Essays on nonlinear evolutionary game dynamics

Ochea, M.I.

Citation for published version (APA):
Chapter 5

Evolution in Iterated Prisoner’s Dilemma Games under Smoothed Best-Reply Dynamics

5.1 Introduction

Studying evolutionary dynamics on iterated prisoner’s dilemma (IPD) demands a selection out of the vast set of the repeated game strategies. This choice turns out to be important for the outcome of the iterated game and the level of cooperation ’evolved’. For instance, although the Tit-For-Tat strategy was an undisputed winner of two round-robin tournaments, Axelrod (1997) stresses the crucial role of the surrounding ecology of submitted iterated rules for the success of this direct reciprocity norm. However, which strategies to choose out of the large set of repeated PD is a question lacking a definite answer in the literature. Kraines and Kraines (2000) define adaptive dynamics\(^1\) on the space of memory one repeated PD strategies and show that this process selects, as long-run outcome, only three classes of repeated rules:

\(^1\)Adaptive dynamics are regularly used in evolutionary biology to model evolution through natural selection on the fitness landscape.
cooperative, alternating and defective. They also claim that, out of the cooperative subset, the Pavlov strategy is the only one that cannot be invaded by a strategy from the same 'cooperative' class (including TFT). Sigmund and Brandt (2006) investigate one such ecology consisting of three strategies: unconditional cooperators (AllC), unconditional defectors (AllD) and reactive players (TFT) and show that Replicator Dynamics exhibits, among others, a rock-scissors-paper pattern of cyclic behavior. However, as we have seen in Chapter 2, such replicator cycles are not robust under Replicator Dynamics, as small payoff perturbations drive all but one strategy near to extinction$^2$.

In this chapter, we contribute to the stream of work on evolution of rules in repeated games by analyzing the repeated Prisoner’s Dilemma with a small number of simple, memory-one strategies. In particular, we extend Sigmund and Brandt (2006) ecology with two additional repeated strategies that seem to have received less attention in the evolutionary IPD game literature: the error-proof, "generous" tit-for-tat which, with a certain probability, re-establishes cooperation after a (possibly by mistake) defection of the opponent and the penitent, "stimulus-response" (thus dubbed Pavlov strategy Kraines and Kraines (1995), Sigmund and Nowak (1993b), Sigmund and Nowak (1993a)) that resets cooperation after the opponent punished for defection. Second, we contrast Replicator Dynamics behavior with a perturbed version of the Best-Reply dynamics, the Logit Dynamics, allowing for an imperfect switching towards a myopic best reply to the existing strategies distribution. Our emphasis is on the possibility of complicated dynamics such as multiplicity of steady states, limit cycles or chaos in the resulting dynamical system. We start with a systematic investigation of all ten $2 \times 2$ repeated game rules interactions and then build-up towards more complex ecologies of 3, 4, and 5 strategies, combining theoretical considerations about the best-reply structure of the resulting normal form games with

$^2$mathematically, a so-called heteroclinic cycle is born, with Replicator Dynamics trajectories lingering longer and longer near the boundary of the simplex.
numerical analysis and simulations. A bifurcation analysis with respect to various model parameters is performed in order to reveal qualitative changes in the set of long-run (non)cooperative behaviors. Preliminary results, in particular for the $4 \times 4$ and $5 \times 5$ IPD ecologies, show that Logit Dynamics displays stable co-existence of repeated strategies but subjected to perpetual oscillations or even chaotic patterns in the distribution of IPD strategies.

The chapter is structured as follows: Section 5.2 introduces the selection of iterated PD strategies and the resulting evolutionary IPD game. The 2, 3 and 4-types ecologies are discussed in Sections 5.3, 5.4 and 5.5, respectively, while Section 5.6 investigates, mainly via computer simulations, the full $5 \times 5$ ecology. The final section is reserved for concluding remarks and future research directions.

### 5.2 An Evolutionary Iterated PD game

We consider a standard $2 \times 2$ Prisoner’s Dilemma stage game where players can either cooperate (C) or defect (D). The payoff matrix of the stage game is given by:

$$
\begin{bmatrix}
C/D & C & D \\
C & b - c, b - c & -c, b \\
D & b, -c & 0, 0
\end{bmatrix}; \quad b > c > 0
$$

where $b$ stands for the benefits of cooperation and $c$ for the costs associated with cooperative behavior. At each time $t$ the state of the play between two opponents is given by an element belonging to set: $\Omega = \{CC, CD, DC, DD\}$. For the iterated PD game we restrict the choice of iterated Prisoner’s Dilemma meta-rules to a set of stochastic *memory-one* strategies (see, for instance, Kraines and Kraines (2000)). Our motivation for such a simple subset of IPD strategies resides, on the one hand, on our attempt to model a particular form of boundedly rational players (namely limited memory agents or forgetting effects) and, on the other hand, to obtain analytical
tractability of the resulting Markov chain. The IPD game starts with a first random move C or D and then proceeds with playing C with probability \((r, s, t, p)\) conditional on the realized state at time \(t-1\) being CC, CD, DC, DD, respectively. Deterministic strategies are particular limits in this stochastic strategy space. In particular we focus on 5 well-known strategies:

- unconditional cooperators \(AllC : (1, 1, 1, 1);\)
- unconditional defectors \(AllD : (0, 0, 0, 0);\)
- conditional cooperators "Tit-for-Tatters" \(TFT : (1, 0, 1, 0);\)
- generous cooperators "Generous-Tit-for-Tat" \(GTFT : (1, m, 1, n)^3;\)
- penitent or Pavlov players "WinStayLoseShift" \(WSLS : (1, 0, 0, 1).\)

Stochastic strategies are \(\varepsilon\)-perturbations of the deterministic ones, where \(\varepsilon\) has the natural interpretation of a probability of mistakes or errors in implementation/execution of the deterministic strategies. Following Kraines&Kraines (2000) the iterated Prisoner’s Dilemma game between two stochastic players \(S_1 = (r, s, t, p)\) and \(S_2 = (x, y, z, w)\) leads to a Markov chain on states CC, CD, DC, DD with transition probabilities given by:

\[
T = \begin{bmatrix}
    \text{state} & CC & CD & DC & DD \\
    CC & rx & sz & ty & pw \\
    CD & r(1-x) & s(1-z) & t(1-y) & p(1-w) \\
    DC & (1-r)x & (1-s)z & (1-t)y & (1-p)w \\
    DD & (1-r)(1-x) & (1-z)(1-s) & (1-t)(1-y) & (1-p)(1-w)
\end{bmatrix}
\]

(5.2)

Each column of the matrix \(T\) in (5.2) contains transition probabilities to states CC, CD, DC, DD, respectively and naturally, its entries add up to one. For instance,

\(^3m\) stands for the probability of cooperating after the opponent defected and \(n\) is the probability of playing cooperate after mutual defection.
in the first column, entry $t_{11}$ gives the probability of next period state staying in $CC$: row player $S_1$ cooperates with probability $r$ after the realized state was $CC$ while column player $S_2$ cooperates with probability $x$ after the same realization, and thus:

$$t_{11} = \text{prob}(CC | CC) = rx.$$ 

Similarly,

$$t_{21} = \text{prob}(CD | CC) = \text{prob}(S_1 \text{ cooperates after } CC) \times \text{prob}(S_2 \text{ defects after } CC) = r \times (1 - \text{prob}(S_2 \text{ cooperates after } CC)) = r(1 - x).$$

One can show that, for strictly positive perturbation parameters, this Markov process is ergodic, i.e. there is positive probability of escaping from any of the states in $\Omega$. Therefore, it has a stationary invariant distribution given by the eigenvector of $T$ corresponding to an eigenvalue 1 (Kemeny and Snell (1975)). The invariant distribution represents the average time the play between two stochastic strategies spends in each state in $\Omega$ and, without discounting future payoffs\footnote{Sigmund and Brandt (2006) do allow for discounting in their 3x3 ecology of behaviors, and construct the iterated game matrix in a different manner, by computing discounted sums of future payoffs, under stochastic strategies; thus, the first move in the game becomes relevant and a repeated game strategy must be characterized, in addition, by the probability of playing C in the first round.}, it enables computation of the average expected payoff resulting from the interaction of two repeated game strategies(Appendix 5.A contains detailed calculations of the invariant distributions and average payoffs). One such average payoff matrix is constructed below for an ecology consisting of the following stochastic versions of the five strategies described above:

- $AllC - (1 - \varepsilon, 1 - \varepsilon, 1 - \varepsilon, 1 - \varepsilon);$
- $AllD - (\varepsilon, \varepsilon, \varepsilon, \varepsilon);$
- $TFT - (1 - \varepsilon, \varepsilon, 1 - \varepsilon, \varepsilon);$
\[ M = \begin{bmatrix}
    \text{AllD} & \text{TFT} & \text{GTFT} & \text{WSLS} & \text{AllC} \\
    \text{AllD} & \varepsilon (b-c) & m_{12} & m_{13} & \frac{1}{2}b - c\varepsilon & b - b\varepsilon - c\varepsilon \\
    \text{TFT} & m_{21} & \frac{1}{2}b - \frac{1}{2}c & m_{23} & \frac{1}{2}b - \frac{1}{2}c & m_{25} \\
    \text{GTFT} & m_{31} & m_{32} & \frac{n}{n+\varepsilon}(b-c) & m_{34} & m_{35} \\
    \text{WSLS} & b\varepsilon - \frac{1}{2}c & \frac{1}{2}b - \frac{1}{2}c & m_{43} & m_{44} & b - \frac{1}{2}c - b\varepsilon \\
    \text{AllC} & b\varepsilon - c + c\varepsilon & m_{52} & m_{53} & m_{54} & (1 - \varepsilon)(b-c)
\end{bmatrix} \quad (5.3)
\]

where \( m_{ij} \)s are complicated algebraic expressions in the stage game parameters \((b, c)\), the errors \((\varepsilon)\) and the ‘generosity’ parameters \( n \) (see Appendix 5.A).

Players are assumed to switch their repeated game strategies based on realized, past average performance. Thus, at time \( t \) the repeated strategy \( i \) will be played with the logistic probability:

\[ x_{i,t} = \frac{e^{\beta(Mx)_{i,t-1}}}{\sum_{i=1}^{5} e^{\beta(Mx)_{i,t-1}}}, \quad \sum_{i=1}^{5} x_{i,t-1} = 1 \quad (5.4)\]

with \( \beta \) denoting the responsiveness to payoff differences between alternative strategies and \( x_{i,t} \)s the time \( t \) fractions of \textit{ALLD}, \textit{TFT}, \textit{GTFT}, \textit{WSLS}, \textit{ALLC} players in the population, respectively. Each entry in the payoff matrix (5.3) defines the long-run\(^6 \) contribution \( m_{ij} \) to the fitness of strategy \( S_i \) due to a particular encounter with strategy \( S_j \). This long-run contribution to fitness is used first to approximate the specific performance of strategy \( S_i \) in a \textit{finitely} repeated PD game and then to

\(^5\)Throughout the Chapter we assume that the ‘generosity’ probabilities \( n \) and \( m \) are an order of magnitude higher than the probability of an error in the strategy execution \( \varepsilon \) (e.g. \( \varepsilon = 0.01, n = 0.1 \)).

\(^6\)i.e. in the invariant distribution of the Markov chain
update the behavior for the next IPD encounter.

In order to disentangle the rich behavior of the full 5-dimensional system (5.4) we start from simple pairwise interactions and then build larger ecologies of behaviors to capture the contribution of each type to the evolution of (non) cooperative outcomes.

5.3 2×2 Ecologies

In this section we discuss each of the \( \binom{5}{2} = 10 \) cases of an ecology with only two repeated strategies. As we will see, depending on the stage game payoff matrix parameterization and of the error parameter \( \varepsilon \), the following interaction structures emerge for the IPD game: Coordination, Hawk-Dove, Prisoner’s Dilemma and dominance solvable games.

5.3.1 AllD vs. TFT

In a 2 × 2 ecology of unconditional defectors (AllD) and reciprocators (TFT) the reduced payoff matrix has the following form:

\[
\begin{bmatrix}
\text{AllD} & \text{TFT} \\
\text{AllD} & \varepsilon(b - c) & -\varepsilon(c - 2b + 2b\varepsilon) \\
\text{TFT} & \varepsilon(b - 2c + 2c\varepsilon) & \frac{1}{2}b - \frac{1}{2}c
\end{bmatrix}
\]  

For \( b > c/(1 - 2\varepsilon) \) and \( \varepsilon > 0 \) the following inequalities hold:

\[
\varepsilon(b - c) > \varepsilon(b - 2c + 2c\varepsilon) \\
\frac{1}{2}b - \frac{1}{2}c > -\varepsilon(c - 2b + 2b\varepsilon)
\]
Thus, the IPD game (5.5) is a Coordination game\(^7\) with two pure strategy equilibria \((AllD, AllD)\) and \((TFT, TFT)\) and, depending on the initial population mixture, the Best-Reply limit \((\beta \to \infty)\) of the Logit Dynamics converges to either of them. When implementation errors are very small \((\varepsilon = 0.01)\), the \((AllD, AllD)\) equilibrium basin of attraction is much smaller compared to the other equilibrium basin. This is due to the fact that \(TFT\) against itself performs much better than \(AllD\) against itself, as can be seen from the payoff matrix (5.5): 
\[
m_{22} = 1/2(b - c) > m_{11} = \varepsilon(b - c).
\]

It turns out that, for large \(\beta\) and small \(\varepsilon\), \(TFT\) can invade and take over the entire population, irrespective of the initial mixture (Fig. (5.1), Panel (a) show time series of the fractions for highly asymmetric starting conditions of 99\% \(AllD\) and only 1\% \(TFT\)). Increasing the probability of mistakes \(\varepsilon\), increases the payoff of defectors against themselves and, in the \(\beta\) large limit, the system displays co-existence of stable steady states. A certain large initial critical mass of defectors (fraction of \(AllD\) 99\% and \(TFT\) 1\%) can eliminate \(TFT\) with the population ending up in a monomorphous \(AllD\) state (Fig. 5.1b). The multiple steady states are created via saddle-node bifurcations: e.g. (Fig. 5.1cd) there is an unique interior fixed point for low values of \(\beta(\varepsilon)\) but as the intensity of choice (mistake probability) reaches certain thresholds \(\beta \approx 140(\varepsilon \approx 0.05)\) two additional steady states are created: one stable (observed in the panels) and one unstable. This behavior is similar to the evolutionary coordination game studied in Chapter 2 (see e.g. Fig. 2.10a).

\(^7\)In the \(\varepsilon \to 0\) limit \(TFT\) weakly dominates \(AllD\) and it is selected by the logit dynamics given when the sensitivity to payoffs difference is high enough (large \(\beta\)). However, taking this limit is problematic as the invariant distribution of the Markov chain is no longer unique, but history-dependent (more precisely the long-run payoff matrix now depends on the first move in the game). Still, for \(\varepsilon = 0\), the payoff matrix could be regarded as the outcome of a particular unfolding of the game history.
Figure 5.1: Unconditional defectors (AllD) vs. Reciprocators (TFT). Different errors in the implementation lead to different long-run steady states: a population of reciprocators (Panel (a)) or a population of defectors (Panel (b)). Panels (c)-(d) display co-existence of stable steady states for large values of $\beta$ and $\varepsilon$. Remaining game parameters: $b = 4, c = 1, \beta = 200$.

### 5.3.2 TFT vs. AllC

For an interaction between unconditional cooperators (AllC) and reciprocators (TFT) the payoff matrix (5.3) reduces to:

$$
\begin{bmatrix}
TFT & \frac{1}{2}(b - c) & b - c - b\varepsilon + 2c\varepsilon - 2c\varepsilon^2 \\
AllC & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c)
\end{bmatrix}
$$

(5.6)
It is straightforward to see that, for small $\varepsilon$ and $b > c$ we have:

\[
\begin{align*}
    b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 &> \frac{1}{2}(b - c) \\
    b - c - b\varepsilon + 2c\varepsilon - 2c\varepsilon^2 &> (1 - \varepsilon)(b - c)
\end{align*}
\]

Consequently, the reduced $2 \times 2$ game (5.6) is of Hawk-Dove type and, thus, it has three equilibria: two asymmetric in pure strategies ($TFT, AllC$) and ($AllC, TFT$) and one symmetric in mixed strategies. The two asymmetric equilibria give rise to a 2-cycle under best-reply dynamics (i.e. logit dynamics with large $\beta$) while the interior equilibrium is unstable. A two-cycle, with population swinging back and forth between the two asymmetric equilibria, is created via a period-doubling bifurcation when either errors in implementation are small (Fig. 5.2c) or players choose a best-reply (large $\beta$, Panel(d)). The intuition for the two-cycle relates to the best-response correspondences in the payoff matrix above: $TFT \in BR\{AllC\}$ and $AllC \in BR\{TFT\}$. In the population game interpretation, when everyone plays $TFT(AllC)$ it is better to switch to $AllC(TFT)$. Still, the amplitude of the cycle fluctuations varies with how responsive players are to payoff differences, with the full-scale two-cycle observed only for large $\beta$.

For intermediated values of $\varepsilon$ (e.g. $\varepsilon \in (\frac{1}{4}, \frac{1}{2})$ for the $b = 2, c = 1$ game parameterization) matrix (5.6) defines a dominance solvable (DS) game with ($TFT, TFT$) as unique pure strategy equilibrium while for large $\varepsilon > \frac{1}{2}$ the game is still dominance solvable, but with ($AllC, AllC$) the unique equilibrium. The transition from a Hawk-Dove to a Dominance Solvable structure in the space of $2 \times 2$ games is illustrated in Fig. 5.2ef for finite and best-reply limit of $\beta$, respectively.
Figure 5.2: Reciprocators(TFT) vs Unconditional Cooperators(AllC). Time series of fractions of TFT and AllC players for small (Panel (a)) and large (Panel (b)) intensity of choice values, respectively. Panels (c)-(d) display a period-doubling bifurcation with respect to $\varepsilon$ and $\beta$, leading to a 2-cycle. Last, Panels (e)-(f) show transition from a Hawk-Dove to Dominance Solvable pattern. Game parameters: $b = 2, c = 1, \varepsilon = 0.01$. 

(a) small $\beta = 15$. Time series

(b) large $\beta = 220$. Time series

(c) $\beta = 15$. Bifurcation diagram ($TFT, \varepsilon$)

(d) $\varepsilon = 0.01$. Bifurcation diagram ($TFT, \beta$)

(e) $\beta = 15$. Game transition HD-DS

(f) $\beta = 100$. Game transition HD-DS-DS
5.3.3 AllD vs. AllC

For a $2 \times 2$ ecology consisting of undiscriminating defective and cooperative types the restricted payoff matrix reads:

$$
\begin{bmatrix}
\text{AllD} & \text{AllC} \\
\text{AllD} & \varepsilon (b - c) & b - b\varepsilon - c\varepsilon \\
\text{AllC} & b\varepsilon - c + c\varepsilon & (1 - \varepsilon) (b - c)
\end{bmatrix}
$$

(5.7)

We have:

$$
\varepsilon (b - c) > b\varepsilon - c + c\varepsilon \\
b - b\varepsilon - c\varepsilon > (1 - \varepsilon) (b - c)
$$

and the Iterated Prisoner’s Dilemma is a PD game itself with $(\text{AllD, AllD})$ as the unique equilibrium. This case illustrates the standard prisoner’s dilemma argument: unless errors in implementation are extremely high (e.g. $\varepsilon \approx 0.5$ or $\beta$ very small, both making the players choice virtually random as in Fig. (5.3)a), the unconditional cooperators are heavily exploited by defectors, a situation leading to full extinction of the cooperative "trait" from the population (Fig. (5.3)b). The result is robust to any initial mixture of the population. For small finite $\beta$ or relatively large mistake probability there is a unique interior steady state of fractions $(x^*_{\text{AllD}}, x^*_{\text{AllC}})$ with $x^*_{\text{AllD}}$ sliding towards 1 as $\beta \to \infty$ and/or $\varepsilon \to 0$. Fig. (5.3)c-d illustrates the continuation of the equilibrium with respect to each of the two parameters.
Figure 5.3: Unconditional defectors (AllD) vs. Unconditional cooperators (AllC). Unless the choice is almost random (Panel (a), $\beta = 0.05$), the system quickly converges to a monomorphic state with defectors only (Panel (b)). Unique steady state converging to an AllD population when $\beta$ is large or $\varepsilon$ gets smaller (Panels (c)-(d)). Game parameters: $b = 2, c = 1, \varepsilon = 0.01$.

### 5.3.4 AllD vs. GTFT

For a sub-ecology of undiscriminating defectors and generous reciprocators the game matrix has the form:

\[ \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \]
\[
\begin{bmatrix}
    AllD & GTFT \\
    AllD & \varepsilon (b - c) & b\varepsilon - c\varepsilon - b\varepsilon^2 + bn - bn\varepsilon \\
    GTFT & b\varepsilon - c\varepsilon + c\varepsilon^2 - cn + cn\varepsilon & \frac{n}{\varepsilon + \varepsilon} (b - c)
\end{bmatrix}
\] (5.8)

which is always a coordination game for a 'reasonable' IPD parameterization.

This behaviors ecology generalizes the case discussed in subsection (5.3.1). Similarly, the system displays co-existence of monomorphic steady states for the \( \beta \to \infty \) limit. Unlike the AllD vs. TFT case the multiplicity of steady states is preserved even for very small \( \varepsilon = 0.01 \) (Fig.5.4a-b) with AllD taking over the population provided that it slightly outnumbers TFT in the initial mixture (Fig.5.4a). The crucial parameter for this robust path-dependence, is the degree of generosity \( n \) in GTFT strategy: "generous" TFT plays cooperate with quite large probability \( (n = 0.3) \) even after the opponent defected and this makes it more prone to exploitation from AllD compared to the typical TFT strategy. Reversing the initial mixture leads to long-run behaviour qualitatively similar to the one observed in the AllD vs. TFT sub-ecology (Fig.5.4b).

Panels (c)-(d) display the fold bifurcation scenarios through which the co-existing monomorphic steady states emerge. For instance, in Panel (d) as the critical threshold \( n \approx 0.3 \) is hit, a new stable fixed point is created with AllD players only for a parameterization \((\varepsilon = 0.01, \beta = 200)\) for which uniqueness of the steady state was observed in the AllD vs. TFT ecology. By carefully tuning in the generosity parameter in the GTFT strategy, one could generate co-existing stable steady states in an AllD – (G)TFT ecology of repeated rules.

\[^{8}\text{AllD is a best reply to itself iff } \varepsilon^2 + n\varepsilon - n < 0 \text{ (satisfied as long as } n \text{ is an order higher than } \varepsilon); \text{ similarly, GTFT is best-response to itself as long as } \varepsilon^2 + n\varepsilon - n < 0 \text{ and } b > c.\]
Figure 5.4: Unconditional defectors (AllD) vs. Generous reciprocators (GTFT). Long run behavior strongly depends on the initial mixture of the population even for very small errors in implementation. Panels (c)-(d): co-existence of monomorphic steady states(created via fold bifurcations) when \( n \) and \( \beta \) are large, even for tiny implementation errors \( \varepsilon \). Baseline game parameters: \( b = 2, c = 1, \varepsilon = 0.01, n = 0.3, \beta = 15 \).

### 5.3.5 AllD-WSLS

In this sub-case, the payoff matrix (5.3) reduces to:

\[
\begin{bmatrix}
\text{AllD} & \varepsilon (b-c) & \frac{1}{2}b-c \varepsilon \\
\text{WSLS} & b\varepsilon - \frac{1}{2}c & (b-c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon)
\end{bmatrix}
\]  

(5.9)
For high enough benefits of cooperation \((b > b^*\))\footnote{\(b^* = 2c(1 - 2\varepsilon + 2\varepsilon^2)/(-4\varepsilon + 4\varepsilon^2 + 1)\)}, this payoff matrix represents a Coordination game if:

\[
\varepsilon (b - c) > b\varepsilon - \frac{1}{2} c
\]

\[
(b - c) (1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) > \frac{1}{2} b - c\varepsilon
\]

In this case two pure strategy equilibria \((AllD, AllD)\) and \((WSLS, WSLS)\) exist. One could notice first that, for small implementation errors \(\varepsilon\), Pavlov fares much better against itself than \(AllD\) does against itself \(((b - c) (1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) > \varepsilon(b - c))\). However, this may not be enough to offset the heavy exploitation Pavlov incurs in mixed encounters when \(AllD\) gets, on average, half from the benefits \(\lim_{\varepsilon \to 0} \frac{1}{2} b - c\varepsilon\) while \(WSLS\) pays half of the costs \(\lim_{\varepsilon \to 0} (b\varepsilon - \frac{1}{2} c)\). This situation is depicted in Fig. (5.5)a where the population ends up in an all-defectors state. If, in addition to small error probabilities, the benefits of cooperation are relatively large \((b > b^*)\), the "own-type" interaction effect may dominate the "cross" interaction effect and the whole population becomes entirely Pavlovian (Fig. 5.5b). However, this results is dependent on the initial mixture of the population as for a given initial threshold of \(AllD\) players, population cannot turn Pavlovian, irrespective of the relative size of the benefits accrued to cooperation \(b\). The coexistence of two stable, monomorphic steady states for high benefits of cooperation is documented in Fig. 5.5c-d where additional fixed points emerge as the parameters of interest - \(\beta\) and \(\varepsilon\) respectively - are changed. When benefits are low enough, \(b < b^*\) the \(2 \times 2\) game becomes dominance solvable (DS) with \((AllD, AllD)\) the unique equilibrium. A bifurcation diagram with respect to the benefits of cooperation \(b\) (Fig. 5.5e) reveals once the threshold \(b^*\) is passed, a transition in the space of \(2 \times 2\) games from a Dominance Solvable game with unique equilibrium to a Coordination game with two equilibria.
Figure 5.5: Unconditional defectors (AllD) vs. Pavlov (WSLS). The selected long-run monomorphic state, depends both on the size of the implementation errors $\varepsilon$ and on the relative benefit of cooperation $b$. Panels (c)-(d): fold bifurcations in $\varepsilon, \beta$ spaces and co-existence of steady states. Panel (e) displays the bifurcation diagram with respect to the benefit of cooperation $b$ and show the transition from a dominance solvable game (low $b$) to a coordination game (high $b$). Panel (f) reports time series converging to an AllD population as WSLS is a strictly dominated strategy for $b < 2c$. Baseline game parameters: $b = 4, c = 1, \varepsilon = 0.01, \beta = 15$. 

(a) $b = 4$. Initial mixture (70%, 30%) 
(b) $b = 4$. Initial mixture (50%, 50%) 
(c) $\varepsilon = 0.01$. Bifurcation diagram 
(d) $\beta = 200$. Bifurcation diagram 
(e) $\beta = 200, \varepsilon = 0.01$. Bifurcation 
(f) $b = 2$. Time series
The $2 \times 2$ payoff matrix for an ecology of generous reciprocators and unconditional cooperators yields:

$$
\begin{bmatrix}
GTFT & AllC \\
GTFT & \frac{n}{n + \varepsilon} (b - c) & b - c - b \varepsilon + 2c \varepsilon - c \varepsilon^2 - cn \varepsilon \\
AllC & b - c - 2b \varepsilon + c \varepsilon + b \varepsilon^2 + bn \varepsilon & (1 - \varepsilon) (b - c)
\end{bmatrix} \quad (5.10)
$$

For $b > \frac{c}{1 - n - \varepsilon}$ and $n + \varepsilon < 1$, the following inequalities hold:

$$
b - c - 2b \varepsilon + c \varepsilon + b \varepsilon^2 + bn \varepsilon > \frac{n}{n + \varepsilon} (b - c)$$

$$b - c - b \varepsilon + 2c \varepsilon - c \varepsilon^2 - cn \varepsilon > (1 - \varepsilon) (b - c)$$

They define a Hawk-Dove best-response structure: $AllC \in BR\{GTFT\}, GTFT \in BR\{AllC\}$ and, thus, generalize the $2 \times 2$ ecology $TFT$ vs. $AllC$ discussed in subsection 5.3.2. In a population of $GTFT$ players it pays off to be an unconditional cooperator because generous reciprocators ($GTFT$) do not exploit the opponent. On the other hand, in a population of AllC but subject to random errors, a strategy sensitive to those errors like $GTFT$ does better than an undiscriminating cooperator.

In contrast with the $TFT$ vs. $AllC$ ecology, the threshold at which $AllC$ invades $GTFT$ is always smaller than 1, i.e. there is always a fraction of $AllC$ players in the population (see. Fig. 5.6b-c). This fraction of permanent $AllC$ players goes up as the generosity parameter $n$ increases. This is due to the fact that with larger $n$, $GTFT - (1 - \varepsilon, n, 1 - \varepsilon, n)$ approaches $AllC - (1 - \varepsilon, 1 - \varepsilon, 1 - \varepsilon, 1 - \varepsilon)$ in the space of stochastic strategies, and, thus, it provides a more favorable environment for $AllC$.

In the small $\beta$ case, and for small relative benefits of cooperation, the payoff differences between the two strategies vanish and we observe coexistence of behaviors.

---

\(^{10}\)If this condition is not satisfied, then the game is dominance solvable with a unique pure strategy equilibrium $(GTFT, GTFT)$. 

120
as long run outcome (Fig. 5.6a).

In sum, the long-run behavior of this ecology can lead to either a steady state with co-existing strategies - for large $n$ and $\beta$ (Panel (d)) or for small $b$ and $\beta$ (Panel(a)) - or to a stable two-cycle for small $n$ and large $\beta$ (Panels, (b)-(c)).

![Panel (a)](image1.png) $b = 2, \beta = 200$. Time series

![Panel (b)](image2.png) $b = 2, \beta = 400$. Time series

![Panel (c)](image3.png) $b = 4, \beta = 200$. Time series

![Panel (d)](image4.png) $b = 2, \beta = 400$. Bifurcation diagram

Figure 5.6: Generous reciprocators (GTFT) vs. Unconditional cooperators (AllC). Polymorphic population for relatively small benefit of cooperation and intensity of choice parameters (Panel (a)) and two-cycle for increased $\beta$ (Panel (b)) or $b$ (Panel (c)). Panel (d): a two-cycle created via period-doubling bifurcation with respect to the 'generosity' parameter $n$. Game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$.

### 5.3.7 GTFT vs. WSLS

For this pair of repeated strategies the payoff matrix (5.3) reduces to:
where \( m_{43} \) and \( m_{34} \) are complicated expressions of all model parameters (see Appendix (5.A) for explicit formulas)). One can show, by algebraic manipulations, that for \( n > \varepsilon (\text{by assumption}) \):

\[
\frac{n}{n + \varepsilon} (b - c) > m_{43} \\
(b - c) (1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) > m_{34}
\]

and this \( 2 \times 2 \) ecology defines a Coordination game with two pure strategy, symmetric Nash equilibria (\(GTFT, GTFT\)) and (\(WSLS, WSLS\)). In the \( \beta \) large limit, the equilibria structure of the coordination game describe the long-run behavior of the logit dynamics. The corresponding basins of attraction relative sizes are significantly biased towards Pavlov players as it takes 70% of \(GTFT\) players in the initial population in order to converge to a log-run all \(GTFT\) state (Fig.5.7a), with anything below this threshold reversing fortunes toward a long run Pavlov monomorphic state. Intuitively, a stimulus-response, past-performance driven strategy like Pavlov, is able to take advantage of the generosity built into \(GTFT\) which reverts, with large probability \(n\), to "nice" behaviour after an opponent defection irrespective of the realized history. This is because \(GTFT\) does not discriminate among possible causes of defection (be it after a DD, or a DC history), while Pavlov re-establishes cooperation only after own defection. For finite, small \(\beta\) the population converges to a unique fixed point with co-existing behaviors of the form \((x^*_{GTFT}, x^*_{WSLS}), x^*_{GTFT} + x^*_{WSLS} = 1\), with Pavlov outnumbering generous reciprocators \((x^*_{WSLS} \text{ slides up in the interval } [1/2, 1] \text{ as we increase } \beta)\). Fig. 5.7c-d depict a saddle-node bifurcation route towards co-existing monomorphic steady states. For low \(\beta\) (given \(n\)) or small \(n\) (fixing \(\beta\) close to the "best-reply" limit) there is a unique polymorphic population steady state.
Figure 5.7: Generous reciprocators (GTFT) vs. Pavlovian (WSLS). Depending on the initial fractions in the populations, in the $\beta$ large limit, the system converges to a monomorphic state with only one type of behavior surviving. Panels (c)-(d): creation of two co-existing steady states via a fold bifurcation when the degree of generosity/intensity of choice is varied. Game parameters: $b = 4, c = 1, \varepsilon = 0.01, n = 0.3, \beta = 100$. 

(a) Time series, initial mixture 70% GTFT-30% WSLS

(b) Time series, initial mixture 50% GTFT-50% WSLS

(c), $n = 0.3$. Bifurcation diagram $(WSLS, \beta)$

(d) $\beta = 100$. Bifurcation diagram $(WSLS, n)$
5.3.8 TFT vs. GTFT

For a population of generous and standard tit-for-tatters, the payoff matrix has the form:

\[
\begin{pmatrix}
TFT & GTFT \\
TFT & \frac{1}{2}b - \frac{1}{2}c & -\frac{ce-bc+be^2-bn+cn+bn\epsilon-2cn\epsilon}{n+3\epsilon-2n\epsilon-2\epsilon^2} \\
GTFT & -\frac{ce-bc+be^2-bn+cn+bn\epsilon-2cn\epsilon}{n+3\epsilon-2n\epsilon-2\epsilon^2} & \frac{n}{n+\epsilon}(b-c)
\end{pmatrix}
\] (5.12)

For \( n > \epsilon \), straightforward manipulations show that:

\[
\frac{b\epsilon - c\epsilon + c\epsilon^2 + bn - cn - 2bn\epsilon + cn\epsilon}{n + 3\epsilon - 2n\epsilon - 2\epsilon^2} > \frac{1}{2}b - \frac{1}{2}c \Leftrightarrow b > b^{**} = \frac{c}{1 - n - \epsilon}
\]

\[
\frac{n}{n + \epsilon}(b - c) > -\frac{ce - b\epsilon + be^2 - bn + cn + bn\epsilon - 2cn\epsilon}{n + 3\epsilon - 2n\epsilon - 2\epsilon^2} \Leftrightarrow b > b^* = \frac{c}{1 - 2\epsilon}
\]

So, for \( b > b^{**} \) the game is dominance solvable with \((GTFT, GTFT)\) as the unique Nash equilibrium\(^{11}\). Consequently, in the \( \beta \to \infty \) limit, the fraction of \( GTFT \) players converges to 1 as there is no selection pressure against a more "generous" reciprocative type as long as the rest of the population proceeds to a mimicking behavior. \( GTFT \) is a generalization of \( TFT \) and a population of \( TFT \) provides a stepstone for "nicer" behavior to flourish. The lack of strongly defective players creates a payoff advantage for \( GTFT \), inducing more switching away from \( TFT \), with population eventually ending up in an all \( GTFT \) state (Fig. 5.8a).

For \( b \in (b^*, b^{**}) \) the best responses are \( TFT \in BR\{TFT\}, GTFT \in BR\{GTFT\} \) and we are in the case of a Coordination game with Logit Dynamics displaying co-existence of behavioral rules, but with \( GTFT \) largely outnumbering \( TFT \) (Fig. 5.8b).

The transition in the space of \( 2 \times 2 \) games is illustrated by the bifurcation diagram

\(^{11}\)similarly, if \( b \in (b^*, b^{**}) \) then we have a coordination game while for \( b < b^* \) the game is again dominance solvable, with \((TFT, TFT)\) unique pure-strategy Nash equilibrium.
in Fig. 5.8f. For a tiny region of the benefit of cooperation parameter \( b \in (1, b^*)^{12} \) the game is dominance solvable with \( GTFT \) strictly dominated by \( TFT \) (see a blow-up of the diagram around \( b = 1 \) in Panel (e) where the equilibrium fraction of GTFT is close to zero before the 2-cycle emerge). Then, for the interval \( b \in (b^*, b^{**}) \) we have a coordination game with two equilibria while for \( b > b^{**} \) a dominance solvable game with \( TFT \) strictly dominated by \( GTFT \).

Finally, for a given game - \( b = 2 \), i.e. a Coordination game- letting the intensity of choice vary also leads to a transition from a unique polymorphic steady state (low values of \( \beta \)) through a 2-cycle for intermediate \( \beta \) and back to a unique monomorphic steady state when 'close' to best-reply dynamics.(Fig. 5.8d). A similar transition to a two-cycle is observed in Fig. 5.8e if we let the degree of generosity parameter \( n \) vary.

\[ ^{12}b < 1 \text{ is not a PD game any longer, as } c = 1. \]
Figure 5.8: Reciprocators(TFT) vs. Generous reciprocators(GTFT). This ecology gives rise to a dominance solvable game for relatively large benefits of cooperation (Panel (a)) and to a Coordination game for intermediate values (Panel (b)). For a fixed Coordination game, Panel (d) shows the transition from a unique polymorphic steady state to a 2-cycle and back to a monomorphic steady state. In Panels (e)-(f) the transition in the space of 2 x 2 games is illustrated for the $\beta$ large limit. Game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.3, \beta = 700$. 
5.3.9 TFT vs. WSLS

For this ecology of strategies the payoff matrix (5.3) boils down to:

\[
\begin{bmatrix}
TFT & WSLS \\
TFT & \frac{1}{2}b - \frac{1}{2}c & \frac{1}{2}b - \frac{1}{2}c \\
WSLS & \frac{1}{2}b - \frac{1}{2}c & (b - c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon)
\end{bmatrix}
\] (5.13)

For small implementation errors \( \varepsilon \), WSLS weakly dominates TFT and, in the large \( \beta \) limit, Logit Dynamics leads to elimination of weakly dominated strategies (Fig. 5.9a). Intuition draws on Pavlov essentially being a performance-dependent strategy (i.e. change status quo when unsatisfied with the past period payoff) relative to TFT’s merely mimicking opponent past move.

For smaller \( \beta \) the payoff differential between the two strategies is small and the system converges to a steady state with co-existing rules (Fig. 5.9b).

![Time series plots](image)

(a) \( \beta = 100 \). Time series  
(b) \( \beta = 1 \). Time series

Figure 5.9: Reciprocators (TFT) vs. Pavlov (WSLS). Panel (a): large values of the intensity of choice, quick convergence to monomorphic population consisting only of Pavlovian players, irrespective of the cooperation benefits size \( b \). Panel (b) shows convergence to a polymorphic state for low \( \beta \). Game parameters: \( b = 2, c = 1, \varepsilon = 0.01, n = 0.3, \beta = 100 \).
5.3.10 WSLS vs. AllC

The payoff matrix for this ecology reads:

\[
\begin{bmatrix}
WSLS & AllC \\
WSLS & (1 - \varepsilon)(1 - 2\varepsilon + 4\varepsilon^2)(b - c) & b - \frac{1}{2}c - b\varepsilon \\
AllC & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c)
\end{bmatrix}
\] (5.14)

If \( b > b^{*13} \) the game (5.14) is of Hawk-Dove type with two asymmetric pure strategy equilibria \((AllC, WSLS)\) and \((WSLS, AllC)\) and one, interior equilibrium, in mixed strategies. One could notice that, in the \( \varepsilon \to 0 \) limit, Pavlov and \( AllC \) perform equally well against their own type but Pavlov does relatively better in a mixed matching \((b - \frac{1}{2}c > b - c)\). However for strictly positive errors, \( AllC \) does better against its own type than Pavlov \((1 - 2\varepsilon + 4\varepsilon^2) < 1 \). This gives a payoff advantage for AllC which ensures that it does not go extinct but survives in small fractions subjected to perpetual oscillations (Fig. 5.10a). For a given game matrix, the two-cycle is created, as usual, via a period-doubling bifurcation when, for instance, the intensity of choice \( \beta \) is increased (Fig. 5.10d). Panel (c) in Fig. 5.10 displays the transition, in the \( 2 \times 2 \) games space, from a dominance solvable (low \( b \)) to a Hawk-Dove game (large \( b \)). The logit dynamics attracting set shifts from a unique monomorphic steady state with Pavlov players only to a 2-cycle alternating between \( WSLS \) and \( AllC \).

\(^{13}b^* = 2c \frac{1 - \varepsilon}{1 - 2\varepsilon} \approx 2c \). Otherwise, the game is dominance solvable with \( WSLS \) surviving elimination of strictly dominated \( AllC \).
Figure 5.10: Pavlov (WSLS) vs. Unconditional cooperators (AllC). Oscillatory coexistence of strategies with Pavlovian behaviour outnumbering unconditional cooperators for small intensity of choice (Panel (a)) and continuous swings between monomorphic states when $\beta$ approaches the "best response" limit (Panel (b)). Panel (c) illustrates the bifurcation in the space of games, as the benefit of cooperation varies, from a unique steady state (equilibrium) to a 2-cycle (2 asymmetric equilibria). Panel (d): as the intensity of choice increases the long-run outcome changes from a polymorphic steady state to a 2-cycle. Remaining game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.3, \beta = 15$. 

(c) $\beta = 500$. Bifurcation diagram

(d) $b = 2$. Bifurcation diagram
5.3.11 Summary

Table 5.1 summarizes the normal form games resulting from 10 different the strategic pairwise interaction of the five repeated PD strategies together with the long-run attractors of the corresponding evolutionary game under the perturbed best-reply dynamics. In the limiting $\beta \to \infty$ case, these long-run outcomes coincide with the attracting sets of the unperturbed best-reply dynamics on the respective $2 \times 2$ game.

<table>
<thead>
<tr>
<th>No.</th>
<th>Game</th>
<th>Bifurcation</th>
<th>Attractors</th>
<th>Path-Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>AllD-TFT</td>
<td>WDS,C</td>
<td>SN</td>
<td>multiple SS</td>
</tr>
<tr>
<td>2</td>
<td>TFT-AllC</td>
<td>HD</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>3</td>
<td>AllD-AllC</td>
<td>DS</td>
<td>none</td>
<td>unique SS</td>
</tr>
<tr>
<td>4</td>
<td>AllD-GTFT</td>
<td>C</td>
<td>SN</td>
<td>multiple SS</td>
</tr>
<tr>
<td>5</td>
<td>AllD-WSLS</td>
<td>DS,C</td>
<td>SN</td>
<td>multiple SS</td>
</tr>
<tr>
<td>6</td>
<td>GTFT-AllC</td>
<td>HD</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>7</td>
<td>GTFT-WSLS</td>
<td>C</td>
<td>SN</td>
<td>multiple SS</td>
</tr>
<tr>
<td>8</td>
<td>TFT-GTFT</td>
<td>DS, C</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>9</td>
<td>TFT-WSLS</td>
<td>WDS</td>
<td>none</td>
<td>unique SS</td>
</tr>
<tr>
<td>10</td>
<td>WSLS-AllC</td>
<td>DS,HD</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
</tbody>
</table>

Table 5.1: Summary of the type of game and dynamical behaviour for all 10 pairwise interaction of iterated Prisoner’s Dilemma strategies under logit dynamics.

We have encountered the following typical pairwise interactions between iterated Prisoner’s Dilemma strategies together with transition from one game form to another as the payoff matrix parameters $(b, c, \varepsilon, n)$ change:

- Coordination game (C): multiplicity of monomorphic (only one type of behaviour surviving) steady states with strong path-dependence

- Hawk-Dove game (HD): asymmetric equilibria; 2-cycle with continuous popu-
lation switching from one monomorphic state to the other

- Prisoner’s Dilemma (PD): uniquely selected long run symmetric, Pareto-inferior, equilibrium (i.e. monomorphic population state)

- Dominance Solvable game (DS), non-PD: unique, monomorphic long-run steady state

- Games with a weakly dominated strategy (WDS): unique, monomorphic long-run steady state

The co-existing steady states and 2-cycles emerging under logit dynamics are usually created via fold/ saddle-node (SN) and period-doubling (PD) bifurcations, respectively. Although the qualitative changes in the space of 2 × 2 games do not make the object of the present chapter, some interesting game transitions were also revealed and it would be worth studying them systematically, as future research.
5.4 3×3 Ecologies

In this section we will discuss some typical examples out of the $\binom{3}{3} = 10$ sub-ecologies of three iterated PD strategies, focusing the analysis on the resulting 3×3 normal form games and on the attracting sets and long-run behavior of the logit evolutionary dynamics.

5.4.1 AllD-TFT-AllC

This ecology has been thoroughly investigated in Sigmund and Brandt (2006) under the Replicator Dynamics. The reduced 3×3 payoff matrix reads:

$$
\begin{bmatrix}
\text{AllD} & \text{TFT} & \text{AllC} \\
\text{AllD} & \varepsilon (b - c) & -\varepsilon (c - 2b + 2\varepsilon) & b - b\varepsilon - c\varepsilon \\
\text{TFT} & \varepsilon (b - 2c + 2c\varepsilon) & \frac{1}{2}b - \frac{1}{2}c & b - c - b\varepsilon + 2c\varepsilon - 2c\varepsilon^2 \\
\text{AllC} & b\varepsilon - c + c\varepsilon & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c)
\end{bmatrix}
$$

After normalization (all elements on the main diagonal set equal to zero by, for instance, deducting the diagonal entry from each column for player I/or from each row for player II) we get:

$$
\begin{bmatrix}
\text{AllD} & \text{TFT} & \text{AllC} \\
\text{AllD} & 0 & \frac{1}{2}(1 - 2\varepsilon)(c - b + 2\varepsilon) & c(1 - 2\varepsilon) \\
\text{TFT} & -c\varepsilon(1 - 2\varepsilon) & 0 & c\varepsilon(1 - 2\varepsilon) \\
\text{AllC} & c(2\varepsilon - 1) & \frac{1}{2}(1 - 2\varepsilon)(b - c + 2b\varepsilon) & 0
\end{bmatrix}
$$

As $\frac{1}{2}(1 - 2\varepsilon)(c - b + 2b\varepsilon) < 0$ and $c(1 - 2\varepsilon) > 0$ this matrix has, in the $\varepsilon \to 0$ limit, a generalized Rock-Paper-Scissors structure\footnote{Sigmund and Brandt (2006) also discover, when mistakes probability $\varepsilon \to 0$, a RSP pattern under Replicator Dynamics, for a similar ecology of IPD game rules, with discounting.}. Namely, AllD ties with its own
type, loses to $TFT$ and wins at $AllC$.

Putting together insights from subsections (5.3.1), (5.3.2) and (5.3.3) we notice that, in a stochastic strategy framework, for $\varepsilon \to 0$, this ecology resembles very much a Rock-Paper-Scissors pattern of behavior: each of the three repeated strategies provide the environment in which another one can emerge and take over in the following order: a population of unconditional defectors ($AllD$) is prone to invasion and supersession by small clusters of reciprocators ($TFT$) which, upon reaching a critical threshold, provide the stepstone for the emergence of unconditional cooperators ($AllC$). These $AllC$ players grow in the population up to the limit when conditions are created for unconditional defectors ($AllD$) to thrive, strike back in the population and, thus, complete the full cycle (see Fig. 5.11e). Any small number of $AllD$ players is able to invade a population of $AllC$ given that $AllD$ strongly dominates $AllC$ (from subsection (5.3.3)) while, in fact, $TFT$ can never invade a population of unconditional cooperators (subsection (5.3.2)). Fig. 5.11c-d depict projections of one such invariant curve onto subspaces of two strategies, together with the Hopf bifurcation route to periodic behavior (see Panels (a),(b), (f) for bifurcation diagrams of $AllD$ and $TFT$ fractions varying the benefit of cooperation $b$, mistake probability $\varepsilon$ and intensity of choice $\beta$ parameters, respectively). The limit cycles born at finite values of $\beta$ turn into a 3-cycle in the vanishing errors and "best-response" limits $\varepsilon \to 0, \beta \to \infty$, according to the best-reply structure of the underlying RSP game.
Figure 5.11: AllD vs. TFT vs. AllC. Bifurcation diagrams with respect to the benefit of cooperation parameter $b$ (Panel (a)), probability of mistake (Panel (b)) and intensity of choice (Panel (f)). Panels (c)-(d): phase plots in AllD-TFT and TFT-AllC subspaces, displaying Rock-Paper-Scissors type of long-run behaviour. Last, Panel (e) shows fractions evolution for large $\beta$. Baseline game parameters: $b = 2$, $c = 1$, $\varepsilon = 0.01$, $n = 0.3$, $\beta = 10$. 

(a) $\varepsilon = 0.01$. Bifurcation diagram

(b) $b = 2$. Bifurcation diagram

(c) $b = 1.4984$. Phase plot

(d) $b = 1.4984$. Phase plot

(e) $\beta = 100$. Time series

(f) $b = 2$. Bifurcation diagram
5.4.2 AllD-GTFT-WSLS

The 3x3 payoff matrix reads:

\[
\begin{bmatrix}
  AllD & GTFT & WSLs \\
  AllD & \varepsilon (b - c) & m_{13} & \frac{1}{2}b - c\varepsilon \\
  GTFT & \frac{n}{n+\varepsilon} (b - c) & m_{31} & m_{34} \\
  WSLs & b\varepsilon - \frac{1}{2}c & m_{43} & (b - c) (1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon)
\end{bmatrix}
\]  

(5.16)

With the help of the pairwise payoff comparisons\(^{15}\) performed in subsections (5.3.4), (5.3.5) and (5.3.7), given \(n > \varepsilon\) (by assumption), we can easily derive the best-response correspondences in game form (5.16) as follows: \(\{AllD\} = BR\{AllD\}\), \(\{GTFT\} = BR\{GTFT\}\) and \(\{WSLS\} = BR\{WSLS\}\). Thus, the game is a 3 \times 3 coordination game, and, in the best-reply limit of our logit dynamic (i.e. \(\beta \to \infty\)) each of these three pure strategy symmetric equilibria could be obtained as long-run outcome if the population starts out with the appropriate distribution of fractions.

See Fig.5.12f for a set of initial conditions from which the \((WSLS, WSLS)\) is selected. For \(\beta\) finite, the situation becomes more intricate as the Logit dynamics appear to break the typical coordination game best-response structure described above and generate Rock-Paper-Scissors patterns of cyclical oscillations irrespective of the initial mixture of the population (Fig.5.12a-d).

As the bifurcation diagram with respect to \(\beta\) (Fig.5.12e) suggests, a unique, fully mixed steady state for low values, destabilizes, for moderate values of the intensity of choice, via a Neimark-Sacker (NS) bifurcation, and RSP-like stable, limit cycles appear. For high values of \(\beta\) a sequence of saddle-node (SN) bifurcations occurs, conducing to three\(^{16}\) co-existing stable steady states, in the \(\beta \to \infty\) limit. This bifurcation scenario is confirmed in Fig. 5.13 where we continue a barycentrical fixed

\(^{15}\)i.e. the best-response functions for the corresponding 2 \times 2 games.

\(^{16}\)Panel (e) displays, for large \(\beta\) only two values of Pavlov fractions steady states, the top one with 100% share of WSLS and the bottom one with 0%. The third steady state supersedes the bottom curve, with zero WSLS fraction, too.
point \((1/3, 1/3, 1/3)\) in the benefit of cooperation/error in strategy execution space (top panels) and detect the conjectured Neimark-Sacker (NS)-Period-Doubling (PD)-Saddle-Node (LP) sequence of bifurcations along the equilibrium curve. The middle and bottom panels of Fig. 5.13 ‘continue’ each detected codimension I singularity with respect to another game or behavioral parameter and displays the resulting bifurcation curves. When a NS/PD/LP curve of codimension I bifurcations is crossed from below a limit cycle/2-cycle/co-existing stable and unstable steady states emerge, respectively. Also, it is worthwhile pointing out the rich selection of codimension II bifurcations occurring along the curves of codimension I bifurcations in this 3x3 ecology of repeated Prisoner’s Dilemma rules under the Logit Dynamics.

The intuition for the emergence, within a Coordination game, of the Rock-Paper-Scissors cycles "\(GTFT\) beats \(AllD\), \(WSLS\) beats \(GTFT\) and \(AllD\) beats \(WSLS\)" is not immediately clear. We know that in the \(\beta \to \infty\) limit the interaction is of Coordination game type with the three co-existing stable steady states created via a sequence of saddle-node bifurcations as shown before. Still, for finite values of the intensity of choice \(\beta\) (Fig.5.12e) the typical Coordination game (each strategy \(E_i\) is a best-reply to itself) behaves dynamically as if a RSP game (each strategy \(E_i\) is a best reply to its successor \(E_{i+1}\)). Nevertheless, even for finite \(\beta\) the Coordination game dynamic behavior can be recovered if the benefits accrued to cooperation \(b\) are high enough (Fig.5.12a).
Figure 5.12: AllD vs. GTFT vs. WSLS. Bifurcation diagrams with respect to the benefit of cooperation parameter $b$ (Panel (a)) and probability of mistakes (Panel (b)). Phase plots in GTFT-WSLS space (Panel (c)) and time series (Panel (d)), displaying Rock-Paper-Scissors type of long-run behaviour for moderate $\beta$. Bifurcation diagram with respect to $\beta$ and time series (Panel (f)) showing convergence, in the $\beta$ large limit to the all Pavlov steady state (for an appropriate initial population mixture). Baseline game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.1, \beta = 15$. 

137
Figure 5.13: AllD vs. GTFT vs. WSLS. Top panels: continuation of an equilibrium in the benefit of cooperation and error probability space. Middle and bottom panels: continuation of the Period-Doubling (PD), Saddle-Node (LP) and Neimark-Sacker (NS) singularities detected in the top two panels with respect to a second parameter: intensity of choice $\beta$, degree of generosity $n$, benefit of cooperation $b$ and error in implementation $\epsilon$.
Baseline game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.1, \beta = 15.$
5.4.3 AllD-GTFT-AllC

The restriction of the full payoff matrix (5.3) to this $3 \times 3$ sub-ecology, can be arranged, after the normalization of diagonal payoffs into the following form:

$$
\begin{bmatrix}
    AllD & GTFT & AllC \\
    AllD & 0 & -\frac{1}{n+\varepsilon} (n - \varepsilon^2 - n\varepsilon) (b - c - bn - b\varepsilon) & c(1 - 2\varepsilon) \\
    GTFT & -c(n - \varepsilon^2 - n\varepsilon) & 0 & c\varepsilon (1 - n - \varepsilon) \\
    AllC & -c(1 - 2\varepsilon) & \frac{\varepsilon}{n+\varepsilon} (1 - n - \varepsilon) (b - c - bn - b\varepsilon) & 0
\end{bmatrix}
$$

(5.17)

It is easy to see that, for $\varepsilon, n \to 0$ this normalized payoff matrix has a generalized Rock-Paper-Scissors structure, i.e. $AllD$ outcompetes $AllC$, $AllC$ outcompetes $GTFT$ and $GTFT$ outcompetes $AllD$. In this case the logit dynamics has a limit cycle as an attracting set for moderate $\beta$ and a three cycle with perpetual switching between the three monomorphic steady states in the $\beta$ large limit. For strictly positive perturbations $\varepsilon, n$ the game is not RSP per se but, nevertheless, the Logit Dynamics generates, for low to intermediate values of the intensity of choice, periodic behavior resembling the described RSP cycles. Reasoning for the emergence of such limit cycles in Fig. 5.14c-d goes along the best-reply correspondences argument already sketched in subsection (5.4.1). Unlike the $AllD - TFT - AllC$ ecology, the shape of the curve, as well as the oscillations bounds, are distorted, with the original ones recovered in the limit $n \to \varepsilon$ when $GTFT$ approaches the $TFT$ strategy. However, in the $\beta \to \infty$ limit system converges to a monomorphic $AllD$ state (Fig. 5.14e-f). The presence of undiscriminating cooperators leads to the extinction of the generous reciprocators.
Figure 5.14: AllD vs. GTFT vs. AllC. Bifurcation diagrams with respect to the benefit of cooperation parameter $b$ (Panel (a)) and degree of generosity $n$ (Panel (b)). Phase plots in AllC-GTFT space together with Rock-Paper-Scissors time series are reported in Panels (c) and (d), respectively. Panels (e),(f) display the long run behaviour in the $\beta$ large limit. Baseline game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.1, \beta = 15,$
### 5.4.4 AllD-TFT-WSLS

The payoff matrix for this sub-ecology takes the following form:

$$
\begin{bmatrix}
AllD & TFT & WSLS \\
AllD & 0 & (\frac{1}{2} - \varepsilon) (c - b + 2b\varepsilon) & (\varepsilon - \frac{1}{2}) (b - 2c - 4b\varepsilon + 4c\varepsilon + 4b\varepsilon^2 - 4c\varepsilon^2) \\
TFT & c\varepsilon (2\varepsilon - 1) & 0 & \frac{1}{2} (b - c) (2\varepsilon - 1)^3 \\
WSLS & c (\varepsilon - \frac{1}{2}) & 0 & 0
\end{bmatrix}
$$

which, in the limit $\varepsilon \to 0$, simplifies to:

$$
\begin{bmatrix}
AllD & TFT & WSLS \\
0 & -\frac{1}{2}(b - c) & c - \frac{1}{2}b \\
0 & 0 & -\frac{1}{2}(b - c) \\
-\frac{1}{2}c & 0 & 0
\end{bmatrix}
$$

For relatively small benefits of cooperation ($c < b \leq 2c$) we see that $AllD$ loses to $TFT$ ($(\frac{1}{2} - \varepsilon) (c - b + 2b\varepsilon) < 0$). Unconditional defectors $AllD$ win against $WSLS$ ($(\varepsilon - \frac{1}{2}) (b - 2c - 4b\varepsilon + 4c\varepsilon + 4b\varepsilon^2 - 4c\varepsilon^2) > 0$) and tie against own-type. Thus, this matrix has a degenerate generalized Rock-Paper-Scissors structure and limit cycles are born via a Hopf bifurcation, for low\(^{17}\) to moderate values of the intensity of choice parameter $\beta$ (Fig. 5.15a-d) with a 3-cycle arising in the $\beta \to \infty$ limit. For strictly positive mistake probability $\varepsilon$ there are still RPS cycles born, but only for moderate values of the intensity of choice, while, for large intensity of choice, the population ends up in an $AllD$ monomorphism (Panel(e)). The intuition for these oscillations could be extracted from results derived in subsections (5.3.1), (5.3.5) and (5.3.9): for small errors of implementation $TFT$ weakly dominates $AllD$, in turn, is it weakly dominated by $WSLS$ and, finally, $AllD$ easily invades a Pavlovian population when benefits accrued to cooperation are small (see Fig. 5.5a). For strictly positive $\varepsilon$ and

\(^{17}\text{If} \beta = 0, \text{the repeated strategy choice is random and the unique steady state is the fully mixed population state.}\)
\[ \beta \to \infty \] the population converges to an All\textit{D} monomorphic state.

If the relative benefits are large \((b > 2c)\), the RSP structure disappears for \(\varepsilon \to 0\), and, for \(\beta\) large the population converges, irrespective of the initial mixture, to a Pavlov-only state (Panel (f)). Notice that, even if the underlying game itself is no longer RSP, there are still oscillations for moderate \(\beta\) due to the evolutionary dynamics.
Figure 5.15: AllD vs. TFT vs. WSLS. Bifurcation diagrams with respect to the benefit of cooperation parameter $b$ (Panel (a)) and probability of mistakes (Panel (b)). Phase plots in AllD-WSLS (Panel (c)) and time series (Panel (d)) displaying Rock-Paper-Scissors type of long-run behaviour. Panels (e)-(f) show long-run behavior and convergence, for large $\beta$, to AllD (Pavlov) monomorphism for $b < 2c$ ($b > 2c$, respectively). Baseline game parameters: $b = 2, c = 1, \varepsilon = 0.01, n = 0.2, \beta = 15$. 

(a) $n = 0.2$. Bifurcation diagram, $b$

(b) $n = 0.2$. Bifurcation diagram, $\varepsilon$

(c) $b = 2.16$. Phase plot

(d) $b = 2.16$. Time series

(e) $b \leq 2\hat{c}$. Bifurcation diagram

(f) $b > 2\hat{c}$. Bifurcation diagram
5.4.5 AllD-TFT-GTFT

The reduced 3 x 3 payoff matrix for an ecology of defectors and (generous) reciprocators reads:

\[
\begin{bmatrix}
\text{AllD} & \text{TFT} & \text{GTFT} \\
\varepsilon (b - c) & -\varepsilon (c - 2b + 2\varepsilon) & b\varepsilon - c\varepsilon - b\varepsilon^2 + bn - bn\varepsilon \\
\varepsilon (b - 2c + 2\varepsilon) & \frac{1}{2}b - \frac{1}{2}c & -\varepsilon - b\varepsilon + b\varepsilon^2 - bn + bn\varepsilon - 2cn\varepsilon \\
b\varepsilon - c\varepsilon + c\varepsilon^2 - cn + cn\varepsilon & \frac{bn - cn - 2b\varepsilon + cn\varepsilon}{n + 3\varepsilon - 2n\varepsilon - 2\varepsilon^2} & \frac{n}{n + \varepsilon} (b - c)
\end{bmatrix}
\]

(5.19)

Using the inequalities from the relevant 2 x 2 sub-ecologies (subsections (5.3.1), (5.3.4) and (5.3.8)) we can show that, for \( b > \frac{c}{1-n} \), there are only two pure strategy equilibria (AllD, AllD) and (GTFT, GTFT). By adding the GTFT strategy to the coordination game AllD vs. TFT in (5.3.1), the steady state (TFT, TFT) is lost. However, TFT is not a dominated strategy and it plays an ‘intermediate layer’ role in ensuring transition from AllD to GTFT. Moreover, for small \( \varepsilon \), TFT breaks the path-dependence observed in the AllD vs. GTFT ecology (see subsection (5.3.4)) and, in the \( \beta \to \infty \) limit, (GTFT, GTFT) is the unique steady state (Fig. 5.16b,f). The intuition relies on the fact that there are only two possible transitions leading to an GTFT—only population: from TFT to GTFT (‘immediate’ in the \( \beta \) large case as TFT is strictly dominated strategy) and from AllD to GTFT (in two stages): in the first stage TFT (\textit{unlike} GTFT) invades and supersedes AllD for any initial fraction of defectors while in the second stage GTFT supersedes TFT according to the strict dominance criterion. Interestingly, for relatively small benefit of cooperation\(^{18}\), oscillatory co-existence of all three strategies (Fig. 5.16c-d) is observed for moderate \( \beta \), while for \( \beta \to \infty \) population converges to an AllD state.

---

\(^{18}\)Actually, the normalized game matrix in the \( \varepsilon \to 0 \) limit, is
\[
\begin{bmatrix}
0 & \frac{1}{2}c - \frac{1}{2}b & c - b + bn \\
0 & 0 & 0 \\
-cn & \frac{1}{2}b - \frac{1}{2}c & 0
\end{bmatrix},
\]
degenerate Rock-Paper-Scissors for \( c < b < \frac{c}{1-n} \).
Figure 5.16: AllD vs. TFT vs. GTFT. Panels (a)-(d): Evolution of strategies fractions for small and large benefit of cooperation coupled with low and high values of the sensitivity to payoff differences, respectively. Remaining (game) parameters set to: $c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $b = 3, \beta = 25$. Time series  
(b) $b = 3, \beta = 300$. Time series  
(c) $b = 1.2, \beta = 29.8$. Limit cycle  
(d) $b = 1.2, \beta = 300$. 5-cycle  
(e) $b < c/(1 - n)$. Bifurcation diagram  
(f) $b > c/(1 - n)$. Bifurcation diagram
5.4.6 AllD-WSLS-AllC

The reduced 3 × 3 payoff matrix for an ecology of defective, stimulus-response and cooperative players takes the following form:

\[
\begin{pmatrix}
\text{AllD} & \text{WSLS} & \text{AllC} \\
\text{AllD} & \varepsilon (b - c) & \frac{1}{2}b - c\varepsilon & b - b\varepsilon - c\varepsilon \\
\text{WSLS} & b\varepsilon - \frac{1}{2}c & (b - c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) & b - \frac{1}{2}c - b\varepsilon \\
\text{AllC} & b\varepsilon - c + c\varepsilon & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c)
\end{pmatrix}
\] (5.20)

We notice first, that AllC is strictly dominated by WSL provided that \(b < 2c\frac{1-\varepsilon}{1-2\varepsilon}\), the necessary and sufficient condition for Pavlov to be a best-reply to itself. In this case the equilibria structure is given by the equilibria of the AllD vs. WSLS game - (AllD, AllD) and (WSLS, WSLS) - in subsection (5.3.5). But, in the limit \(\varepsilon \rightarrow 0\), the ecology AllD−WSLS was dominance solvable for \(b < 2\varepsilon\) with WSLS the strictly dominated strategy. Combining the two restrictions, we obtain (AllD, AllD) as the only remaining Nash equilibrium in our 3x3 ecology for small benefits of cooperation \(b < 2c\).

If \(b > 2c\frac{1-\varepsilon}{1-2\varepsilon}\), then AllC is a best-reply to WSLS, but WSLS is not a best reply to AllC (as \(b - b\varepsilon - c\varepsilon > b - \frac{1}{2}c - b\varepsilon\)) so both the Coordination and the Hawk-Dove patterns of the AllD – WSLS and WSLS – AllC ecologies (see subsection (5.3.10) are broken. Again we are only left with (AllD, AllD) as the unique pure strategy equilibrium.(Fig. 5.17a). We observe the same phenomenon as with the AllD – GFTF – AllC ecology, i.e. the presence of undiscriminating players AllC leads to the extinction of the discriminating type Pavlov.

Interestingly, if we relax the \(\beta \rightarrow \infty\) assumption, the logit dynamics easily generates periodic (Fig. 5.17b-c) or chaotic behavior with irregular switching between the three strategies. Fig. (5.17d) shows a plot of the largest Lyapunov exponent \(\lambda\), with \(\lambda > 0\) implying chaos. In this case complicated dynamics arises through a period-doubling route to chaos.
Figure 5.17: AllD vs. WSLS vs. AllC. Time series evolution for small and high values of β (Panels (a)-(b)). Bifurcation diagram with respect to the benefit of cooperation $b$ (intensity of choice $\beta$) and evidence for chaotic behavior in Panels (c)-(d) and (e)-(f), respectively. Game parameters: $b = 4, c = 1, \varepsilon = 0.01, n = 0.3$. 
5.4.7 TFT-WSLS-AllC

The $3 \times 3$ payoff matrix for an ecology of reciprocators, Pavlovians and cooperators is given by:

$$
\begin{bmatrix}
TFT & WSLS & AllC \\
\frac{1}{2}b - \frac{1}{2}c & \frac{1}{2}b - \frac{1}{2}c & b - c - b\varepsilon + 2c\varepsilon - 2c\varepsilon^2 \\
\frac{1}{2}b - \frac{1}{2}c & (b - c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) & b - \frac{1}{2}c - b\varepsilon \\
b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c)
\end{bmatrix}
$$

(5.21)

By comparing entries in this payoff matrix, we notice the best-reply correspondences: $\{AllC\} = BR\{TFT\}, \{WSLS\} = BR\{AllC\}$. Also, $\{AllC\} = BR\{WSLS\}$ for $b > 2c$ and $\{WSLS\} = BR\{WSLS\}$ if $b < 2c$. Consequently, apart from mixed strategies equilibria, the pure strategy equilibria structure $\{(WSLS, AllC), (AllC, WSLS)\}$ from subsection (5.3.10) is preserved for large benefit of cooperation ($b > 2c$) while the one from subsection (5.3.9) (i.e. Coordination Game, with $WSLS$ weakly dominating $TFT$), is preserved for relatively small $b$. Moreover, $TFT$ remains weakly dominated by $WSLS$ even in this enlarged ecology. These two situations are illustrated in Fig. (5.18): a Hawk-Dove stable 2-cycle for large $b$ (see Panels (a) and (b) for the unboundedly and boundedly rational cases) and a Coordination structure with a stable steady state for small $b$ (Panels (c) and (d) for large and small $\beta$, respectively).
Figure 5.18: TFT vs. WSLS vs. AllC. Depending on the size of the cooperation benefit two situations may arise: a Hawk-Dove pattern with complete population swings from one monomorphic state to the other (Panel (b)) or a Coordination Game with most of the initial conditions leading to a monomorphic Pavlov population (Panel (d)). Panels (a) and (c) display the boundedly rational (i.e. small intensities of choice values) versions of the two benchmarks. Remaining game parameters: $c = 1, \varepsilon = 0.01, n = 0.3.$
5.4.8 TFT-GTFT-WSLS

The payoff matrix for this 3 × 3 ecology of behavioral rules reads:

\[
\begin{bmatrix}
TFT & GTFT & WSLS \\
\frac{1}{2}b - \frac{1}{2}c & -\frac{c - b + \frac{b^2}{n+3\varepsilon} + \frac{cn + bn - 2cn\varepsilon}{n+3\varepsilon}}{n+3\varepsilon} & \frac{1}{2}b - \frac{1}{2}c \\
\frac{b - c + \frac{b^2}{n+3\varepsilon} + \frac{bn + cn - 2bn\varepsilon + c\varepsilon}{n+3\varepsilon}}{n+3\varepsilon} & \frac{n}{n+\varepsilon} (b - c) & m_{34} \\
\frac{1}{2}b - \frac{1}{2}c & m_{43} & (b - c) \left(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon\right)
\end{bmatrix}
\]

(5.22)

In subsection (5.3.8) it was shown that GTFT strictly dominates TFT while subsection (5.3.9) showed that WSLS and GTFT are best responses to themselves, in the context of the corresponding 22 ecologies. Building on these results and investigating the relevant additional inequalities\(^{19}\) in the above matrix, we can derive the best-reply correspondences for the integrated 3×3 ecology as follows: \{GTFT\} = BR{TFT}, \{GTFT\} = BR{GTFT} and \{WSLS\} = BR{WSLS}. The resulting symmetric pure strategies equilibria are: \{GTFT, GTFT\} and \{WSLS, WSLS\}. Fig. 5.19a-b displays, in the β large case, this path-dependence and the convergence to one of these two equilibria as we start with different population mixtures. For imperfect switching to the best-reply, i.e. for lower values of β, oscillatory dynamics may arise as well (Fig. 5.19a-b).

\(^{19}\) \[\frac{n}{n+\varepsilon} (b - c) > m_{43} \quad \text{and} \quad \frac{n}{n+\varepsilon} (b - c) > -\frac{c - b + \frac{b^2}{n+3\varepsilon} + \frac{bn + cn - 2cn\varepsilon}{n+3\varepsilon}}{n+3\varepsilon} \quad \text{and} \quad (b - c) \left(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon\right) > m_{34} \quad \text{from subsection (5.3.7)}\]
Figure 5.19: TFT vs. GTFT vs. WSLS. Dependence on the population initial mixture for rational choice (Panels (a)-(b)) and oscillatory dynamics for low intensity of choice (Panel (c)). Remaining game parameters: $b = 4, c = 1, \varepsilon = 0.01, n = 0.3$.  

(a) Time series, $b = 4$, $\beta = 100$, initial distribution: TFT-70%, GTFT-10%, WSLS-20%  
(b) Time series, $b = 4$, $\beta = 100$, initial distribution: TFT-10%, GTFT-30%, WSLS-60%  
(c) Time series, $b = 4$, $\beta = 15$, initial distribution: TFT-1/3, GTFT-1/3, WSLS-1/3  
(d) Time series, $b = 4$, $\beta = 15$, initial distribution: TFT-10%, GTFT-20%, WSLS-70%
5.4.9 Summary

Table 5.2 summarizes the qualitative long-run behavior of logit dynamics on various $3 \times 3$ sub-ecologies of PD repeated strategies. There are two basic routes to periodic behavior: the first one is the limiting case $\varepsilon \to 0$ when five out of the ten normal form games unveil a (degenerate) generalized Rock-Paper-Scissors structure. The Logit Dynamics on this generalized RSP game gives rise to stable limit cycles created via a supercritical Neimark-Sacker (Hopf) bifurcation (see also Chapter 2) with the three monomorphisms connected via a 3-cycle in the $\beta \to \infty$ best-reply limit. In the second route to oscillating behavior, when we maintain the strictly positive error parameter $\varepsilon$, the game forms are not of RPS type, but, nevertheless, for moderate values of the intensity of choice stable oscillations created via a supercritical NS bifurcation are observed. However, for such non-RPS games, in the limiting case $\beta \to \infty$, the cyclical structure disappears with one of the strict equilibria selected as the long run outcome.

A somewhat different scenario occurs for the $AllD - GTFT - WSLS$ $3 \times 3$ coordination game, where a sequence of Neimark-Sacker and Fold bifurcations unfolds as the intensity of choice slides from low to high values: the long-run behavior varies from a unique polymorphic steady state for low $\beta$, through limit cycles to three co-existing steady states corresponding to the underlying game Nash equilibria in the best-reply limit.

In the transition to the $AllC$ ecologies, in the second half of the table, we have found one $3 \times 3$ example of chaotic dynamics created through a period-doubling route to chaos. Chaos appeared for finite intensity of choice $\beta$, in the $AllD - Pavlov - AllC$ ecology, whose underlying payoff matrix has a non-RSP structure even for the limiting case $\varepsilon \to 0$. The typical RSP cyclical dominance structures seem to preclude the onset of chaos.

In the last four ecologies in Table 5.2 with no defectors ($AllD$), the long-run, $\beta \to \infty$ outcomes consist, generally, of simple behavior, steady states or 2-cycle with
continuous population swings between one monomorphic state to another, with the third strategy extinct from the population.

<table>
<thead>
<tr>
<th>No.</th>
<th>3×3</th>
<th>Bifurcation</th>
<th>Attractors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Moderate β</td>
<td>β → ∞</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>AllD-TFT-AllC</td>
<td>NS</td>
<td>limit cycle</td>
</tr>
<tr>
<td>2</td>
<td>AllD-GTFT-WSLS</td>
<td>NS,LP</td>
<td>limit cycle</td>
</tr>
<tr>
<td>3</td>
<td>AllD-GTFT-AllC</td>
<td>NS</td>
<td>limit cycle</td>
</tr>
<tr>
<td>4</td>
<td>AllD-TFT-WSLS</td>
<td>NS</td>
<td>limit cycle</td>
</tr>
<tr>
<td>5</td>
<td>AllD-TFT-GTFT</td>
<td>NS</td>
<td>limit cycle</td>
</tr>
<tr>
<td>6</td>
<td>AllD-WSLS-AllC</td>
<td>PD</td>
<td>2-cycle, chaos</td>
</tr>
<tr>
<td>7</td>
<td>TFT-WSLS-AllC</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>8</td>
<td>TFT-GTFT-WSLS</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>9</td>
<td>GTFT-WSLS-AllC</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
<tr>
<td>10</td>
<td>TFT-GTFT-AllC</td>
<td>PD</td>
<td>2-cycle</td>
</tr>
</tbody>
</table>

Table 5.2: Summary of the dynamical behaviour for all 10 3x3 interaction of iterated Prisoner’s Dilemma strategies under logit dynamics.

5.5 4×4 ecologies

Before studying the full 5×5 ecology, in this section we discuss the long-run behavior of the five 4-types interaction under the Logit Dynamics.

5.5.1 No TFT

This ecology appends the AllD – GTFT – WSLs subset (subsection (5.4.2)) with unconditional cooperators. In the β large limit, AllC destroys the Coordination game structure of AllD – GTFT – WSLs and the population converges to an AllD monomorphism, irrespective of the initial conditions (Fig. 5.20a). This is due, on the one
hand, to the Hawk-Dove nature of the $WSLS - AllC$ and $GTFT - AllC$ interactions—which breaks the best-response correspondences in (5.4.2) $\{WSLS\} = BR\{WSLS\}$ and $GTFT = BR\{GTFT\}$ - and, on the other hand, to the unconditional defectors $AllD$ "easiness" of invading undiscriminating cooperators $AllC$ relative to $WSLS$ and $GTFT$. Briefly, adding "cooperative" players leads to the extinction of the two discriminating types ($GTFT$ and $WSLS$) and, consequently, the overall level of long-run cooperation drops. However, for moderate $\beta$, Pavlov seems to be favoured by the evolutionary dynamics\(^\text{20}\) (see Panel (e) for the evolution of an initially balanced population) and rich dynamics of the $WSLS$ fraction unfold if either the mistake probability ($\varepsilon$) or the intensity of choice is altered (Fig. 5.20b-c). Co-existence of attractors - one chaotic (Panel (d) plots the largest Lyapunov exponent as numerical evidence for chaos) and an $AllD$ monomorphic steady state - is detected for intermediate values of $\beta$ while the entire population becomes $AllD$ for large values of the intensity of choice. Panels (e)-(f) show plots of fractions evolution on the two attractors: a fully mixed population converges to the chaotic attractor (Panel (e)) while a population with no reciprocators (Panel (f)) ends up in a defectors $AllD$ only state. This also illustrates the critical role $TFT$ players have in the emergence and success of Pavlovian ($WSLS$) types.

\(^{20}\)Also, from subsection (5.3.7) ($WSLS; WSLS$) equilibrium is more likely to emerge because of a larger basin of attraction compared to ($GTFT; GTFT$); still this dependence of initial distribution (even though basins are largely asymmetric) seems to be lost once we investigate the enlarged, no-$TFT$ ecology.
Figure 5.20: AllD vs. GTFT vs. WSLS vs. AllC. Evolution towards an AllD population for $\beta$ large limit. Panels (b)-(c) plot bifurcation diagrams with respect to the error in strategy implementation $\varepsilon$ and intensity of choice $\beta$ parameters, while numerical evidence, i.e. positive Lyapunov exponent, for chaos in Panel (d). Last, evolution of fractions on the two co-existing attractors, for the boundedly rational choice, is displayed in Panels (e) and (f). Baseline game parameters: $b = 3, c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $\beta = 100$. Time series
(b) $\beta = 22$. Bifurcation diagram
(c) $b = 3$. Bifurcation diagram
(d) Largest Lyapunov exponent
(e) $\beta = 22$. Initial mix (1/4,1/4,1/4,1/4)
(f) $\beta = 22$. Initial mix (0.5,0,0.1,0.4)
5.5.2 No AllD

In the absence of hard defectors AllD, the system converges to a 2-cycle where the population displays either full swings back and forth between a Pavlovian state and an AllC state (for large $\beta$), or bounded fluctuations with Pavlov outnumbering AllC (moderate values of the intensity of choice) (Fig. 5.21a-b). The intuition for the 2-cycle as unique long-run attractors goes along the following lines: First, both GTFT and WSLS dominate TFT("strictly" or "weakly", respectively) and so TFT is not to be expected in the long run of logit dynamics. Second, each of the GTFT and WSLS generates a Hawk-Dove pattern (i.e. 2-cycle) in an ecology with AllC. Then, which of the two 2-cycles (GTFT – AllC or WSLS – AllC) will eventually emerge, depends on the evolutionary competition between GTFT, WSLS and AllC. The game matrix restricted to this ecology reads:

$$
\begin{bmatrix}
  GTFT & WSL & AllC \\
  \frac{n}{n+\varepsilon} (b - c) & m_{34} & b - c - b\varepsilon + 2c\varepsilon - c\varepsilon^2 - cn\varepsilon \\
  m_{43} & (b - c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) & b - \frac{1}{2}c - b\varepsilon \\
  b - c - 2b\varepsilon + c\varepsilon + b\varepsilon^2 + bn\varepsilon & b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 & (1 - \varepsilon)(b - c) \\
\end{bmatrix}
$$

The following inequalities are helpful for constructing the best-response correspondences:

$$
b - c - 2b\varepsilon + c\varepsilon + b\varepsilon^2 + bn\varepsilon > \frac{n}{n+\varepsilon} (b - c) , \text{ if } b > \frac{c}{1 - n - \varepsilon}
$$

$$
b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2 > (b - c)(1 - 4\varepsilon^3 + 6\varepsilon^2 - 3\varepsilon) , \text{ if } b > 2c\frac{1 - \varepsilon}{1 - 2\varepsilon}
$$

We see that for large enough benefits of cooperation, $b > 2c\frac{1 - \varepsilon}{1 - 2\varepsilon}$, adding unconditional cooperators AllC, breaks the initial Coordination game structure of the GTFT – WSLS ecology and there are only two asymmetric pure strategy equilibria left, namely (WSLS, AllC) and (AllC, WSLS). There are never co-existing
two 2-cycles in this ecology. The $WSLS - AllC$ 2-cycle depicted in Fig. 5.21a-b for the "boundedly rational" and the "best-reply" repeated PD strategy choices, respectively.

For $\frac{c}{1-n-\varepsilon} < b < 2c\frac{1-\varepsilon}{1-2\varepsilon}$ we have $AllC = BR\{GFTT\}, WSLS = BR\{WSLS\}, WSLS = BR\{AllC\}$ and the population converges, for $\beta \to \infty$, to a Pavlovian $WSLS$ monomorphism.

Last, for $c < b < \frac{c}{1-n-\varepsilon}$, we have $WSLS = BR\{GFTT\}$ so $AllC$ is a strictly dominated strategy. We are back in the $GFTT - WSLS$ ecology where Pavlov strictly dominates $GFTT$ and, consequently, is the unique monomorphic long-run outcome of Logit Dynamics (Panels (c), (d) for high and low values of $\beta$ respectively). We can conclude from this analysis that co-existence of two 2-cycles does not occur.
Figure 5.21: TFT vs. GTFT vs. WSLS vs. AllC. Panels (a)-(b): for large benefits of cooperation, $b = 4$, fractions converge to a (WSLS, AllC) 2-cycle for both high and low values of the intensity of choice. Panels (c)-(d): for low benefit of cooperation, $b = 1.1$, system converges to a monomorphic Pavlovian state. Remaining game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $b = 4, \beta = 15$. Time series 
(b) $b = 4, \beta = 200$. Time series 
(c) $b = 1.1, \beta = 600$. Time series 
(d) $b = 1.1, \beta = 100$. Time series
5.5.3 No GTFT

Unlike the previous case, the 4×4 ecology without $GTFT^{21}$ exhibits path-dependence. Hence, for moderate values of the intensity of choice $\beta \approx 27$, the system displays co-existence of attractors. Depending on the initial population mix, the long-run evolutionary outcome can be either the cyclical competition between $AllD, TFT$ and $AllC$ with asymmetric amplitudes (time series Fig. 5.22b) as in subsection (5.4.1), or a 1-piece chaotic attractor (time series in Fig. 5.22c) with Pavlov having a dominant position, as in, for instance, subsection (5.4.6). In both cases, the other strategies remain present in the population, but in small numbers.

For high values of $\beta$ the path-dependence is lost, the only remaining attractor being a full-ranged Rock-Paper-Scissors 3-cycle (Panel (d)) in $AllD, TFT$ and $AllC$, with stimulus-response players $WSLS$ going extinct. The bifurcation diagram in Panel (a) organizes the long-run outcomes of this ecology and displays the two co-existing attractors: the chaotic one in the top and the cyclical one in the bottom curves.

---

$^{21}$Imhof et al. (2007) also discuss, in a finite population framework, an ecology of $AllD, TFT, WSLS$ and $AllC$ behaviors, under the frequency-dependent, mutation-selection Moran process. They show that $AllD$ is selected for stage game payoffs (i.e. $b/c$ ratio) below a certain threshold, while $WSLS$ wins the evolutionary competition otherwise.
Figure 5.22: AllD vs. TFT vs. WSLS vs. AllC. Panel (a) bifurcation diagram of Pavlov fractions with respect to the intensity of choice parameter; Panels (b)-(c): depending on initial distribution of fractions the long-run population state may be either ongoing oscillations among AllD, TFT, AllC(Panel (b)) or co-existence, although in a possibly chaotic fashion, of WSLS and AllC players with WSLS largely outnumbering unconditional cooperators (Panel (c)). Last, Panel (d) displays the large $\beta$ limit of the long-run dynamics of fractions. Remaining game parameters: $b = 4, c = 1, \varepsilon = 0.01, n = 0.3$. 
5.5.4 No WSLS

With the stimulus-response WSLS rule put aside, the evolutionary dynamics becomes very complicated. In the $\beta$ large limit, co-existence of the four strategies in a cyclical fashion, is observed for high benefits of cooperation $b$ (Fig. 5.23a) while, for low benefits $b$, GTFT is driven extinct and full-scale cyclical dynamics with the remaining three strategies emerge (Panel (b)).

For finite $\beta$, a stable limit cycle is born via a Neimark-Sacker bifurcation (Fig. 5.23a) for small $b$, while a period-doubling route to chaos emerges for high benefits accrued to cooperation (Panel (d)). Finally, in Fig. 5.23e a period-doubling route to chaos is depicted (positive Lyapunov exponent plot in Panel (f)) as the generosity parameter $n$ built into the GTFT strategy is fine-tuned.
Figure 5.23: AllD vs. TFT vs. GTFT vs. AllC. Time series of the corresponding fractions, for large $\beta$, and large/small benefit of cooperation (Panel (a) and (b), respectively). Bifurcations diagrams for high and low $b$ are also shown in Panel (c)-(d). Last, Panel (e) displays a bifurcation diagram of GTFT fraction with respect to the "generosity" parameter $n$ while Panel (f) shows some numerical evidence for chaos. Remaining game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $b = 2$, large $\beta = 100$. Time series

(b) $b = 6$, large $\beta = 100$. Time series

(c) $b = 2$. Bifurcation diagram $\beta$

(d) $b = 6$. Bifurcation diagram $\beta$

(e) $b = 6, \beta = 15$. Bifurcation diagram $n$

(f) $\beta = 15$. Largest Lyapunov exponent
5.5.5 No AllC

There are two cases to be distinguished for an AllD – TFT – GTFT – WSLS ecology. These two situations emerge from the discussion of the corresponding $3 \times 3$ sub-ecologies (see subsections (5.4.2), (5.4.4) and (5.4.5), namely low ($b \leq 2c$) and high benefits of cooperation ($b > 2c$).

For low $b \leq 2c$ and in the $\beta \to \infty$ limit, the system converges to a unique monomorphic GTFT state (Fig. 5.24a). However, all four behavioral rules co-exist, either in the cyclical (Panel (a) or even chaotic (Panels (b)-(d)) manner, for intermediate values of the intensity of choice.

As the responsiveness to payoffs differentials increases, quasi-periodic behavior emerges through a Neimark-Sacker bifurcation, which, if $\beta$ is pushed even further, breaks into a chaotic attractor (see Fig. 5.24b for a plot of the positive largest Lyapunov exponent). The strange attractor depicted in Panel (d) emerges via one such Neimark-Sacker, "breaking of the invariant circle" route to chaos.

When benefits of cooperation are relative large ($b > 2c$) there is path-dependence producing the co-existence of two monomorphisms GTFT and WSLS, respectively. Panel (e) shows, for $b = 4$, the bifurcation diagram of GTFT fraction with respect to $\beta$ and we can see the two steady states created for large responsiveness to payoff differences(at the top and bottom of the panel). For instance, an initial fractions distribution biased towards Pavlov converges to a Pavlov-only state (Panel (f)). Alternatively, an even set of initial conditions favors the GTFT as the long-run, good-fated type of behavior.
Figure 5.24: AllD vs. TFT vs. GTFT vs. WSLS. Panels (a): bifurcation diagram with respect to the intensity of choice showing quasi-periodic behavior. Panels (c)-(d): time series, phase portrait and Lyapunov exponent plot as numerical evidence for chaotic time series when the intensity of choice $\beta$ increases. Finally, Panels (e)-(f) show the long-run behavior, in the best-reply limit, for large benefits of cooperation: this is, co-existence of monomorphic GTFT and Pavlov WSLS states. Baseline game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$.
5.5.6 Summary

Table 5.3 summarizes the long-run behavior of all $4 \times 4$ ecologies both the best-reply ($\beta$ large) and in the boundedly rational ($\beta$ small) choice cases. In comparison to the $3 \times 3$ ecologies reviewed in the previous section, having more strategies not only enlarges the set of possible outcomes but also makes the behavior much more complicated. In particular there is now strong path-dependence in most of the ecologies, leading to the co-existence of complicated attractors, periodic and chaotic ones. We identified two routes to complex dynamics in these $4 \times 4$ ecologies: the period-doubling and breaking of the invariant circle route to chaos. However, the path dependence is lost for some ecologies (subsections (5.5.1), (5.5.2), (5.5.3)) when $\beta$ approaches the "full rationality" limit. $WSLS$ fares well in an $AllD$ environment with no $AllC$ players, but poorly, almost going extinct, in an $AllC$ environment with $AllD$ strategists around.

<table>
<thead>
<tr>
<th>4x4</th>
<th>Bifurcations</th>
<th>small $\beta$</th>
<th>large $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No AllD</td>
<td>PD</td>
<td>2-cycle</td>
<td>2-cycle or unique SS</td>
</tr>
<tr>
<td>No TFT</td>
<td>PD, NS</td>
<td>co-existence SS and chaos</td>
<td>unique SS (AllD)</td>
</tr>
<tr>
<td>No GTFT</td>
<td>PD, NS</td>
<td>co-existence RSP and chaos</td>
<td>unique SS (AllD)</td>
</tr>
<tr>
<td>No WSLS</td>
<td>NS</td>
<td>limit cycles, chaos</td>
<td>4-cycle</td>
</tr>
<tr>
<td>No AllC</td>
<td>NS, PD</td>
<td>limit cycles, chaos</td>
<td>multiple SS (GTFT, WSLS)</td>
</tr>
</tbody>
</table>

Table 5.3: Summary of the dynamical behaviour for all $4x4$ interaction of iterated Prisoner’s Dilemma strategies under logit dynamics.
5.6 5×5 Ecology

In this section, we report simulations about the behavior of Logit Dynamics on the full 5×5 ecology AllD – TFT – GTFT – WSLS – AllC that generated the game matrix (5.3). Depending on the relative size of cooperation benefits we distinguish again two situations.

First, for small benefits, \( b \leq 2c \) there is co-existence of all five strategies, albeit in a fluctuating manner, for low responsiveness to payoffs differences (Fig. 5.25a). When \( \beta \) reaches the best-response limit the long-run outcome is an AllD-TFT-AllC 3-cycle (Fig. 5.25b).

If the benefits of cooperation are large, Pavlov wins the evolutionary competition for small \( \beta \) (Panel (c)) but goes extinct in the \( \beta \to \infty \) limit, with a 4-cycle in the remaining strategies as long run behavior (Panel (d)). Last, Panels (e), (f) display bifurcation diagrams of AllC and Pavlov fractions respectively with respect to the intensity of choice. It is apparent from the AllC bifurcation diagram in Panel (e) that an initially stable steady state for very low \( \beta \), destabilizes for moderate values, with continuous swings between high and low fractions emerging for large \( \beta \). As far as Pavlov is concerned the corresponding bifurcation diagram shows its way to extinction when players choose a best-reply (Fig. 5.25f).

The chaotic patterns in the two diagrams for intermediate values of \( \beta \) are confirmed in Fig. 5.26a-f that displays two dimensional projections of a chaotic attractor together with the fractions evolution on the attractor for \( b = 2.4805 \) and \( \beta = 15 \). Finally, Fig. 5.27c-d illustrates the "breaking of an invariant circle" route to chaos: a quasiperiodic attractor for \( \beta = 0.05 \) breaks into a 6-piece quasi-periodic and then chaotic attractor (Panels (d)-(e)) at \( \beta \approx 10.05 \) and \( \beta = 10.2 \). Eventually these pieces join together to form a 1-piece strange attractor (Fig. 5.27f) when the intensity of choice \( \beta \) reaches the threshold \( \beta = 10.8 \).
Figure 5.25: AllD vs. TFT vs. GTFT vs. WSLS vs. AllC. Panels (a)-(b) show the long-run outcome for low benefits of cooperation $b$ in the $\beta$ low and high limit, respectively. Similar time series are displayed in Panels (c)-(d) for the high benefit $b$ case. Panels (e)-(f) show, for high $b$, bifurcation diagrams of AllC and Pavlov fractions. Baseline game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$. 
Figure 5.26: AllD vs. TFT vs. GTFT vs. WSLS vs. AllC. Panels (a)-(d): two-dimensional projections of a strange attractor for $b = 2.4805$ and $\beta = 15$. Panels (e)-(f) evolution of fractions on the strange attractor. Remaining game parameters: $c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $b = 2.48, \beta = 15$. Strange attractor
(b) $b = 2.48, \beta = 15$. Strange attractor
(c) $b = 2.48, \beta = 15$. Strange attractor
(d) $b = 2.48, \beta = 15$. Strange attractor
(e) $b = 2.48, \beta = 15$. Time series
(f) $b = 2.48, \beta = 15$. Time series
Figure 5.27: AllD vs. TFT vs. GTFT vs. WSLS vs. AllC. Panels (a)-(b): bifurcation diagram with respect to intensity of choice $\beta$ and numerical evidence of chaos (plot of the largest Lyapunov exponent). Panels (c)-(f) show the evolution of a quasiperiodic attractor into a strange attractor as the intensity of choice increases. Remaining game parameters: $b = 2.48, c = 1, \varepsilon = 0.01, n = 0.3$. 

(a) $b = 2.48$. Bifurcation diagram $\beta$  
(b) $b = 2.48$. Lyapunov exponent  
(c) $\beta = 9.05$. 1-piece attractor  
(d) $\beta = 10.05$. 6-piece quasiperiodic  
(e) $\beta = 10.2$. 6-piece chaotic  
(f) $\beta = 10.8$. 1-piece chaotic
5.6.1 Numerical Bifurcation Curves

The visual analysis of the bifurcation diagram in Fig. 5.25 can be confirmed by a rigorous continuation procedure, i.e. the computation of curves of equilibria along with their detected codimension I bifurcations as one parameter is varied using the Cl_Matcont bifurcation software package\textsuperscript{22} (Dhooge et al. (2003)). One such fixed point - (0.4972065; 0.4178546; 0.06911757; 0.01521537; 0.01521537) for the game parameterization $b = 1.1; c = 1; \varepsilon = 0.01; n = 0.3; \beta = 14$ - is continued in Fig. 5.28 top left panel, as the benefit of cooperation $b$ is increased. Three singularities are detected by the continuation package in the meaningful region $[b > c = 1]$ of $b$ : two Neimark-Sacker (NS) points at $b = 1.270046$ and $b = 2.636476$\textsuperscript{23} and one Period-Doubling (PD) point at $b = 3.158022$. These two points are next 'continued' with respect to another parameter as depicted with codimension II singularities\textsuperscript{24} detected along them. The bifurcation curves describe in a systematic way the qualitative changes in the behavior of the dynamical system in a certain parameters space. For instance, in the two bottom panels in Fig. 5.28 the plotted curves of Neimark-Sacker singularities partition the parameters space in regions with qualitatively similar behavior: when crossing a NS curve from below the system loses stability of the steady state and a stable limit cycles arises around the unstable steady state. Such an analysis reveals that the behavior envisaged by simulations is robust to perturbations of the repeated Prisoner’s Dilemma game (as parametrized by the $b/c$ ratio), the probability of mistake in implementing a particular strategy $\varepsilon$, and the generosity parameter $n$. Note that, compared to the $3 \times 3$ ecologies, there is no fold bifurcation as coordination on equilibria is more difficult to achieve with more strategies. Instead, increasing the strategy space generates oscillatory behavior more easily.

\textsuperscript{22}Cl_Matcont for maps, see package documentation at http://www.matcont.ugent.be/

\textsuperscript{23}normal form coefficients are -2.741700 and -5.440686 meaning that the NS bifurcations are supercritical, i.e. the limit cycles are born stable.

\textsuperscript{24}e.g. a Chenciner CH codim II bifurcation gives rise to much more complicated patterns of behavior (see Kuznetsoy (1995) pp. 482).
Figure 5.28: AllD vs. TFT vs. GTFT vs. WSLS vs. AllC. Top left Panel continues a numerically computed fixed point, in the benefit of cooperation space. Detected codimension I singularities are then continued with respect to another game/behavioral parameter. The resulting curves of Period-Doubling (PD) and Neimark-Sacker (NS) points, with respect to the intensity of choice $\beta$, probability of error in strategy execution $\varepsilon$, and degree of generosity $n$, are plotted in the top-right, middle and bottom panels, respectively. The occurrence of codimension II singularities can be observed along all these curves. Unless free to float, game parameters set to: $b=2, c=1, \varepsilon=0.01, n=0.3, \beta=15$
5.7 Conclusions

In an evolutionary set-up, we append an ecology of iterated Prisoner’s Dilemma (IPD) game strategies, consisting of unconditional cooperators (AllC), unconditional defectors (AllD) and reactive players (TFT) with two repeated strategies that have received less attention in the evolutionary IPD game literature, the error-proof, "generous" tit-for-tat (GTFT) which, with a certain probability, re-establishes cooperation after a (possibly by mistake) defection of the opponent and the penitent, "stimulus-response" (WSLS) strategy that resets cooperation after the opponent punished for defection. Stable oscillations in the frequency of both the forgiving (GTFT) and repentant (Pavlov) strategy along with chaotic behavior emerge under a perturbed version of best-response dynamics, the logit dynamics. We have performed a detailed analysis of the relatively simple $2 \times 2$ and $3 \times 3$ case with the dynamic behaviour now fairly well understood. However, the enlarged $4 \times 4$ and $5 \times 5$ exhibit complicated behavior via period-doubling and breaking of an invariant circle routes to chaos. Finite intensity of choice $\beta$ leads to rich dynamics, path-dependence and co-existence of cyclical and chaotic attractors for a wide selection of $4 \times 4$ ecologies. If we turn to the best-reply limit of the logit dynamics, various $3-$and $4-$cycles are exhibited in specific $4 \times 4$ and in the $5 \times 5$ ecologies. Last, in terms of individual strategies performance, we discovered first that the presence of unconditional cooperators turns out detrimental to the discriminating types (TFT, GTFT and WSLS) in some $4 \times 4$ interactions leading the population to an AllD monomorphism. Second, there is mixed evidence for a Pavlovian meta-rule in an evolutionary repeated PD with a population of rational players: on the one hand, the stimulus-response strategy does very well and wins the evolutionary competition in a $4 \times 4$ environment with hard defectors (AllD) but no undiscriminating cooperators (AllC), but it almost goes extinct in the full $5 \times 5$ repeated game. Nevertheless, with boundedly rational players, the fractions of Pavlov maintain high values even within the complete ecology of five rules.
5.A Iterated PD Game-stationary distributions and average payoffs

For each pairwise interaction of the iterated PD strategies $S_i \times S_j$, the stationary distribution $\Omega = \{\tau_{CC}, \tau_{CD}, \tau_{DC}, \tau_{DD}\}$ gives the average fraction of time system spends in each of the four states $CC, CD, DC, DD$, respectively. Average expected payoffs are computed as weighted average of the stage game matrix payoffs with weights obtained from the stationary distribution:

$$\pi_i(S_i, S_j) = \begin{bmatrix} b - c \\ -c \\ b \\ 0 \end{bmatrix}^T \begin{bmatrix} \tau_{CC} \\ \tau_{CD} \\ \tau_{DC} \\ \tau_{DD} \end{bmatrix}$$

$i, j \in \{AllD, TFT, GTFT, WSLS, AllC\}$

Using the transition probability matrix (5.2) we can derive the stationary distribution together with the respective average expected payoffs as summarized in the following table (some expression $m_{ij}$ are too long and given at the end of the table):
<table>
<thead>
<tr>
<th>Interaction</th>
<th>Stationary Distribution</th>
<th>Average Payoff$^{25}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AllD vs. AllD</td>
<td>$\varepsilon^2$</td>
<td>$m_{11} = \varepsilon(b-c)$</td>
</tr>
<tr>
<td></td>
<td>$-\varepsilon(\varepsilon - 1)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\varepsilon(\varepsilon - 1)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\varepsilon - 1)^2$</td>
<td></td>
</tr>
<tr>
<td>AllD vs. TFT</td>
<td>$-2\varepsilon^2(\varepsilon - 1)$</td>
<td>$m_{12} = -\varepsilon(c - 2b + 2b\varepsilon)$</td>
</tr>
<tr>
<td></td>
<td>$2\varepsilon^3 - 2\varepsilon^2 + \varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2\varepsilon(\varepsilon - 1)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-2\varepsilon^3 + 4\varepsilon^2 - 3\varepsilon + 1$</td>
<td></td>
</tr>
<tr>
<td>AllD vs. GFT</td>
<td>$\frac{\varepsilon(1-\varepsilon)(n+\varepsilon)}{n-m+m\varepsilon-n\varepsilon+1}$</td>
<td>$m_{13}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\varepsilon mc - \varepsilon - m + \varepsilon^2 + 1}{n-m+m\varepsilon-n\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{(\varepsilon-1)^2(n+\varepsilon)}{n-m+m\varepsilon-n\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{(1-\varepsilon)(mc - \varepsilon - m + \varepsilon^2 + 1)}{n-m+m\varepsilon-n\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td>AllD vs. WSLS</td>
<td>$\frac{1}{2}\varepsilon$</td>
<td>$m_{14} = \frac{1}{2}b - c\varepsilon$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} - \frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} - \frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td>AllD vs. AHC</td>
<td>$-\varepsilon(\varepsilon - 1)$</td>
<td>$m_{15} = b - b\varepsilon - c\varepsilon$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\varepsilon - 1)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\varepsilon(\varepsilon - 1)$</td>
<td></td>
</tr>
</tbody>
</table>

$^{25}$Stationary distributions are valid for the general formulation of $GTFT - (1, m, 1, n)$. However, expressions for the average payoffs become very complicated and are therefore computed under the restriction $m = n$ in the general form of $GTFT$, with the impact of the asymmetry ($m \neq n$) in the two probabilities of restoring cooperation (i.e. after a $CD$ or $DD$ history) left for further research.
<table>
<thead>
<tr>
<th>Interaction</th>
<th>Stationary Distribution</th>
<th>Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>TFT vs ABD</td>
<td>$-2\varepsilon^2 (\varepsilon - 1)$</td>
<td>$m_{21}$</td>
</tr>
<tr>
<td></td>
<td>$2\varepsilon (\varepsilon - 1)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2\varepsilon^3 - 2\varepsilon^2 + \varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-2\varepsilon^3 + 4\varepsilon^2 - 3\varepsilon + 1$</td>
<td></td>
</tr>
<tr>
<td>TFT vs TFT</td>
<td>$1/4$</td>
<td>$m_{22} = \frac{1}{2}b - \frac{1}{2}c$</td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>TFT vs GTFT</td>
<td>$\frac{(1-\varepsilon)(n+\varepsilon)(m+\varepsilon-2m\varepsilon)}{2\varepsilon^2 \varepsilon - 1} (m+\varepsilon-2)(\varepsilon+n-2n\varepsilon)$</td>
<td>$m_{23}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{m\varepsilon+9n^2+9e^2-12n^2+4e^4+6m\varepsilon^2+4m\varepsilon^3-10n\varepsilon^2+4m\varepsilon^4+mn-4m\varepsilon+4m\varepsilon^2}{2\varepsilon^2 (\varepsilon-1)(n+\varepsilon)}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{m\varepsilon+9n^2+9e^2-12n^2+4e^4+6m\varepsilon^2+4m\varepsilon^3-10n\varepsilon^2+4m\varepsilon^4+mn-4m\varepsilon+4m\varepsilon^2}{2\varepsilon^2 (\varepsilon-1)(m+\varepsilon-2)}$</td>
<td></td>
</tr>
<tr>
<td>TFT vs WSLS</td>
<td>$1/4$</td>
<td>$m_{24} = \frac{1}{2}b - \frac{1}{2}c$</td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$1/4$</td>
<td></td>
</tr>
<tr>
<td>TFT vs ABC</td>
<td>$-(\varepsilon - 1) (2\varepsilon^2 - 2\varepsilon + 1)$</td>
<td>$m_{25}$</td>
</tr>
<tr>
<td></td>
<td>$-2\varepsilon^2 \frac{\varepsilon-1}{2\varepsilon-2\varepsilon^2} (2\varepsilon^2 - 2\varepsilon + 1)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2\varepsilon (\varepsilon - 1)^2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-2\varepsilon^2 (\varepsilon - 1)$</td>
<td></td>
</tr>
<tr>
<td>Interaction</td>
<td>Stationary Distribution</td>
<td>Payoffs</td>
</tr>
<tr>
<td>------------------</td>
<td>----------------------------------------------------------------------------------------</td>
<td>---------</td>
</tr>
<tr>
<td>GTFT vs. AllD</td>
<td>$-\varepsilon (\varepsilon - 1) \frac{n+\varepsilon}{n-m+me-\varepsilon+1}$</td>
<td>$m_{31}$</td>
</tr>
<tr>
<td></td>
<td>$(\varepsilon - 1)^2 \frac{n+\varepsilon}{n-m+me-\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\varepsilon \frac{me-\varepsilon-m+\varepsilon^2+1}{n-m+me-\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$- (\varepsilon - 1) \frac{me-\varepsilon-m+\varepsilon^2+1}{n-m+me-\varepsilon+1}$</td>
<td></td>
</tr>
<tr>
<td>GTFT vs. TFT</td>
<td>$(1-\varepsilon)(n+\varepsilon)(m+\varepsilon-2me)$</td>
<td>$m_{32}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{me+5me+9\varepsilon^2-12\varepsilon^3+4\varepsilon^4-6me^2+4me^3-10me^2+4me^3+mn-4mne+4mne^2}{2\varepsilon(\varepsilon - 1)^2(n+\varepsilon)}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{me+5me+9\varepsilon^2-12\varepsilon^3+4\varepsilon^4-6me^2+4me^3-10me^2+4me^3+mn-4mne+4mne^2}{2\varepsilon^2 \frac{m+\varepsilon-2}{m+\varepsilon-2}(m+\varepsilon-2)(n+\varepsilon-2)}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{me+5me+9\varepsilon^2-12\varepsilon^3+4\varepsilon^4-6me^2+4me^3-10me^2+4me^3+mn-4mne+4mne^2}{2\varepsilon(\varepsilon - 1)(m+\varepsilon-2)}$</td>
<td></td>
</tr>
<tr>
<td>GTFT vs. GTFT</td>
<td>$\frac{2m-2me+me-2m\varepsilon}{2m-2me+me-2m\varepsilon}$</td>
<td>$m_{33}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{n\varepsilon^3-2\varepsilon^2-4n\varepsilon+mn^2+me^2+2n^2\varepsilon-2mn+2mne}{n+\varepsilon-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{n\varepsilon^3-2\varepsilon^2-4n\varepsilon+mn^2+me^2+2n^2\varepsilon-2mn+2mne}{n+\varepsilon-2}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{\varepsilon^2 \varepsilon^3-2\varepsilon^2+4n\varepsilon+mn^2+me^2+2n^2\varepsilon-2mn+2mne}{m+\varepsilon-2}$</td>
<td></td>
</tr>
<tr>
<td>GTFT vs. WLS</td>
<td>$\frac{(n+\varepsilon-2n\varepsilon-\varepsilon^2+2n^2\varepsilon-2mn+2mne)(\varepsilon - 1)}{4me-7e-n+4ne+15\varepsilon^2-12\varepsilon^3+4\varepsilon^4-8me^2+4me^3-8me^2+4me^3+mn-4mne+4mne^2}$</td>
<td>$m_{34}$</td>
</tr>
<tr>
<td></td>
<td>$(n+\varepsilon-2n\varepsilon-\varepsilon^2+2n^2\varepsilon-2mn+2mne)(\varepsilon - 1)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{-2e-2me-ne-3\varepsilon^2+3\varepsilon^4+4me^3-2me^3+2me^2+mn-2mne^2}{(n+\varepsilon-2n\varepsilon-\varepsilon^2+2n^2\varepsilon-2mn+2mne)(\varepsilon - 1)}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{-2e-2me-ne-3\varepsilon^2+3\varepsilon^4+4me^3-2me^3+2me^2+mn-2mne^2}{(n+\varepsilon-2n\varepsilon-\varepsilon^2+2n^2\varepsilon-2mn+2mne)(\varepsilon - 1)}$</td>
<td></td>
</tr>
<tr>
<td>GTFT vs. AIC</td>
<td>$\frac{-\varepsilon-1}{n+\varepsilon} \frac{(n\varepsilon - 2\varepsilon + \varepsilon^2 + 1)}{n+\varepsilon}$</td>
<td>$m_{35}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{\varepsilon}{n+\varepsilon} \frac{(n\varepsilon - 2\varepsilon + \varepsilon^2 + 1)}{n+\varepsilon}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\varepsilon (\varepsilon - 1) \frac{m+\varepsilon-2}{n+\varepsilon}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\varepsilon^2 \frac{m+\varepsilon-2}{n+\varepsilon}$</td>
<td></td>
</tr>
<tr>
<td>Interaction</td>
<td>Stationary Distribution</td>
<td>Payoffs</td>
</tr>
<tr>
<td>---------------------</td>
<td>------------------------------------------------------------------------------------------</td>
<td>--------------------------</td>
</tr>
<tr>
<td>WSLS vs. AllD</td>
<td>$\frac{1}{2}\varepsilon$</td>
<td>$m_{41} = b\varepsilon - \frac{1}{2}c$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} - \frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2} - \frac{1}{2}\varepsilon$</td>
<td></td>
</tr>
<tr>
<td>WSLS vs. TFT</td>
<td>$\frac{1}{4}$</td>
<td>$m_{42} = \frac{1}{2}(b - c)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td>WSLS vs. GFT</td>
<td>$\frac{1}{4}\varepsilon - 2\varepsilon^2 - 2\varepsilon_2 - mn + 2 mn_2$ $(\varepsilon - 1)$</td>
<td>$m_{43}$</td>
</tr>
<tr>
<td></td>
<td>$4 n_2 - 7 \varepsilon - n + 4 n_2 + 15 \varepsilon^2 - 12 \varepsilon^3 + 4 \varepsilon^4 - 8 n_2^2 + 4 n_2\varepsilon^3 - 8 n_2 \varepsilon^2 + 4 n_2 \varepsilon^4 + mn - 4 mn_\varepsilon + 4 mn_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-(2 \varepsilon - 2 n_2 - n - 3 \varepsilon^2 + 4 \varepsilon^3 - 2 \varepsilon_2 + 2 n_2^2 + mn - 2 mn_2)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_{43}$</td>
<td></td>
</tr>
<tr>
<td>WSLS vs. WSLS</td>
<td>$\frac{1}{4}\varepsilon - 7 \varepsilon - n + 4 n_2 + 15 \varepsilon^2 - 12 \varepsilon^3 + 4 \varepsilon^4 - 8 n_2^2 + 4 n_2\varepsilon^3 - 8 n_2 \varepsilon^2 + 4 n_2 \varepsilon^4 + mn - 4 mn_\varepsilon + 4 mn_2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-(2 \varepsilon - 2 n_2 - n - 3 \varepsilon^2 + 4 \varepsilon^3 - 2 \varepsilon_2 + 2 n_2^2 + mn - 2 mn_2)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_{43}$</td>
<td></td>
</tr>
<tr>
<td>WSLS vs. AllC</td>
<td>$\frac{1}{2} - \frac{1}{2}\varepsilon$</td>
<td>$m_{45}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{4}$</td>
<td></td>
</tr>
</tbody>
</table>

177
<table>
<thead>
<tr>
<th>Interaction</th>
<th>Stationary Distribution</th>
<th>Payoffs</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC vs. AllD</td>
<td>(-\varepsilon (\varepsilon - 1)) (\varepsilon^2) (-\varepsilon (\varepsilon - 1))</td>
<td>(m_{51} = b\varepsilon - c + c\varepsilon)</td>
</tr>
<tr>
<td>AIC vs. TFT</td>
<td>(-2\varepsilon^3 + 4\varepsilon^2 - 3\varepsilon + 1) (2\varepsilon (\varepsilon - 1)^2) (2\varepsilon^3 - 2\varepsilon^2 + \varepsilon) (-2\varepsilon^2 (\varepsilon - 1))</td>
<td>(m_{52} = b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2)</td>
</tr>
<tr>
<td>AIC vs. GTFT</td>
<td>(-\frac{\varepsilon-1}{ne-ne+1} (ne - 2\varepsilon + \varepsilon^2 + 1)) (\varepsilon (\varepsilon - 1) \frac{m+\varepsilon-2}{ne-ne+1}) (\varepsilon \frac{ne - 2\varepsilon + \varepsilon^2 + 1}{ne-ne+1})</td>
<td>(m_{53} = b - c - 2b\varepsilon + c\varepsilon + b\varepsilon^2 + bn\varepsilon)</td>
</tr>
<tr>
<td>AIC vs. WSLS</td>
<td>(-2\varepsilon^3 + 4\varepsilon^2 - 3\varepsilon + 1) (2\varepsilon (\varepsilon - 1)^2) (2\varepsilon^3 - 2\varepsilon^2 + \varepsilon) (-2\varepsilon^2 (\varepsilon - 1))</td>
<td>(m_{54} = b - c - 2b\varepsilon + c\varepsilon + 2b\varepsilon^2)</td>
</tr>
<tr>
<td>AIC vs. AIC</td>
<td>((\varepsilon - 1)^2) (-\varepsilon (\varepsilon - 1)) (-\varepsilon (\varepsilon - 1)) (\varepsilon^2)</td>
<td>(m_{55} = (1 - \varepsilon)(b - c))</td>
</tr>
</tbody>
</table>

Table 5.4: Stationary distributions and average expected payoffs for an iterated PD game with an ecology of repeated rules consisting of AllD, TFT, GTFT, WSLS and AllC
where,

\[ m_{13} = b \varepsilon - c \varepsilon - b \varepsilon^2 + bn - bn \varepsilon \]
\[ m_{21} = \varepsilon (b - 2c + 2c \varepsilon) \]
\[ m_{23} = \frac{-c \varepsilon - b \varepsilon^2 - bn + cn + bn \varepsilon - 2cn \varepsilon}{n + 3n \varepsilon - 2n \varepsilon - 2 \varepsilon^2} \]
\[ m_{25} = b - c - b \varepsilon + 2c \varepsilon - 2c \varepsilon^2 \]
\[ m_{31} = b \varepsilon - c \varepsilon + c \varepsilon^2 - cn + cn \varepsilon \]
\[ m_{32} = \frac{b \varepsilon - c \varepsilon^2 + bn - cn - 2bn + cn \varepsilon}{n + 3n \varepsilon - 2n \varepsilon - 2 \varepsilon^2} \]
\[ m_{34} = -m_{341}/m_{342}, \text{ with} \]
\[ m_{341} = 3b \varepsilon - 3c \varepsilon - bn^2 + cn^2 - 5be^2 + 2b \varepsilon^3 + 8c \varepsilon^2 - 7c \varepsilon^3 + 2c \varepsilon^4 + bn - cn - 4bn^2 \varepsilon^2 + 2cn^2 \varepsilon^2 - 6bn \varepsilon + 3cn \varepsilon + 10bn \varepsilon^2 + 4bn^2 \varepsilon - 4bn \varepsilon - 4n^2 \varepsilon^3 - 2cn^2 \varepsilon - 3cn \varepsilon^2 + 4cn \varepsilon^3 \]
\[ m_{342} = 4n^2 \varepsilon^2 - 4n^2 \varepsilon + n^2 + 8n \varepsilon^3 - 16n \varepsilon^2 + 8n \varepsilon - n + 4 \varepsilon^4 - 12 \varepsilon^3 + 15 \varepsilon^2 - 7 \varepsilon \]
\[ m_{35} = b - c - b \varepsilon + 2c \varepsilon - c \varepsilon^2 - cn \varepsilon \]
\[ m_{43} = m_{431}/m_{432} \]
\[ m_{431} = 3c \varepsilon - 3b \varepsilon + bn^2 - cn^2 + 8b \varepsilon^2 + 7b \varepsilon^3 + 2b \varepsilon^4 - 5c \varepsilon^2 + 2c \varepsilon^3 - bn + cn + 2bn^2 \varepsilon^2 - 4cn^2 \varepsilon^2 + 3bn \varepsilon - 6cn \varepsilon - 6bn \varepsilon^2 - 3bn^2 \varepsilon + 4bn \varepsilon^3 + 10cn \varepsilon^2 + 4cn^2 \varepsilon - 4cn \varepsilon^3 \]
\[ m_{44} = (b - c) (1 - 4 \varepsilon^3 + 6 \varepsilon^2 - 3 \varepsilon) \]
\[ m_{45} = b - \frac{1}{2} c - b \varepsilon \]