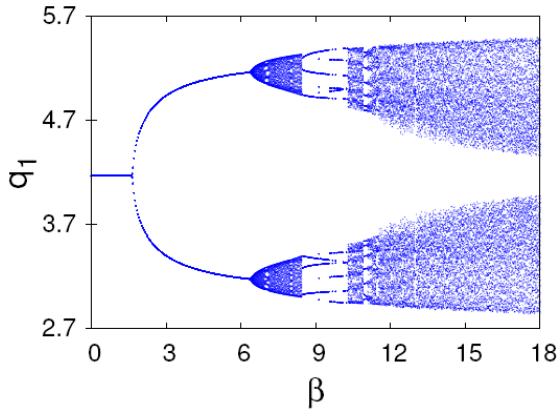
Figure 3.3: Cournot duopoly game with an ecology of Adaptive and Weighted Fictitious Play behaviors. Periodic and chaotic time series of the quantities evolution for different values of the intensity of choice (Panels (a)-(b)); evolution of the WFP heuristic share in the population (Panel (c)). Panel (d) displays aquasi-periodic attractor in the space of quantities chosen by the duopolists. Game parameters: $a = 17, b = 0.8, c = 10, d = 1.1, k = 1.1, \alpha = 0.1, \gamma = 0.9, \delta = 0.1$.



(a) Bifurcation diagram, $\alpha - q_1, \gamma = 0.9$

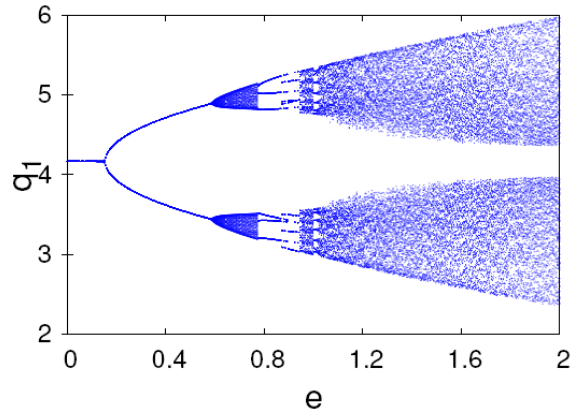


(b) Bifurcation diagram, $\gamma - q_1, \alpha = 0.1$



(c) Bifurcation diagram,

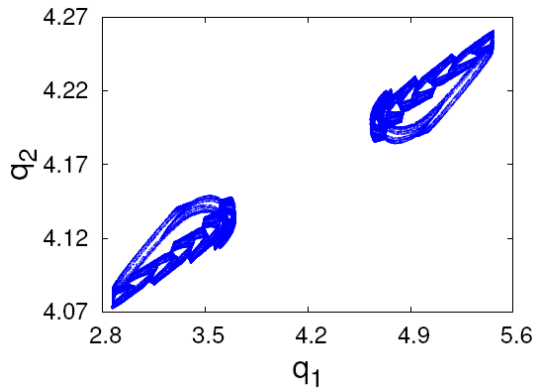
$\beta - q_1, \alpha = 0.1[AE], \gamma = 0.9[WFP]$



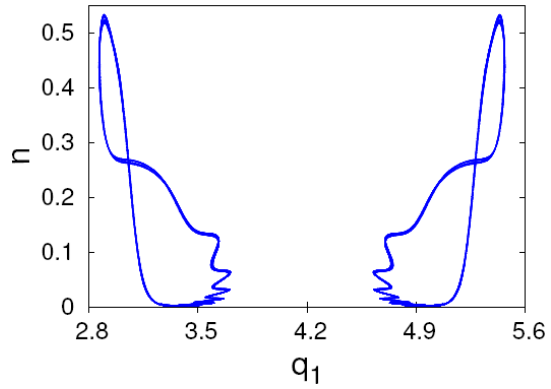
(d) Bifurcation diagram,

$e - q_1, \alpha = 0.1[AE], \gamma = 0.9[WFP]$

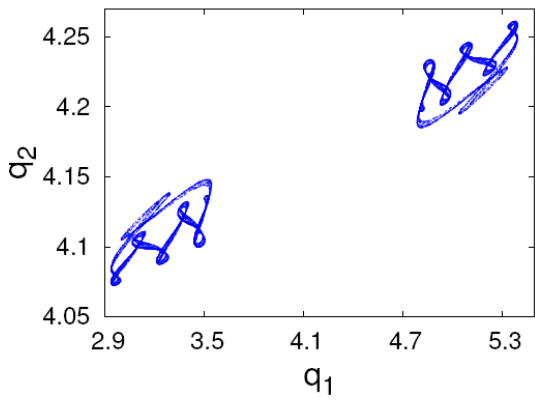
Figure 3.4: Cournot duopoly game with an ecology of Adaptive and Weighted Fictitious Play behaviors. Panels (a)-(d): bifurcation diagrams of the equilibrium quantity picked by the Adaptive type (q_1) with respect to degree of expectation adaptiveness (α), weighting parameter in the WFP rule (γ), intensity of choice (β) and costs associated with WFP (e), respectively. Game parameters: $a = 15, b = 0.7, c = 10, d = 1.1, k = 1.1, \alpha = 0.1, \gamma = 0.9, \beta = 12, \delta = 0.1$.



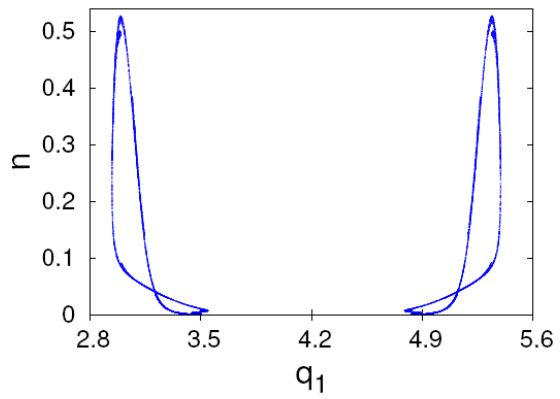
(a) Strange attractors,
 $q_1 - q_2, \beta = 13.5, \delta = 0.5$



(b) Strange attractors,
 $q_1 - n, \beta = 13.5, \delta = 0.5$



(c) Strange attractors,
 $q_1 - q_2, \beta = 10.8, \delta = 0.1$

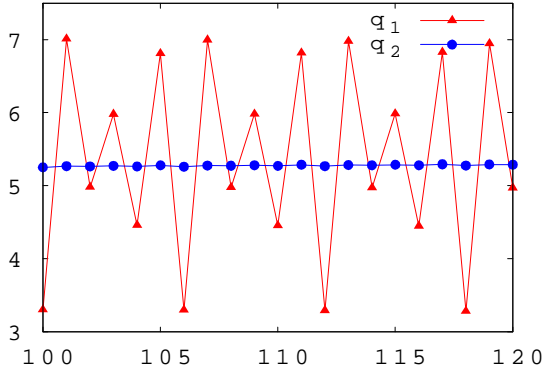


(d) Strange attractors,
 $q_1 - n, \beta = 10.8, \delta = 0.1$

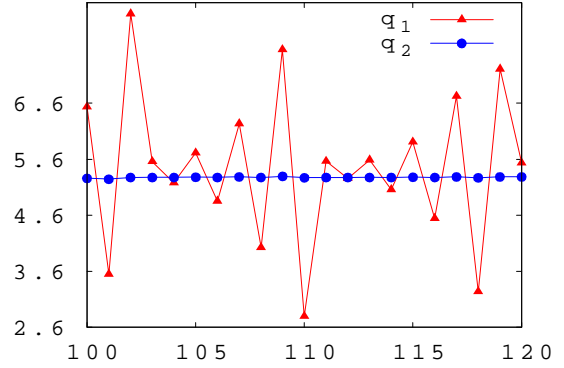
Figure 3.5: Cournot duopoly game with an ecology of Adaptive and Weighted Fictitious Play behaviors. Long-run strange attractors in the quantity-quantity and quantity-share of Adaptive players, for low (Panels (a)-(b)) and high (Panel (c)-(d)) degree of synchronicity (δ) in the updating process. Game and behavioral parameters: $a = 15, b = 0.7, c = 10, d = 1.1, k = 1.1, \alpha = 0.1, \gamma = 0.9, \delta = 0.1$.

3.4.4 Naive vs. Fictitious Play

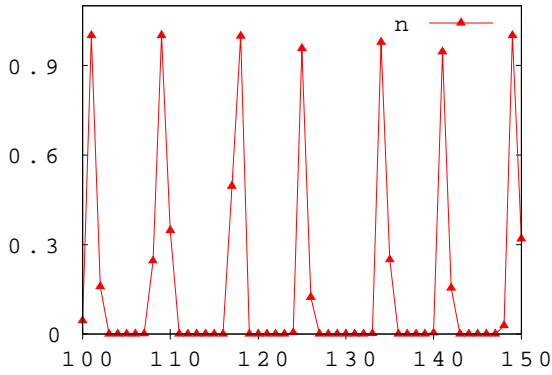
In this subsection we report numerical simulations of the long-run behavior of the evolutionary competition between costless naive expectations and costly fictitious play for a linear inverse demand-quadratic costs quantity-setting duopoly. This ecology combines, in a sense, two limiting results from the previous subsections: Cournot or naive expectations - as $\alpha \rightarrow 0$ - along with standard fictitious play expectations - as $\gamma \rightarrow 1$. It is known that fictitious play converges in a homogenous population of fictitious players to Cournot-Nash equilibrium while naive expectations alone generate continuous oscillations. Hence, a natural question to ask is whether the presence of fictitious players could stabilize a population of naive players. Fig. (3.6) presents numerical simulations of the competition between the two heuristics and show persistent fluctuations of the naive players quantity around the Cournot-Nash equilibrium. The share of fictitious players (Fig. 3.6c) lingers close to extinction and spikes occasionally taking over almost the entire population. Similar to the previous two subsections, what is driving this results are the costs associated with the fictitious play rule: when the entire population follows the FP rule, the CNE is stabilized and so there is an incentive to switch to the cheaper rule, i.e. adaptive heuristic. As more and more players become adaptive the CNE loses stability, fluctuations emerge and ,thus, they create an advantageous environment for long-memory players to step in. Depending on the intensity of selection (as captured by parameter β) the fluctuations could be periodic (Panel(a)) or even chaotic: see Panel (d) in Fig. 3.6 for numerical plot of the largest Lyapunov exponent. Fig. 3.7a-d depicts the evolution of such an attractor as the sensitivity to material payoffs β varies. In order to escape the long transient of the fictitious play dynamics, the phase portraits are plotted after skipping 100,000 points of the series. While the fictitious player quantities stay within a small neighborhood the Cournot-Nash equilibrium, the strategies of the Cornout(naive) player display chaotic fluctuations of large amplitudes.



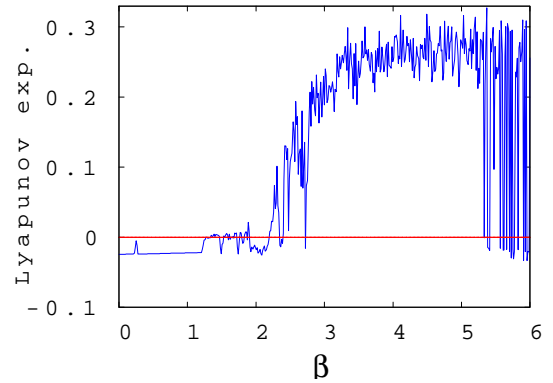
(a) $\beta = 2$. Time series



(b) $\beta = 6$. Time series



(c) $\beta = 6$. Evolution of FP share



(d). Largest Lyapunov exponent

Figure 3.6: Cournot duopoly game with Naive and Fictitious Play Heuristics. Pattern of play evolution for naive and fictitious players, for different values of the intensity of choice in Panels (a) and (b), respectively. Panel (c) shows the time evolution of Fictitious Play fraction while Panel (d) provides numerical evidence for chaotic dynamics for large values of β . Game and behavioral parameters: $a = 17, b = 0.8, c = 10, d = 1.1, k = 1, \alpha = 0, \delta = 0.1$.

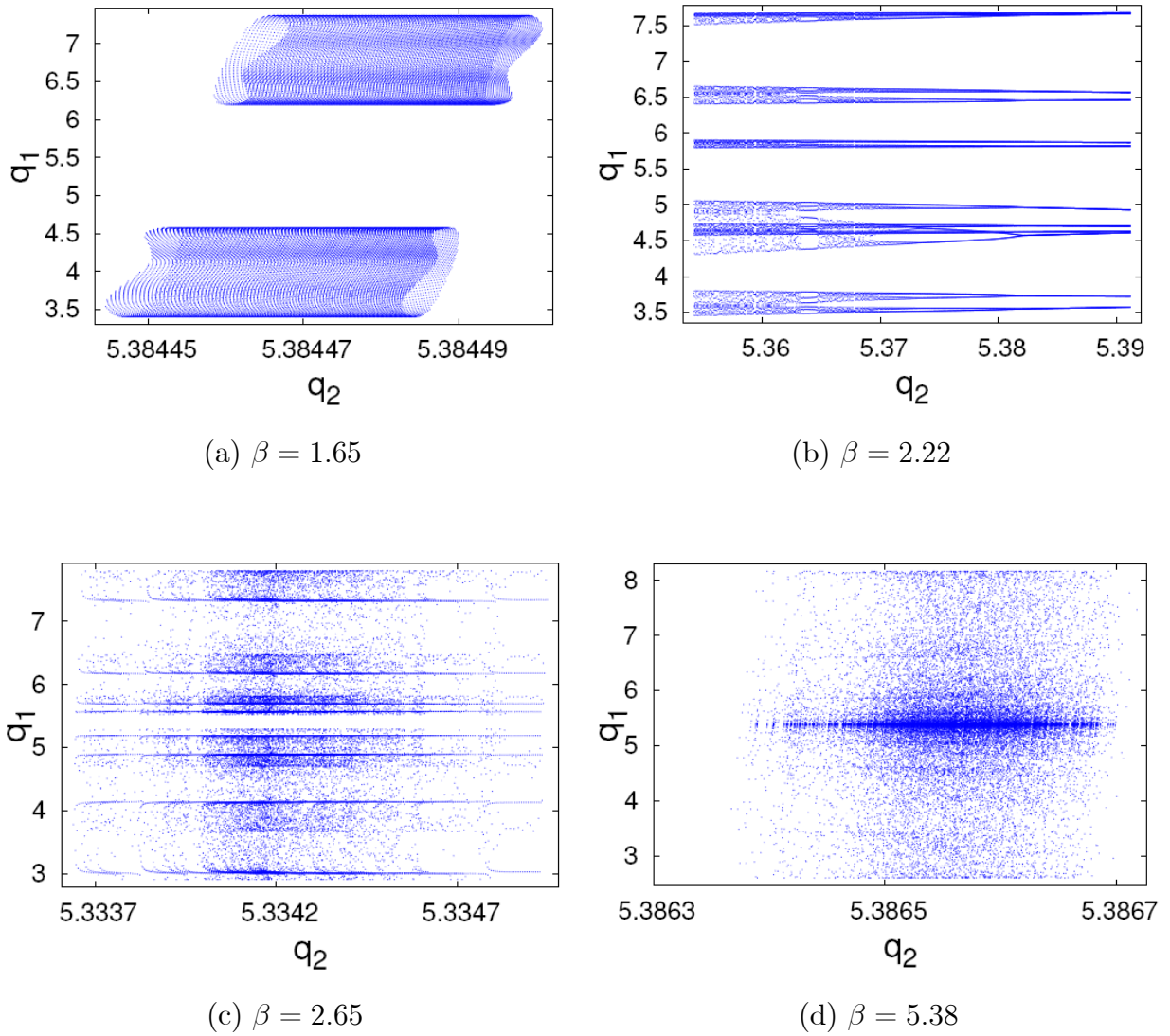


Figure 3.7: Cournot duopoly game with Naive and Fictitious Play Heuristics. Panels (a)-(d): the evolution of the phase portrait into a strange attractor as the intensity of choice increases. Notice that the quantity picked by the fictitious player wanders around the Cournot-Nash equilibrium quantity while the output choice of the naive players exhibits more wide chaotic fluctuations around the same CNE. Game and behavioral parameters: $a = 17, b = 0.8, c = 10, d = 1.1, e = 1, \alpha = 0, \delta = 0.1$.

3.5 Concluding Remarks

We studied evolutionary Cournot games with heterogenous learning rules about opponent strategic quantity or price choice. The evolutionary selection of such 'learning' heuristics is driven by a logistic-type evolutionary dynamics. While our focus is on the interplay between adaptive and 'fictitious play' expectations, other rules ecologies (e.g. weighted fictitious play vs. rational or Nash expectations) are explored, as well. Analytical and simulation results about the behaviour of the corresponding evolutionary Cournot games, when various parameters of interest change, are presented. In a Cournotian setting, the main finding is that a population of (weighted) fictitious players (i.e best-responders to the empirical distribution of past history of opponent choices) could be destabilized away from the Cournot-Nash equilibrium play by the presence of the adaptive expectation formation rule (i.e. players who place large weight on the last period choice and heavily discard more remote past observations). The interaction between a costly weighted fictitious play and the cheap adaptive expectations yields complicated quantity dynamics. The key mechanism driving dynamics is the endogenous switching between learning rules based on realized fitness: The fictitious play drive the system near the Cournot-Nash equilibrium where the simple, adaptive rule performs relatively well and, due, to its cost advantage, can invade and take over the more sophisticated predictor. However, the adaptive rule alone destabilizes the interior equilibrium and, when far from the CNE it pays off to bear the costs of the more involved rule reinitializing the evolutionary cycle.