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**Essays on nonlinear evolutionary game dynamics**

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*Citation for published version (APA):*

Ochea, M. I. (2010). *Essays on nonlinear evolutionary game dynamics*. Amsterdam: Thela Thesis.

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# Chapter 4

## On the Stability of the Cournot Solution: An Evolutionary Approach

### 4.1 Introduction

Within a linear cost-linear demand set-up, Theocharis (1960) shows that once the duopoly assumption is relaxed and the number of players increases, the Cournot Nash equilibrium loses stability and bounded oscillations arise already for triopoly quantity-setting games. For more than 3 players oscillations grow quickly unbounded but they are stabilized by the non-negativity price and demand constraint. Hahn (1962) puts forth an alternative proof of this results and derives stability conditions for asymmetric cost functions and nonlinear cost and demand functions. Fisher (1961) and McManus (1964) recover stability of the equilibrium when firms use, instead of the discrete-time best response dynamic of Theocharis (1960), a continuous adjustment process to some optimally derived output target. However, firms do these adjustments incompletely (i.e. with different speeds of adjustment) at each time instance. In this class of adjustment processes, increasing the marginal costs acts as

a stabilizer of the Cournot-Nash equilibrium, irrespective of the number of players. In Okuguchi (1970) both the actual output and expectations of rivals' output follow a continuous adjustment process to some target and restrictions on the speeds of adjustment (of both actual and the rival's expected output) are derived in relation to the rate of marginal costs increase such that stability is regained for an arbitrary number of players. Adaptive expectations generalize Cournot (naive) expectations. Thus, in a multi-player oligopoly game, where players follow a discrete-time best-response to adaptively formed expectations about rivals output, Szidarovszky et al. (1994) find suitable restrictions on the coefficients of adaptive expectations such that the unique Cournot-Nash equilibrium is stabilized.

In this short Chapter we take a different route and relax the assumption of homogenous expectations, while preserving the linear structure and best-reply adjustment dynamics of Theocharis (1960). The questions we address is whether Theocharis (1960) classical instability result still persist under the evolutionary selection of heterogenous expectations rule. We show that, in an evolutionary environment, the instability is robust to a heterogenous ecology of heuristics. In particular, we focus on the interplay between a simple adaptive rule and a more involved, but costly, rational rule with agents choosing the best-performing rule according to a logistic updating mechanism. The threshold number of players that triggers instability may vary with the costs of the rational expectations rule or with the adaptive expectations coefficient.

In Section 4.2 the model is introduced, results and simulations are reported in Section 4.3 while Section 4.4 contains some concluding remarks.

## 4.2 The model

We start with the Theocharis (1960) simple linear inverse demand-linear costs homogenous,  $n$ -player oligopoly game. Linear inverse demand is given by

$$\mathcal{P} = a - b \sum_{i=1}^n q_i, \quad (4.1)$$

and linear costs by

$$C_i(q_i) = cq_i. \quad (4.2)$$

Each period  $n$  players are drawn from a large populations to play a  $n$ -player quantity-setting game. Furthermore, assume that the population of oligopolists consists of two behavioural types: adaptive players (share  $1 - p$  in the population) playing quantity  $q_1$  and rational (share  $p$ ) picking quantity  $q_2$ . From the perspective of an individual player the aggregate output of the remaining  $n - 1$  opponents, given behavioral types, could be computed for different market structures as follows:

In the triopoly case the two randomly drawn opponents may be, with probability  $p^2$  of rational type (producing  $q_2$  each), with probability  $(1 - p)^2$  of adaptive type (and thus choosing  $q_1$  each) and, with probability  $2p(1 - p)$  of different types (thus picking  $q_1$  and  $q_2$ , respectively). Thus, the output of an *average* opponent in a triopoly game is given by:

$$p^2 2q_2 + 2p(1 - p)(q_2 + q_1) + (1 - p)^2 2q_1 \quad (4.3)$$

Similarly, for the quadropoly Cournot game we get

$$p^3 3q_2 + 3p^2(1 - p)(2q_2 + q_1) + 3p(1 - p)^2(q_2 + 2q_1) + (1 - p)^3 3q_1 \quad (4.4)$$

and for quintopoly:

$$p^4 4q_2 + 4p^3(1-p)(3q_2 + q_1) + 6p^2(1-p)^2(2q_2 + 2q_1) + 4p(1-p)^3(3q_2 + 3q_1) + (1-p)^4 4q_1 \quad (4.5)$$

More generally, in the  $n$ -player quantity-setting game the output of an average opponent is given by:

$$\begin{aligned} & \sum_{k=0}^{n-1} \left[ \binom{n}{k} p^k (1-p)^{n-1-k} ((n-1-k)q_1 + kq_2) \right] \\ = & q_2 \sum_{k=0}^{n-1} \left[ \binom{n}{k} p^k (1-p)^{n-1-k} k \right] + q_1 \sum_{k=0}^{n-1} \left[ \binom{n}{k} p^k (1-p)^{n-1-k} (n-k-1) \right] \\ = & q_2(n-1)p + q_1(n-1)(1-p) \end{aligned} \quad (4.6)$$

The aggregate expected output of the  $n-1$  opponents is equivalent to the output of  $n-1$  *averaged-across-types* opponents. Thus, each player has to form expectations about the output chosen by the *average type* in the population.

The adaptive type (AE) forms adaptive expectations about average the opponent:

$$q_1^e(t+1) = \alpha q_1^e(t) + (1-\alpha)[q_1(t)(1-p) + q_2(t)p]. \quad (4.7)$$

The rational type (RE) has perfect foresight:

$$q_2^e(t+1) = q_1(t+1)(1-p) + q_2(t+1)p. \quad (4.8)$$

Both AE and RE types best-respond to expectations:

$$\begin{aligned} q_1(t+1) &= R_1(q_1^e(t+1)) = \frac{a-c-b(n-1)q_1^e(t+1)}{2b} \\ q_2(t+1) &= R_2(q_2^e(t+1)) = \frac{a-c-b(n-1)q_2^e(t+1)}{2b} \end{aligned} \quad (4.9)$$

$$q_2(t) = R_2(q_2^e(t)) = \frac{a - c - b(n-1)q_2^e(t)}{2b} = \frac{a - c - b(n-1)[q_1(t)(1-p) + q_2(t)p]}{2b} \quad (4.10)$$

$$q_2(t) = \frac{a - c - bq_1(t)(n-1)(1-p)}{2b(\frac{1}{2}p(n-1) + 1)} \quad (4.11)$$

After substituting (4.9) and (4.11) into (4.7) we obtain the adaptive expectations dynamics as:

$$q_1^e(t+1) = \alpha q_1^e + (1-\alpha)[R_1(q_1^e(t))(1-p) + \frac{a - c + bR_1(q_1^e(t))(n-1)(1-p)}{2b(\frac{1}{2}p(n-1) + 1)}p] \quad (4.12)$$

Next, we compute the expected profits as:

$$\begin{aligned} \Pi_1(t) &= q_1(t)[a - b[q_1(t) + (n-1)[q_1(t)(1-p) + q_2(t)p]] - c] \\ \Pi_2(t) &= q_2(t)[a - b[q_2(t) + (n-1)[q_1(t)(1-p) + q_2(t)p]] - c] \end{aligned}$$

The share  $p$  of rational players updates according to asynchronous updating (only a fraction  $1 - \delta$  revise strategies according to the logistic probability) based on the realized performance measure:

$$p(t+1) = \delta p(t) + \frac{1 - \delta}{1 + e^{(\beta(\Pi_1(R(q_1^e(t)), R(q_2^e(t)), p(t)) - \Pi_2(R(q_1^e(t)), R(q_2^e(t)), p(t)) + k))}} \quad (4.13)$$

where  $k$  stands for the costs associated with the rational expectations predictor (see the discussion regarding asynchronous logistic updating in Chapter 3). Equations (4.12) and (4.13) define a two dimensional dynamical system  $(q_1^e(t+1), p(t+1)) = \Phi(q_1^e(t), p(t))$  which has an interior steady state at:

$$(q_1^{e*} = \frac{a - c}{b(1+n)}, p^* = \frac{1}{1 + \exp(\beta k)})$$

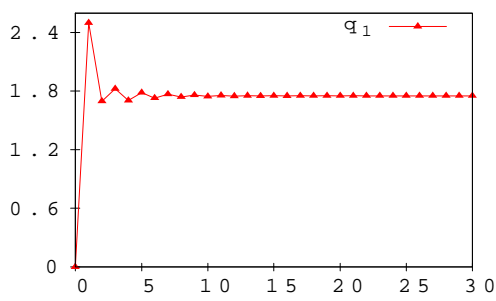
Expressed in terms of actual quantity choices the steady state is:

$$q_1^* = q_2^* = \frac{a - c}{b(1 + n)}, p^* = \frac{1}{1 + \exp(\beta k)} \quad (4.14)$$

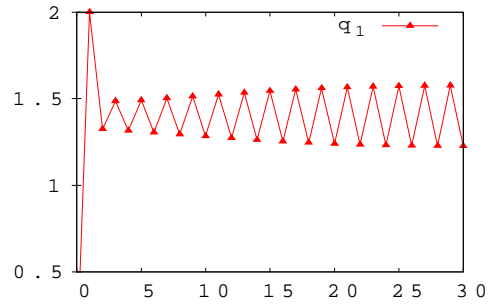
Before proceeding to the stability analysis of this equilibrium, we present in Fig. 4.1 simulations of the system behavior for varying number of players. As the game changes from triopoly to a large number of players the interior Cournot-Nash equilibrium (4.14) loses stability and bounded, regular or irregular oscillations arise. We identify a period-doubling route to strange attractors as the number of players increases.<sup>1</sup> A snapshot of such a strange attractor is captured in Fig. 4.1e along with numerical evidence (strictly positive largest Lyapunov exponent) for chaos in Panel (f).

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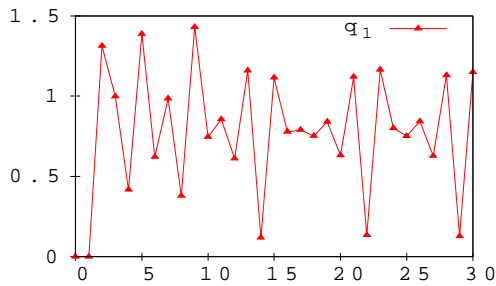
<sup>1</sup>This bifurcation diagram should be interpreted under the caveat that  $n$  is not a continuous parameter (the number of players in the Cournot game only takes positive integers value) . Still, the bifurcation diagram can provide some information about qualitative changes in the behavior of the system, *given* the number of players.



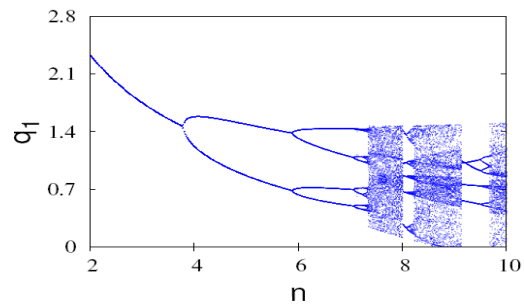
(a) Time series, triopoly [ $n = 3$ ]



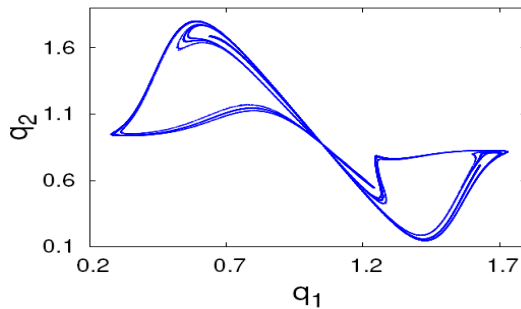
(b) Time series, quadropoly [ $n = 4$ ]



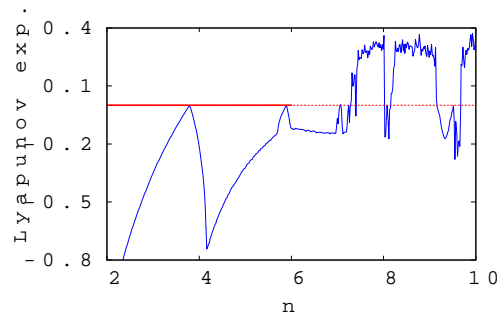
(c) Time series, octopoly [ $n = 8$ ]



(d) Bifurcation diagram  $q_1 - n$



(e) Phase plot, octopoly [ $n = 8$ ]



(f) Largest Lyapunov exponent

Figure 4.1: Linear  $n$ -player Cournot game with an ecology of Adaptive and Rational players. Panels (a)-(c) display converging, oscillating and chaotic time series of the quantity  $q_1$  chosen by the Adaptive type for 3, 4 and 8-player game, respectively. Instability sets in already for the quadropoly game (Panel (b)). Panel (d) depicts a period-doubling route to chaotic dynamics in equilibrium quantity  $q_1$  as we increase the number of players  $n$ . A typical phase portrait for an 8-player game is shown in Panel (e) while Panel (f) provides numerical evidence for chaos (i.e. positive largest Lyapunov exponent) as number of players increases. Game and behavioral parameters:  $a = 17, b = 1, c = 10, k = 1, \alpha = 0.1, \beta = 2.8, \delta = 0.1$ .



### 4.3 Results

We start with the following:

**Proposition 7** *The Cournot-Nash equilibrium  $(q_1^{e*} = \frac{a-c}{b(1+n)}, p^* = \frac{1}{1+\exp(\beta k)})$  loses (local) stability as the number of player  $n$  reaches a critical threshold  $n^{PD} = \frac{2p^* - \alpha - 3}{2p^* + \alpha - 1}$  and a 2-cycle in the equilibrium quantities is born via a period-doubling bifurcation.*

**Proof.** We can rewrite system in deviations from the Cournot-Nash steady-state

$$Q(t) = \frac{a-c}{b(1+n)} - q_1^{e*}(t) :$$

$$Q(t+1) = Q(t) \frac{(1 + \alpha - (n + p(t)) + n(\alpha + p(t)))}{p(t)(n-1) + 2} \quad (4.15)$$

$$p(t+1) = \frac{1 - \delta}{1 + \exp(\beta(\frac{-Q(t)(n^2-1)(Q(t)bn^2+2cpn+4c-Q(t)b-2cp)}{4n^2p^2-8np^2+16np+4p^2-16p+16} + k))} \quad (4.16)$$

with the interior steady state:  $(Q^* = 0, p^* = \frac{1}{1+\exp(\beta k)})$ . The Jacobi matrix at this steady state is:

$$\begin{bmatrix} \frac{1}{p^*(n-1)+2} (\alpha - n - p^* + n(\alpha + p^*) + 1) & 0 \\ \frac{(4c-2cn+2cnp^*)\beta p^*(\delta-1)(p^*-1)((p^*)^2-1)}{4n^2(p^*)^2-8n^2p^*+4n^2+16np^*-16n+16} & \delta \end{bmatrix},$$

with corresponding eigenvalues  $\lambda_1 = \frac{1}{np^* - p^* + 2} (\alpha - n - p^* + n\alpha + np^* + 1)$  and  $\lambda_2 = \delta$ . The period-doubling bifurcation ( $\lambda_1 = -1$ ) condition reads:

$$\frac{1}{np^* - p^* + 2} (\alpha - n - p^* + n\alpha + np^* + 1) = -1,$$

which yields,

$$n^{PD} = \frac{2p^* - \alpha - 3}{2p^* + \alpha - 1}. \quad (4.17)$$

■

We notice first that the Cournot-Nash equilibrium is stable if the rational predictor is costless. The threshold (4.17) can be re-arranged as  $\beta^{PD} = \frac{1}{k} \ln \left( \frac{(\alpha+1)(1+n)}{n-\alpha(1+n)+3} \right)$

and observe that there is no finite value of the intensity of choice satisfying this equality for costless RE (i.e.  $k = 0$ ).

### 4.3.1 Best-response dynamics limit, $\beta \rightarrow \infty$

We can now derive Theocharis (1960) unstable triopoly result (i.e. bounded oscillations for  $n = 3$  and exploding oscillations for  $n = 4$ ) in the limit of naive expectations [ $\alpha \rightarrow 0$ ] and best-response dynamics [ $\beta \rightarrow \infty$ ]. First we compute  $\lim_{\beta \rightarrow \infty} p^* = 0$  and then evaluate the period-doubling threshold (4.17) at  $\alpha \rightarrow 0$  yielding

$$\lim_{\beta \rightarrow \infty, \alpha \rightarrow 0} n^{PD}(p^*(k)) = 3. \quad (4.18)$$

The intuition behind the original "two is stable, three is unstable" result had to do with the slope of a reaction curves (4.9)  $\left| \frac{\partial R}{\partial q} \right| = \left| -\frac{n-1}{2} \right|$ . For a duopoly game this slope is, in absolute values, always smaller than 1, but it hits the instability threshold exactly at  $n = 3$  and is larger than 1 starting for a quadropoly game. This is exactly what our threshold  $n^{PD}$  boils down to when we let players choose a best reply [ $\beta \rightarrow \infty$ ] to Cournot (i.e. naively-determined) expectations [ $\alpha \rightarrow 0$ ].

Some further comparative statics can be informative about the role expectations play in (de) stabilizing a Cournot-Nash equilibrium. Ceteris paribus, the number of players' instability threshold  $n^*$  increases in the degree of expectation adaptiveness [ $\alpha$ ]:

$$\partial n^{PD} / \partial \alpha = 4 \frac{1 - p^*}{(\alpha + 2p^* - 1)^2} > 0.$$

Intuitively, a larger  $\alpha$  implies that the unstable adaptive rule place a higher weight on remote observations in the past and thus it acts as an effective stabilizing force. This, in turn, requires an even larger slope of the reaction curve (i.e. larger threshold  $n^*$ ) to de-stabilize the interior CN equilibrium. Similarly, we can show that

$n^*$  decreases in the costs associated with the perfect foresight rule:

$$\partial n^{PD}(p^*(k))/\partial k < 0,$$

$$\text{as } \partial n^{PD}(p^*)/\partial p^* = 4\frac{\alpha+1}{(2p+\alpha-1)^2} > 0 \text{ and } \partial p^*/\partial k < 0.$$

For a given number of players, at the steady state, higher costs of rational expectations  $k$ , increase the payoff differential between the rational and the naive heuristics. Hence, more agents switch to the simple, yet destabilizing adaptive strategy. Thus, the system becomes more unstable and it takes a less steeper reaction curve (i.e. fewer players) to hit the period-doubling threshold (4.17).

### 4.3.2 Costly Rational Expectations, $k > 0$ and finite $\beta$

In the sequel we derive the threshold number of firms for which instability arises when there are positive costs associated with the perfect foresight and the other behavioural rule approaches either the Cournot ( $\alpha = 0$ ) or adaptive expectations (e.g.  $\alpha = 0.2$ ). Recall from (4.17) that  $n^{PD} = \frac{2p^*-\alpha-3}{2p^*+\alpha-1}$  and  $p^* = \frac{1}{1+\exp(\beta k)}$ . For small costs of the rational predictor ( $k = 1$ ) the period-doubling threshold is hit at:

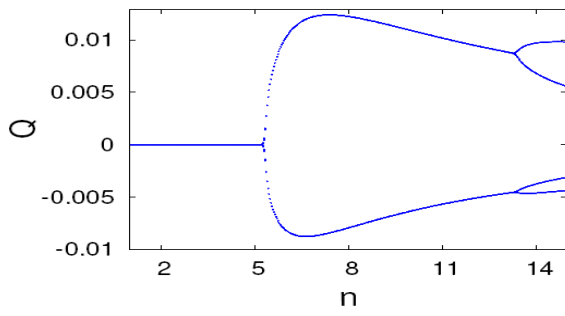
$$n_{k=1}^{PD} = \frac{\alpha + 3e^\beta + \alpha e^\beta + 1}{e^\beta - \alpha - \alpha e^\beta - 1} \quad (4.19)$$

Table 4.1 reports unstable market structures in evolutionary Cournot games (i.e. number of players' period-doubling, rounded thresholds (4.19)) for different adaptive expectations rule  $\alpha$  and intensity of choice  $\beta$ .

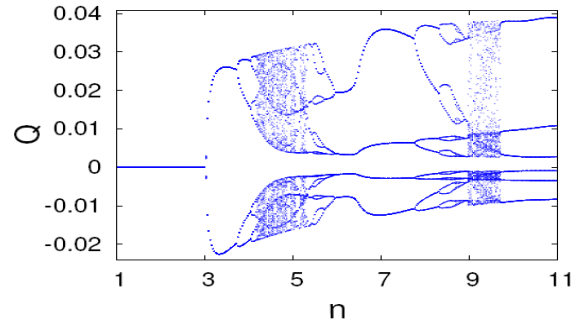
$\alpha/\beta$	1 - "low"	5 - "high"
0 ("Cournot" expectation)	5 (quintopoly)	3 (triopoly)
0.2 ("adaptive" expectations)	10 (decapoly)	4 (quadropoly)

Table 4.1: Linear  $n$ -player Cournot game with costly Rational expectations. Unstable market structures for different expectations and behavioral parameterization. Game and behavioral parameters:  $a = 17, b = 1, c = 10, k = 1, \delta = 0.1$ .

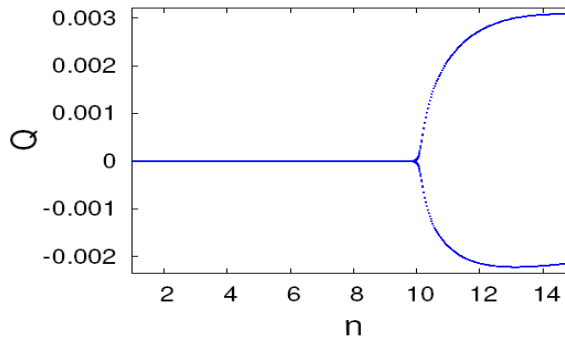
These analytically derived thresholds are confirmed by the numerically-computed bifurcations curves illustrated Fig. 4.2a-d. Higher values of the intensity of choice may destabilize a given market structure: larger  $\beta$  effectively implies turning on the evolutionary selection mechanism built into the model and population switching into/away from the adaptive, unstable heuristic. The two bottom panels in Fig. 4.2 display, for a quintopoly game, a strange attractors in the quantities chosen by the two types in the population (Panel (e)) and fractions of rational players-deviations from steady state quantity (Panel (f)).



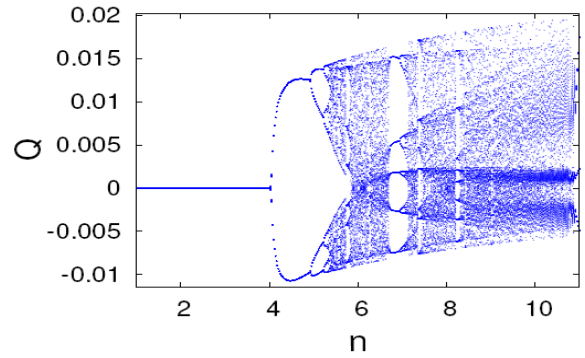
(a)  $\alpha = 0, \beta = 1$ .  $n^{PD} = \text{Quintopoly}$



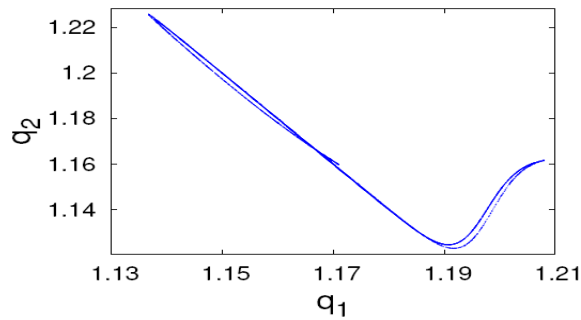
(b)  $\alpha = 0, \beta = 5$ .  $n^{PD} = \text{Triopoly}$



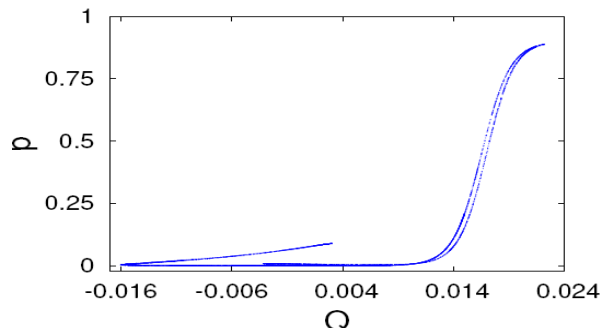
(c)  $\alpha = 0.2, \beta = 1$ .  $n^{PD} = \text{Decapoly}$



(d)  $\alpha = 0.2, \beta = 5$ .  $n^{PD} = \text{Quadropoly}$



(e)  $n = 5, \alpha = 0.2, \beta = 10.65$



(f)  $n = 5, \alpha = 0.2, \beta = 10.65$

Figure 4.2: Linear  $n$ -player Cournot game with costly Rational expectations. Bifurcation diagrams of the deviations  $Q$  from equilibrium quantity  $q_1$  with respect to number of firms  $n$  for Naive [ $\alpha = 0$ ] (Panel (a), (b)) and Adaptive [ $\alpha = 0.2$ ] (Panel (c), (d)) expectations. Panels (a)-(d): The first period-doubling bifurcation threshold decreases in the intensity of choice parameter  $\beta$ . Panels (e)-(f) display projection of a strange attractor onto the quantity-quantity and deviations from CNE quantity-fractions of RE players, respectively. Game and behavioral parameters:  $a = 17, b = 1, c = 10, k = 1, \delta = 0.1$ .

## 4.4 Conclusions

We constructed an evolutionary version of Theocharis (1960) seminal work on the stability of the Cournotian equilibrium in multi-player quantity-setting games. The assumption of homogenous naive expectations is relaxed and players are allowed to choose between a costless, adaptive rule and a costly, rational rule. The results are consistent with the original homogenous-expectations analysis of Theocharis, but the instability thresholds vary with the costs of the sophisticated rule and the degree of adaptiveness in the expectation formation process. One implication of the model is that by fine-tuning these two parameters a particular Cournot market structure can be (de)stabilized. For instance, the classical unstable Cournot triopoly may be stabilized via making the rational expectations predictor freely available. On the other hand, when rational expectations are costly a period-doubling bifurcation route to chaos arises when the number of players increases.