Small steps in dynamics of information
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A.1 Completeness of \( \text{IE}_K \) w.r.t. IE-models

This section provides the completeness proof for the implicit/explicit language with respect to IE-models (Theorem 2.1). Non-defined concepts like satisfiability of a formula and a set of formulas, a \( \Lambda \)-consistent (inconsistent) set and a maximal \( \Lambda \)-consistent set (for \( \Lambda \) an axiom system) are completely standard, and can be found in chapter 4 of Blackburn et al. (2001).

For a completeness proof, the key observation is that an axiom system \( \Lambda \) is strongly complete with respect to a class of models if and only if every \( \Lambda \)-consistent set is satisfiable in a structure of the given class (Proposition 4.12 of Blackburn et al. (2001)). Then, we will use the canonical model technique to show that every \( \text{IE} \)-consistent set is satisfiable in a pointed IE-model. Proofs of Lindenbaum’s Lemma and Existence Lemma are standard. For the Truth Lemma, we will prove the case for our access and rule set formulas.

Lemma A.1 (Lindenbaum’s Lemma) For any \( \text{IE} \)-consistent set of formulas \( \Sigma \), there is a maximal \( \text{IE} \)-consistent set \( \Sigma^+ \) such that \( \Sigma \subseteq \Sigma^+ \).

Definition A.1 (Canonical model for \( \text{IE} \)) Recall that \( I\mathcal{E} \) denotes the implicit/explicit language. The canonical model of the axiom system \( \text{IE} \) is the model

\[
M^{\text{IE}} = \langle W^{\text{IE}}, R^{\text{IE}}, V^{\text{IE}}, A^{\text{IE}}, R^{\text{IE}} \rangle
\]

where:

- \( W^{\text{IE}} \) is the set of all maximal \( \text{IE} \)-consistent set of formulas;
- \( R^{\text{IE}}_{wu} \) iff for all \( \varphi \) in \( I\mathcal{E} \), \( \varphi \in u \) implies \( \diamond \varphi \in w \) (or, equivalently, \( R^{\text{IE}}_{wu} \) iff for all \( \varphi \) in \( I\mathcal{E} \), \( \square \varphi \in w \) implies \( \varphi \in u \));

and, for all \( w \in W^{\text{IE}} \)
Appendix A. Technical appendix

• \( \mathcal{V}^\text{IE}(w) := \{ p \in P \mid p \in w \} \);

• \( \mathcal{A}^\text{IE}(w) := \{ \gamma \in L_p \mid A \gamma \in w \} \);

• \( \mathcal{R}^\text{IE}(w) := \{ \rho \in R \mid R \rho \in w \} \);

Lemma A.2 (Existence Lemma) For every world \( w \in W^\text{IE} \), if \( \Diamond \varphi \in w \), then there is a world \( u \in W^\text{IE} \) such that \( R^\text{IE}wu \) and \( \varphi \in u \).

Lemma A.3 (Truth Lemma) For all \( w \in W^\text{IE} \), we have \( (M^\text{IE}, w) \models \varphi \iff \varphi \in w \).

Proof. We prove the case of access and rule set formulas. For the first,

\[
(M^\text{IE}, w) \models A \gamma \iff \gamma \in \mathcal{A}^\text{IE}(w) \quad \text{by semantic interpretation}
\]

\[
\iff A \gamma \in w \quad \text{by definition of } \mathcal{A}^\text{IE}
\]

The case of rule set formulas is similar.

By the mentioned Proposition 4.12 of Blackburn et al. (2001), all we have to do is show that every \( \text{IE} \)-consistent set is satisfiable in a pointed \( \text{IE} \)-model, so take any such set \( \Sigma \). By Lindenbaum’s Lemma, we can extend it to a maximal \( \text{IE} \)-consistent set \( \Sigma^+ \); by the Truth Lemma, we have \( (M^\text{IE}, \Sigma^+) \models \Sigma \), so \( \Sigma \) is satisfiable in the canonical model of \( \text{IE} \) at \( \Sigma^+ \). It is only left to show that \( M^\text{IE} \) is indeed a model in \( \text{IE} \), that is, we have to show that it satisfies coherence for formulas and rules.

Remember that any maximal \( \text{IE} \)-consistent set \( \Phi \) is closed under logical consequence, that is, if \( \varphi \) and \( \varphi \rightarrow \psi \) are in \( \Phi \), so is \( \psi \).

• Coherence for formulas. Suppose \( \gamma \in \mathcal{A}^\text{IE}(w) \); we want to show that for all \( u \) such that \( R^\text{IE}wu \) we have \( \gamma \in \mathcal{A}^\text{IE}(u) \). Note that \( A \gamma \rightarrow \Box A \gamma \) (axiom \( \text{Coh}_{L_p} \)) is in \( w \).

By definition, \( \gamma \in \mathcal{A}^\text{IE}(w) \) implies \( A \gamma \in w \); by the logical consequence closure, we have \( \Box A \gamma \in w \). Take any \( u \) such that \( R^\text{IE}wu \); by definition of \( R^\text{IE} \) we have \( A \gamma \in u \), and therefore \( \gamma \in \mathcal{A}^\text{IE}(u) \).

• Coherence for rules. Similar to the case of formulas, using the \( \text{Coh}_R \) axiom.

A.2 Completeness of \( \text{IE}_K \) w.r.t. \( \text{IE}_K \)-models

This section provides the completeness proof for the implicit/explicit language with respect to \( \text{IE}_K \)-models (Theorem 2.2).

We know already that \( \text{IE} \) is complete with respect to models in \( \text{IE} \) (Theorem 2.1). In order to show that \( \text{IE}_K \) is complete with respect to \( \text{IE}_K \), we just have to show that the canonical model for \( \text{IE}_K \) satisfy equivalence, truth for formulas and truth for rules.
A.3. Closure of deduction operation

Definition A.2 (Canonical model for IE_K) The canonical model for IE_K, M^{IE_K} = (W^{IE_K}, R^{IE_K}, V^{IE_K}, A^{IE_K}, R^{IE_K}), is defined just as the canonical model for IE (Definition A.1), but the worlds are maximal IE_K-consistent sets of formulas instead of maximal IE-consistent ones. □

Here is the proof for the three properties.

• Equivalence. Axioms T, 4 and 5 are canonical for reflexivity, transitivity and euclideanity, respectively, so R^{IE_K} is an equivalence relation.

• Truth for formulas. We want to show that γ ∈ A^{IE_K}(w) implies (M^{IE_K}, w) ⊩ γ. Suppose γ ∈ A^{IE_K}(w); then we get A γ ∈ w. By axiom Tth_{LP} we have γ ∈ w; by the Truth Lemma, (M^{IE_K}, w) ⊩ γ.

• Truth for rules. Similar to the case of formulas, with axiom Tth_{R}.

A.3 Closure of deduction operation

This section proves that IE_K-models are closed under the deduction operation (Proposition 2.1).

Let M be a model in IE_K. To show that M ≃ (Definition 2.16) is also in IE_K, we will show that it satisfies coherence and truth for formulas, coherence and truth for rules, and equivalence. Equivalence and both properties of rules are immediate since neither the accessibility relation nor the rule set function are modified. For the properties of formulas, we have the following.

• Coherence for formulas. Suppose γ ∈ A′(w) and pick any u ∈ W such that Rwu in M ≃; we will show that γ ∈ A′(u).

From the definition of A′, we know that γ was added by the operation or was already in A(w). In the first case, γ should be cn(σ) and therefore pm(σ) ⊆ A(w) and σ ∈ R(w). But then, by coherence for formulas and rules of M and the fact that Rwu, we have pm(σ) ⊆ A(u) and σ ∈ R(u); therefore, the operation also adds cn(σ) (our γ) to the access set of u, that is, γ ∈ A′(u). In the second case, by coherence for formulas of M and Rwu, we have γ ∈ A(u) and therefore γ ∈ A′(u).

• Truth for formulas. Suppose γ ∈ A′(w); we will show that (M ≃, w) ⊩ γ.

Again, from the definition of A′, we know that γ was added by the operation or was already in A(w). In the first case, γ should be cn(σ) and therefore pm(σ) ⊆ A(w) and σ ∈ R(w). By truth for formulas of M we have (M, w) ⊩ (∧ γ∈pm(σ) γ); by truth for rules of M we have (M, w) ⊩ (∧ γ∈pm(σ) γ) → cn(σ). Therefore, we have (M, w) ⊩ cn(σ), i.e., (M, w) ⊩ γ. In the second case, by truth for formulas of M we also get (M, w) ⊩ γ.
Now, $\gamma$ is a propositional formula so its truth value depends only on the valuation at $w$. But since the valuations at $M$ and $M_{\rightarrow r}$ are the same, we get $(M_{\rightarrow r}, w) \vDash \gamma$, as required.

A.4 Closure of structural operations

This section proves that $\text{IE}_K$-models are closed under the three structural operations of Definition 2.18: reflexivity, monotonicity and cut (Proposition 2.2).

The proposition already argues for equivalence and the two properties of formulas, coherence and truth. It is only left to prove coherence and truth for rules for each one of the three operations.

**Coherence** In the case of the reflexivity operation, the coherence property follows immediately, since the original model $M$ already has the property and the new rule $\varsigma_\delta$ is added to the rule set of all worlds in $M_{\text{Ref}_i}$. For the monotonicity operation, we just need to check coherence for the new rule $\varsigma'$. Recall that it is added only to worlds that already have $\varsigma$. But if a world $w$ has $\varsigma$ in $M$, then, by coherence for rules of $M$, every world $R$-reachable from $w$ also has $\varsigma$ in $M$, and therefore it will have $\varsigma'$ in $M_{\text{Mon}_i}$. The case of cut is similar: $\varsigma'$ is added to those worlds that have $\varsigma_1$ and $\varsigma_2$, but then every world $R$-accessible from $w$ also has $\varsigma_1$ and $\varsigma_2$ in $M$, so they will have $\varsigma'$ in $M_{\text{Cut}_{\varsigma_1\varsigma_2}}$.

**Truth** Note that for this property, it is enough in all three cases to show that the added rules are truth-preserving in $M$ because the truth-value of the translation, a purely propositional formula, depends just on the valuation, which is preserved by the operations. So let $R$ be the rule set function of the original model $M$, $R'$ be the rule set function of the corresponding new model, and pick any world $w \in W$.

- **Reflexivity.** Recall that $\varsigma_\delta$ is given by $\{\delta\} \Rightarrow \delta$, and pick any rule $\rho \in R'(w)$. If $\rho$ is already in $R(w)$, we have $(M, w) \vDash \text{tr}(\rho)$ since $M$ is in $\text{IE}_K$. Otherwise, $\rho$ is $\varsigma_\delta$, and we obviously have $(M, w) \vDash \delta \rightarrow \delta$.

- **Monotonicity.** Recall that $\varsigma'$ is given by $\text{pm}(\varsigma) \cup \{\delta\} \Rightarrow \text{cn}(\varsigma)$, and pick any $\rho \in R'(w)$. If $\rho$ is already in $R(w)$, we have $(M, w) \vDash \text{tr}(\rho)$. Otherwise, $\rho$ is $\varsigma'$, and then $\varsigma \in R(w)$. Since $M$ is in $\text{IE}_K$, we have $(M, w) \vDash (\wedge_{\gamma \in \text{pm}(\varsigma)} \gamma) \rightarrow \text{cn}(\varsigma)$ and therefore $(M, w) \vDash ((\wedge_{\gamma \in \text{pm}(\varsigma)} \gamma) \land \delta) \rightarrow \text{cn}(\varsigma)$.

- **Cut.** Recall that $\varsigma'$ is given by $(\text{pm}(\varsigma_2) \setminus \{\text{cn}(\varsigma_1)\}) \cup \text{pm}(\varsigma_1) \Rightarrow \text{cn}(\varsigma_2)$, and pick any $\rho \in R'(w)$. If $\rho \in R(w)$, we have $(M, w) \vDash \text{tr}(\rho)$ since $M$ is in $\text{IE}_K$. Otherwise, $\rho$ is $\varsigma'$ and we have $\{\varsigma_1, \varsigma_2\} \subseteq R(w)$.

Suppose $\wedge_{\gamma \in \text{pm}(\varsigma')} \gamma$ is true at $w$ in $M$; then, every premise of $\varsigma'$ is true at $w$ in $M$. This includes every premise of $\varsigma_1$ and every premise of $\varsigma_2$ except
structural operation and deduction

This section provides a sketch for the proof of the validities of Table 2.8. Take any pointed IE-model \((M, w)\). The main idea of the proof is that, under the appropriate circumstances, different sequences of operations produce models that are exactly the same from \(w\)'s point of view, and therefore satisfy the same formulas of our language. For example, we will argue that if we have \(\delta \in A(w)\) and \(\varsigma \in R(w)\), then the pointed models \((M_{\text{Mon}} \delta, \varsigma) \hookrightarrow \varsigma'\) are the same from \(w\)'s point of view (third entry for monotonicity in Table A.1 below). Note how a stronger identity between models does not hold because we cannot verify what happens in worlds that are not reachable from \(w\). Therefore, we will state this “identity from \(w\)'s perspective” in terms of an extended notion of bisimulation that asks for related worlds to have the same access and rule set.

Definition A.3 (Bisimulation) Take two IE-models \(M_1 = \langle W_1, R_1, V_1, A_1, R_1 \rangle\) and \(M_2 = \langle W_2, R_2, V_2, A_2, R_2 \rangle\). A non-empty relation \(B \subseteq (W_1 \times W_2)\) is a bisimulation between \(M_1\) and \(M_2\) (in symbols, \(M_1 \leftrightarrow B M_2\)) if and only if \(B\) is a standard bisimulation between \(\langle W_1, R_1, V_1 \rangle\) and \(\langle W_2, R_2, V_2 \rangle\) and, if \(Bw_1w_2\), then \(A_1(w_1) = A_2(w_2)\) and \(R_1(w_1) = R_2(w_2)\).

We will write \((M_1, w_1) \leftrightarrow_B (M_2, w_2)\) when \(M_1 \leftrightarrow B M_2\) and \(Bw_1w_2\).

The validity of the formulas stated in Table 2.8 follows from the bisimilarities between models stated in Table A.1, where models of the form \((M_{\text{STR}})\hookrightarrow \sigma\) are the result of applying the structural operation STR and then the deduction operation with rule \(\sigma\), and similar for models of the form \((M_{\text{Mon}})\hookrightarrow \sigma\). In all cases, the bisimulation is the identity relation over worlds reachable from \(w\).

Now for the proof. The involved operations (structural ones and deduction) preserve worlds, accessibility relations and valuations. Then, in order to show that the identity relation over worlds reachable from \(w\) is indeed a bisimulation, we just need to show that such worlds have the same access and rule set in both models.

Consider as an example the third bisimilarity for monotonicity; we will work with \(w\) first. For access sets, take any \(\gamma\) in the access set of \(w\) in \((M_{\text{Mon}})\hookrightarrow \sigma\); by definition, either it was already in that of \(w\) in \(M_{\text{Mon}}\) or else it was added by
Appendix A. Technical appendix

Reflexivity with $\varsigma_\delta$ the rule $[\delta] \Rightarrow \delta$
If $\sigma \neq \varsigma_\delta$, then $\left( (M_{\text{Ref}_{\delta}})_{\gamma, \delta}, w \right) \leftrightarrow \left( (M_{\gamma, \delta})_{\text{Ref}_{\delta}}, w \right)$
If $\varsigma_\delta \in R(w)$, then $\left( (M_{\text{Ref}_{\delta}})_{\gamma, \varsigma_\delta}, w \right) \leftrightarrow \left( M_{\gamma, \varsigma_\delta}, w \right)$
If $\delta \in A(w)$, then $\left( (M_{\text{Ref}_{\delta}})_{\gamma, \varsigma_\delta}, w \right) \leftrightarrow \left( (M_{\gamma, \varsigma_\delta})_{\text{Ref}_{\delta}}, w \right)$

Monotonicity with $\varsigma'$ the rule $\text{pm}(\varsigma) \cup [\delta] \Rightarrow \text{cn}(\varsigma)$
If $\sigma \neq \varsigma'$, then $\left( (M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma}, w \right) \leftrightarrow \left( (M_{\gamma, \varsigma})_{\text{Mon}_{\varsigma}}, w \right)$
If $\varsigma' \in R(w)$, then $\left( (M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma'}, w \right) \leftrightarrow \left( M_{\gamma, \varsigma'}, w \right)$
If $\delta \in A(w)$ and $\varsigma \in R(w)$, then $\left( (M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma'}, w \right) \leftrightarrow \left( (M_{\gamma, \varsigma'})_{\text{Mon}_{\varsigma}}, w \right)$

Cut with $\varsigma'$ the rule $(\text{pm}(\varsigma_2) \setminus \text{cn}(\varsigma_1)) \cup \text{pm}(\varsigma_1) \Rightarrow \text{cn}(\varsigma_2)$
If $\sigma \neq \varsigma'$, then $\left( (M_{\text{Cut}_{\varsigma_1}})_{\gamma, \varsigma}, w \right) \leftrightarrow \left( (M_{\gamma, \varsigma})_{\text{Cut}_{\varsigma_1}}, w \right)$
If $\varsigma' \in R(w)$, then $\left( (M_{\text{Cut}_{\varsigma_1}})_{\gamma, \varsigma'}, w \right) \leftrightarrow \left( M_{\gamma, \varsigma'}, w \right)$.
If $(\text{pm}(\varsigma_1) \cup \text{cn}(\varsigma_1)) \in A(w)$ and $\varsigma_1 \in R(w)$, then $\left( M_{\text{Cut}_{\varsigma_1}} \gamma, \varsigma', w \right) \leftrightarrow \left( (M_{\gamma, \varsigma'})_{\text{Cut}_{\varsigma_1}}, w \right)$.

Table A.1: Bisimilarities for deduction and structural operations

The deducational operation with $\varsigma'$. In the first case, $\gamma$ is in the access set of $w$ in $M$, since structural operations do not modify access sets; then it is also in the access set of $w$ in $M_{\gamma, \varsigma}$, and in that of $w$ in $(M_{\gamma, \varsigma})_{\text{Mon}_{\varsigma}}$. In the second case, $\gamma$ should be $\text{cn}(\varsigma')$, but then we have the premises of $\varsigma'$ (and hence those of $\varsigma$) in the access set of $w$ in $M_{\text{Mon}_{\varsigma}}$. Then, they are already in that of $w$ in $M$ and, by hypothesis, we have $\varsigma$ in the rule set of $w$ in $M$, so $\text{cn}(\varsigma)$, which is nothing but $\text{cn}(\varsigma')$, is in the access set of $w$ in $M_{\gamma, \varsigma}$ and hence it is in that of $w$ in $(M_{\gamma, \varsigma})_{\text{Mon}_{\varsigma}}$.

For the other direction, take $\gamma$ in the access set of $w$ in $(M_{\gamma, \varsigma})_{\text{Mon}_{\varsigma}}$. Then it is in that of $w$ in $M_{\gamma, \varsigma}$ and therefore either it was already in that of $w$ in $M$, or else it was added by the deducational operation. In the first case, $\gamma$ is preserved through the monotonicity and the deducational operations, and therefore it is in the access set of $w$ at $(M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma}$. In the second case, $\gamma$ should be $\text{cn}(\varsigma)$, and then we should have $\text{pm}(\varsigma)$ and $\varsigma$ in the corresponding sets of $w$ in $M$. By hypothesis, we have $\delta$ in the access set of $w$ in $M$, so we have all the premises of $\varsigma'$ in the access set of $w$ in $M$; therefore they are also in that of $w$ at $M_{\text{Mon}_{\varsigma}}$. Since we have $\varsigma$ in the rule set of $w$ in $M$, we have $\varsigma'$ in that of $w$ in $M_{\text{Mon}_{\varsigma}}$ too. Hence, we have $\text{cn}(\varsigma')$, which is nothing but $\text{cn}(\varsigma)$, in the access set of $w$ in $(M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma}$.

The argument for rules is similar, and hence $w$ has the same access and rule sets in $(M_{\text{Mon}_{\varsigma}})_{\gamma, \varsigma}$ and $(M_{\gamma, \varsigma})_{\text{Mon}_{\varsigma}}$.
A.6 Closure of explicit observation operation

Now suppose a world $u$ is reachable from $w$ through the accessibility relation at $M_{\text{Mon}_\delta,\zeta}$. Since $R$ is not modified by the operations, $u$ is reachable from $w$ at $M$ and therefore $u$ is reachable from $w$ at $M_{\xrightarrow{\delta,\zeta}}$ too. Now we use the coherence properties: since $\delta \in A(w)$ and $\zeta \in R(w)$, we have $\delta$ and $\zeta$ in the corresponding sets of $u$, and then we can apply the argument used for $w$ to show that $u$ has the same information and rule set on both models.

These bisimulations allow us to prove the validities of Table 2.8. For example, recall the two formulas for monotonicity:

\[
\begin{align*}
&\text{Monotonicity with } \zeta' \text{ the rule } pm(\zeta) \cup \{\delta\} \Rightarrow cn(\zeta) \\
&\langle Mon_{\delta,\zeta} \rangle \langle \xleftarrow{\sigma} \rangle \varphi \leftrightarrow \langle \xleftarrow{\sigma} \rangle \langle Mon_{\delta,\zeta} \rangle \varphi \quad \text{for } \sigma \neq \zeta' \\
&\langle Mon_{\delta,\zeta} \rangle \langle \xleftarrow{\zeta'} \rangle \varphi \leftrightarrow \left(\langle \xleftarrow{\zeta'} \rangle \varphi \lor (A \delta \land R \zeta \land \langle \xleftarrow{\sigma} \rangle \langle Mon_{\delta,\zeta} \rangle \varphi)\right)
\end{align*}
\]

As mentioned in the text, the first formula indicates that the operation does not affect deduction with a rule different from the new one, and it follows from the first bisimilarity for monotonicity:

\[\text{if } \sigma \neq \zeta', \text{ then } (M_{\text{Mon}_{\delta,\zeta} \xleftarrow{\sigma}}, w) \leftrightarrow (M_{\xleftarrow{\sigma}}, w)\]

The second formula indicates how deduction with the generated rule changes after the structural operation, and it expresses the disjunction of two cases. If the rule created by the monotonicity operation was already in the original rule set, then the monotonicity operation is irrelevant, and just deduction is needed. This follows from the second bisimilarity for this structural operation:

\[\text{if } \zeta' \in R(w), \text{ then } (M_{\text{Mon}_{\delta,\zeta} \xleftarrow{\sigma}}, w) \leftrightarrow (M_{\xleftarrow{\zeta'}}, w)\]

But if the created rule was not in the original set, then we need some requirements, as the third bisimilarity for the operation shows:

\[\text{if } \delta \in A(w) \text{ and } \zeta \in R(w), \text{ then } (M_{\text{Mon}_{\delta,\zeta} \xleftarrow{\sigma}}, w) \leftrightarrow (M_{\xleftarrow{\zeta'}}, Mon_{\delta,\zeta}, w)\]

A.6 Closure of explicit observation operation

This section proves that IE$_K$-models are closed under the explicit observation operation of Definition 2.20 (Proposition 2.3).

Let $M = \langle W, R, V, A, R \rangle$ be a model in IE$_K$. To show that $M_{\chi}$ is also in IE$_K$, we will show that it satisfies equivalence, coherence and truth for formulas and coherence and truth for rules.

Equivalence is immediate since the new model is a sub-model of the original one. For the other properties, suppose $\chi$ is a formula. Coherence for formulas
Appendix A. Technical appendix

of $M_{\chi^+}$ follows from that of $M$ and the fact that $\chi$ is added uniformly to all preserved worlds. Coherence for rules of $M_{\chi^+}$ follows simply from that of $M$. Truth for formulas of $M_{\chi^+}$ follows from that of $M$ since the truth of formulas in $A$-sets depends only on the atomic valuation of each world, which is not modified by the operation, and from the fact that the preserved worlds are precisely those in which $\chi$ is true. Truth for rules of $M_{\chi^+}$ simply relies on that of $M$, again because the truth-value of the translation of a rule depends only on the unmodified atomic valuation of each world. The argument for the case in which $\chi$ is a rule is similar.

A.7 Explicit observation and deduction

This section provides a sketch for the proof of the validities of Table 2.10.

Just as the case of structural operations and deduction, the validity of the formulas follows from bisimilarities, this time the ones stated in Table A.2.

<table>
<thead>
<tr>
<th>If $\chi$ is a formula:</th>
</tr>
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<tbody>
<tr>
<td>If $\chi \notin \text{pm}(\sigma)$, then $(M_{\chi^+}, w) \leftrightarrow (M_{\sigma^+}, w)$</td>
</tr>
<tr>
<td>If $\chi \in \text{pm}(\sigma)$ and $\chi \in A(w)$, then $(M_{\chi^+}, w) \leftrightarrow (M_{\sigma^+}, w)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>If $\chi$ is a rule:</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $\chi \neq \sigma$, then $(M_{\chi^+}, w) \leftrightarrow (M_{\sigma^+}, w)$</td>
</tr>
<tr>
<td>If $\chi = \sigma$ and $\chi \in R(w)$, then $(M_{\chi^+}, w) \leftrightarrow (M_{\sigma^+}, w)$</td>
</tr>
</tbody>
</table>

Table A.2: Bisimilarities for deduction and explicit observation operations

The proof is also similar to the case of structural operations and deduction, keeping in mind that explicit observations remove worlds, therefore modifying accessibility relations.

A.8 Awareness as a full language

This section provides the proofs of Lemmas 4.1 and 4.2.

Lemma 4.1 states that if an agent $i$ has the formula $\varphi$ at her disposal, that is, if $(M, w) \models [i]^p \varphi$, then she has at her disposal all atoms in it, that is, $(M, w) \models [i]p$ for every $p \in \text{atm}(\varphi)$. In other words, the formula

$[i] \varphi \rightarrow [i] p$

is valid for every $p \in \text{atm}(\varphi)$. 

We will prove the equivalent statement:

\[(M, w) \models [i]![i] q \quad \text{implies} \quad \text{atm}(q) \subseteq \text{PA}_i(w)\]

The equivalence of the statements follows from the semantic interpretation of formulas of the form \([i]p\) (Definition 4.5).

The proof is by induction on \(q\). The base case is immediate: if \(q\) is an atom \(p\) and \((M, w) \models [i]p\), then the semantic interpretation gives us \(p \in \text{PA}_i(w)\), hence \(\text{atm}(p) \subseteq \text{PA}_i(w)\). For the inductive cases we have the following.

\(q \) as \([i]p\). Suppose \((M, w) \models [i]([i]p)\). From Definition 4.3 we have the validity \([i]([i]p) \leftrightarrow [i]p\) so \((M, w) \models [i]p\); then \(p \in \text{PA}_i(w)\) and hence \(\text{atm}([i]p) \subseteq \text{PA}_i(w)\).

\(q \) as \(A_i \psi\). Suppose \((M, w) \models [i](A_i \psi)\). Definition 4.3 gives us \([i](A_i \psi) \leftrightarrow [i] \psi\) so \((M, w) \models [i] \psi\). But \(\psi\) is a sub-formula of \(A_i \psi\); then, by inductive hypothesis, \(\text{atm}(\psi) \subseteq \text{PA}_i(w)\), and hence \(\text{atm}(A_i \psi) \subseteq \text{PA}_i(w)\).

\(q \) as \(R_i \rho\). Suppose \((M, w) \models [i](R_i \rho)\). Definition 4.3 gives us \([i](R_i \rho) \leftrightarrow [i] \rho\) so \((M, w) \models [i] \rho\). But \(\rho\) is a sub-formula of \(R_i \rho\); then, by inductive hypothesis, \(\text{atm}(\rho) \subseteq \text{PA}_i(w)\), and hence \(\text{atm}(R_i \rho) \subseteq \text{PA}_i(w)\).

\(q \) as \(\neg \psi\). Suppose \((M, w) \models [i](\neg \psi)\). Definition 4.3 gives us \([i](\neg \psi) \leftrightarrow [i] \neg \psi\) so \((M, w) \models [i] \neg \psi\). Since \(\neg \psi\) is a sub-formula of \(\neg \psi\), inductive hypothesis gives us \(\text{atm}(\psi) \subseteq \text{PA}_i(w)\) and hence \(\text{atm}(\neg \psi) \subseteq \text{PA}_i(w)\).

\(q \) as \(\psi \lor \psi'\). Suppose \((M, w) \models [i](\psi \lor \psi')\). Definition 4.3 gives us \([i](\psi \lor \psi') \leftrightarrow ([i] \psi \lor [i] \psi')\) so \((M, w) \models [i] \psi \lor [i] \psi'\), that is, \((M, w) \models [i] \psi\) and \((M, w) \models [i] \psi'\). Since \(\psi\) and \(\psi'\) are sub-formulas of \(\psi \lor \psi'\), inductive hypothesis gives us \(\text{atm}(\psi) \cup \text{atm}(\psi') \subseteq \text{PA}_i(w)\) and hence \(\text{atm}(\psi \lor \psi') \subseteq \text{PA}_i(w)\).

\(q \) as \(\square_i \psi\). Suppose \((M, w) \models [i](\square_i \psi)\). Definition 4.3 gives us \([i](\square_i \psi) \leftrightarrow [i] \psi\) so \((M, w) \models [i] \psi\). But \(\psi\) is a sub-formula of \(\square_i \psi\); then, by inductive hypothesis, \(\text{atm}(\psi) \subseteq \text{PA}_i(w)\), and hence \(\text{atm}(\square_i \psi) \subseteq \text{PA}_i(w)\).

This completes the proof.

Lemma 4.2 states that if agent \(i\) has all atoms in \(\{p_1, \ldots, p_n\} \) at her disposal, that is, if \((M, w) \models [i]p_k\) for every \(k \in \{1, \ldots, n\}\), then she has at her disposal any formula built from such atoms, that is, \((M, w) \models [i]q\) for any formula \(q\) built from \(\{p_1, \ldots, p_n\}\). In other words, the formula

\[
\left( \bigwedge_{k \in \{1, \ldots, n\}} [i]p_k \right) \rightarrow [i]q
\]

is valid for every \(q\) built from \(\{p_1, \ldots, p_n\}\).
Again, we will prove an equivalent statement, this time:

\[ \{p_1, \ldots, p_n\} \subseteq \text{PA}_i(w) \implies (M, w) \vDash [i] \varphi \]

for every \( \varphi \) built from \( \{p_1, \ldots, p_n\} \). Again, the equivalence of the statements follows from the semantic interpretation of formulas of the form \([i]p\).

So suppose \( \{p_1, \ldots, p_n\} \subseteq \text{PA}_i(w) \); we will proceed by induction on \( \varphi \). The base case is immediate: if \( \varphi \) is any \( p \) in \( \{p_1, \ldots, p_n\} \), then we obviously have \((M, w) \vDash [i]p\). For the inductive cases we have the following.

**\( \varphi \) as \([i]p \).** In this case \( p \) should be in \( \{p_1, \ldots, p_n\} \) so \((M, w) \vDash [i]p\). But by Definition 4.3 we have \([i]([i]p) \iff [i]p\), so \((M, w) \vDash [i]([i]p)\).

**\( \varphi \) as \(A_j \psi \).** Since \( A_j \psi \) is built from atoms in \( \{p_1, \ldots, p_n\} \), we have \( \text{atm}(A_j \psi) \subseteq \{p_1, \ldots, p_n\} \), that is, \( \text{atm}(\psi) \subseteq \{p_1, \ldots, p_n\} \). Since \( \psi \) is a sub-formula of \( A_j \psi \), inductive hypothesis gives us \((M, w) \vDash [i]\psi\). But by Definition 4.3 we have \([i](A_j \psi) \iff [i]\psi\), so \((M, w) \vDash [i](A_j \psi)\).

**\( \varphi \) as \(R_j \rho \).** We have \( \text{atm}(R_j \rho) \subseteq \{p_1, \ldots, p_n\} \), that is, \( \text{atm}(\rho) \subseteq \{p_1, \ldots, p_n\} \). Since \( \rho \) is a sub-formula of \( R_j \rho \), inductive hypothesis gives us \((M, w) \vDash [i]\rho\). But by Definition 4.3 we have \([i](R_j \rho) \iff [i]\rho\), so \((M, w) \vDash [i](R_j \rho)\).

**\( \varphi \) as \(\neg \psi \).** We have \( \text{atm}(\psi) \subseteq \{p_1, \ldots, p_n\} \). Since \( \psi \) is a sub-formula of \( \neg \psi \), inductive hypothesis gives us \((M, w) \vDash [i]\psi\). But by Definition 4.3 we have \([i](\neg \psi) \iff [i]\neg [i]\psi\), so \((M, w) \vDash [i](\neg \psi)\).

**\( \varphi \) as \(\psi \lor \psi' \).** We have \( \text{atm}(\psi) \cup \text{atm}(\psi') \subseteq \{p_1, \ldots, p_n\} \). Since \( \psi \) and \( \psi' \) are both sub-formulas of \( \psi \lor \psi' \), inductive hypothesis yields \((M, w) \vDash [i]\psi \land [i]\psi'\). Definition 4.3 gives us \([i](\psi \lor \psi') \iff ([i]\psi \land [i]\psi')\) so we have \((M, w) \vDash [i](\psi \lor \psi')\).

**\( \varphi \) as \(\square_j \psi \).** We have \( \text{atm}(\psi) \subseteq \{p_1, \ldots, p_n\} \). Since \( \psi \) is a sub-formula of \( \square_j \psi \), inductive hypothesis gives us \((M, w) \vDash [i]\psi\). But by Definition 4.3 we have \([i](\square_j \psi) \iff [i]\psi\), so \((M, w) \vDash [i](\square_j \psi)\).

Note how this case does not involve availability at worlds other than \( w \). It simply says that if \( \square_j \psi \) is a formula built from atoms the agent has locally available at \( w \), then all atoms in \( \psi \) should be locally available. By inductive hypothesis, \( \psi \) should be locally available, \([i]\psi\), and therefore by Definition 4.3, \( \square_j \psi \) is also locally available, \([i](\square_j \psi)\).

This completes the proof.
A.9 Upgrade and locally well-preorders

This section provides the proof of Proposition 5.11

We need to show that if the relation $\leq$ is a locally well-preorder, so is the relation $\leq'$ defined as

$$\leq' := (\leq; \chi ?) \cup (\neg \chi ?; \leq) \cup (\neg \chi ?; \sim; \chi ?)$$

In words, we have $w \leq' u$ if and only if in $M$ (1) $w \leq u$ and $u$ is a $\chi$-world, or (2) $w$ is a $\neg \chi$-world and $w \leq u$, or (3) $w$ is a $\neg \chi$-world, $u$ is a $\chi$-world and the two worlds are in the same comparability class. Note that the only case in which we do not have $w \leq' u$ is when, in $M$, $w$ is a $\chi$-world and $u$ is a $\neg \chi$-world.

The key observation is that a locally well-preorder is a locally connected and conversely well-founded preorder. We will prove that if $\leq$ satisfies such properties, so does $\leq'$ defined as above.

For **reflexivity**, pick any $w \in W$. Since $\leq$ is reflexive, we have $w \leq w$. Now, $w$ is either a $\chi$-world or a $\neg \chi$-one. In the first case we get $w \leq' w$ from part (1) of the definition of $\leq'$; in the second case we get it from part (2).

For **transitivity**, suppose $w \leq' u$ and $u \leq' v$ and consider $w$. If it is a $\chi$-world, then so is $u$ (otherwise there would not be a link from $w$ to $u$) and hence so is $v$ too; therefore, by part (1) of the definition, $w \leq' v$. If it is a $\neg \chi$-world, part (2) of the definition gives us $w \leq' v$. Hence, $\leq'$ is transitive.

For **local connectedness**, first we will show that, for every $u_1, u_2$ in $W$, we have $u_1 \sim u_2$ if and only if $u_1 \sim' u_2$.

($\Rightarrow$) Suppose $u_1 \sim u_2$. If we have $u_1 \leq' u_2$, then we have $u_1 \sim' u_2$ and we are done. Otherwise, $u_1$ should be a $\chi$-world in $M$ and $u_2$ should be a $\neg \chi$-world in $M$; this together with $u_1 \sim u_2$ gives us $u_2 \leq' u_1$ by part (3) of the definition, and hence we have $u_1 \sim' u_2$.

($\Leftarrow$) If $u_1 \sim' u_2$, then we have $u_1 \leq' u_2$ or $u_2 \leq' u_1$. Consider the first case, and let us review the three possibilities. If we have $u_1 \leq' u_2$ because of part (1) of the definition of $\leq'$, then we have $(u_1, u_2) \in (\leq; \chi ?)$; hence $u_1 \leq u_2$ and therefore $u_1 \sim u_2$. If it is because of part (2), then we have $(u_1, u_2) \in (\neg \chi ?; \leq)$; hence $u_1 \leq u_2$ and therefore $u_1 \sim u_2$. If it is because of part (3), then we have $(u_1, u_2) \in (\neg \chi ?; \sim; \chi ?)$; hence $u_1 \sim u_2$. In the three possibilities we get the required $u_1 \sim u_2$. The second case is analogous.

Now, to show local connectedness, take any $w \in W$ and pick $u_1, u_2$ in $V_w$ under $\leq'$. By definition of $V_w$ we have $w \sim' u_1$ and $w \sim' u_2$; by the just proved property we get $w \sim u_1$ and $w \sim u_2$; by local connectedness of $\leq$ we have $u_1 \sim u_2$ and then by the just proved property again we get the required $u_1 \sim' u_2$.

For **converse well-foundedness** we proceed by contradiction. Suppose that there is an infinite ascending chain $u_1 <' u_2 <' \cdots$. These worlds are either $\chi$ or...
$\neg \chi$-worlds in the original model. Since the chain is infinite, there must be an infinite sub-chain of either $\chi$ or $\neg \chi$-worlds (we cannot have an alternation from a $\chi$-world to a $\neg \chi$-one because of the definition of $\leq$). But inside these areas, the new relation is the old one, contradicting the converse well-foundedness of $\leq$. Then, such infinite chain cannot exists, and therefore $\leq'$ is conversely well-founded.

This completes the proof.

A.10 Product update and locally well-preorders

This section provides the proofs of Proposition \[5.12\]

We will show that if $\leq$ and $\preceq$ are two locally well-preorders over $W$ and $E$ respectively, so is the relation $\leq'$ over $W \times E$ given by

$$(w_1, e_1) \leq' (w_2, e_2) \text{ iff } \begin{cases} (e_1 < e_2 \text{ and } w_1 \sim w_2) \text{ or } (e_1 \equiv e_2 \text{ and } w_1 \leq w_2) \end{cases}$$

Recall that a locally well-preorder is a locally connected and a conversely well-founded preorder.

For reflexivity, take any $(w, e) \in W \times E$. By reflexivity of $\leq$ and $\preceq$, we have $w \leq w$ and $e \equiv e$. Then $w \leq w$ and $e \equiv e$ and hence $(w, e) \leq' (w, e)$ from (2) of the definition of $\leq'$.

For transitivity, suppose $(w_1, e_1) \leq' (w_2, e_2)$ and $(w_2, e_2) \leq' (w_3, e_3)$. According to the definition of $\leq'$, each one of these two inequalities has two possible reasons, and this gives us four cases. We will prove two of them in detail; the other two can be proved in a similar way.

1. Suppose that both $(w_1, e_1) \leq' (w_2, e_2)$ and $(w_2, e_2) \leq' (w_3, e_3)$ hold because of part (1) of the definition of $\leq'$. Then we have

$$e_1 < e_2, \quad w_1 \sim w_2, \quad e_2 < e_3, \quad w_2 \sim w_3.$$

By unfolding the definitions of $<$ and $\sim$ we get

$$e_1 \preceq e_2, \quad e_1 \not\preceq e_2, \quad \begin{cases} w_1 \leq w_2 \\ w_2 \leq w_1 \end{cases}, \quad e_2 \preceq e_3, \quad e_3 \not\preceq e_2, \quad \begin{cases} w_3 \leq w_2 \\ w_2 \leq w_3 \end{cases}.$$

For the action part, recall that $\preceq$ is transitive. Then we have $e_1 \preceq e_3$. We also have $e_3 \not\preceq e_1$ because otherwise from $e_1 \preceq e_2$ we will get $e_3 \preceq e_2$, contradicting part of the assumptions. Then we have $e_1 < e_3$. 

For the static part we have again four possibilities. If we have \( w_1 \leq w_2 \) and \( w_2 \leq w_3 \), then we get \( w_1 \leq w_3 \) by transitivity of \( \leq \), and hence \( w_1 \sim w_3 \). If we have \( w_1 \leq w_2 \) and \( w_3 \leq w_2 \), then we should have \( w_1 \leq w_3 \) or \( w_3 \leq w_1 \) because \( \leq \) is locally connected; hence \( w_1 \sim w_3 \). In the other two cases, a similar reasoning shows that \( w_1 \sim w_3 \) holds too.

Then, part (1) of the definition of \( \leq' \) gives us \( (w_1, e_1) \leq' (w_3, e_3) \).

2. Suppose that while \( (w_1, e_1) \leq' (w_2, e_2) \) holds because of part (1) of the definition of \( \leq' \), \( (w_2, e_2) \leq' (w_3, e_3) \) holds because of part (2). Then

\[
e_1 < e_2, \quad w_1 \sim w_2, \quad e_2 \equiv e_3, \quad w_2 \leq w_3.
\]

By unfolding the definitions we get

\[
e_1 \leq e_2, \quad e_1 \neq e_2, \quad \begin{cases} w_1 \leq w_2, \\ w_2 \leq w_1, \end{cases} \quad e_2 \leq e_3, \quad e_3 \leq e_2, \quad w_2 \leq w_3
\]

For the action part, recall that \( \leq \) is transitive. Then we have \( e_1 \leq e_3 \). We also have \( e_3 \neq e_1 \) because otherwise from \( e_2 \leq e_3 \) we will get \( e_2 \leq e_1 \), contradicting part of the assumptions. Then we have \( e_1 < e_3 \).

For the static part we have two possibilities. If we have \( w_1 \leq w_2 \), then together with \( w_2 \leq w_3 \) we get \( w_1 \leq w_3 \); hence \( w_1 \sim w_3 \). If we have \( w_2 \leq w_1 \) then from \( w_2 \leq w_3 \) we should have \( w_1 \leq w_3 \) or \( w_3 \leq w_1 \) because \( \leq \) is locally connected; hence \( w_1 \sim w_3 \).

Then, part (1) of the definition gives us \( (w_1, e_1) \leq' (w_3, e_3) \).

For local connectedness, first we will show that, for every \( w_1, w_2 \) in \( W \) and \( e_1, e_2 \) in \( E \), we have \( w_1 \sim w_2 \) and \( e_1 \equiv e_2 \) if and only if \( (w_1, e_1) \sim' (w_2, e_2) \).

\( (\Rightarrow) \) If \( w_1 \sim w_2 \) and \( e_1 \equiv e_2 \), then we have \( w_1 \leq w_2 \) or \( w_2 \leq w_1 \), and \( e_1 \equiv e_2 \) or \( e_2 \equiv e_1 \). This gives us four cases. For example, suppose \( w_1 \leq w_2 \) and \( e_1 \equiv e_1 \). If we have \( e_1 \equiv e_2 \) we get \( e_1 \equiv e_2 \); then by part (2) of the definition we get \( (w_1, e_1) \leq' (w_2, e_2) \) and hence \( (w_1, e_1) \sim' (w_2, e_2) \). If we do not have \( e_1 \equiv e_2 \), then we have \( e_2 \equiv e_1 \) and from \( w_1 \leq w_2 \) we already have \( w_1 \sim w_2 \); then by part (1) of the definition we have \( (w_2, e_2) \leq' (w_1, e_1) \) and hence \( (w_1, e_1) \sim' (w_2, e_2) \). The other three cases can be proved in a similar way.

\( (\Leftarrow) \) If \( (w_1, e_1) \sim' (w_2, e_2) \), then \( (w_1, e_1) \leq' (w_2, e_2) \) or \( (w_2, e_2) \leq' (w_1, e_1) \). In the first case, we have either possibility (1), which gives us \( e_1 \equiv e_2 \) and \( w_1 \sim w_2 \), hence \( e_1 \equiv e_2 \) and \( w_1 \sim w_2 \), or else possibility (2), which gives us \( e_1 \equiv e_2 \) and \( w_1 \leq w_2 \), hence \( e_1 \equiv e_2 \) and \( w_1 \sim w_2 \). The second case is similar.

Now, to show local connectedness, take any \( (w, e) \in (W \times E) \) and pick \( (w_1, e_1), (w_2, e_2) \) in \( V_{(w, e)} \) under \( \leq' \). By definition of \( V_{(w, e)} \) we have \( (w, e) \sim' (w_1, e_1) \) and \( (w, e) \sim' (w_2, e_2) \); by the just proved property we get \( w \sim w_1, e \equiv e_1, w \sim w_2 \).
and $e \approx e_2$; by local connectedness of $\leq$ and $\preceq$ we have $w_1 \sim w_2$ and $e_1 \approx e_2$ and then by the just proved property again we get the required $(w_1, e_1) \sim' (w_2, e_2)$.

For converse well-foundedness we proceed by contradiction. Suppose that there is an infinite ascending chain $(w_1, e_1) <' (w_2, e_2) <' \cdots$. Consider the infinite chain $e_1, e_2, \ldots$: if there is an infinite number of pairs $e_i$ and $e_{i+1}$ for which the plausibility order is strict, that is, if $e_i < e_{i+1}$ happens infinitely often, then we have an infinite ascending chain in $E$, contradicting the converse well-foundedness of $\preceq$. On the other hand, if $e_i < e_{i+1}$ only happens finitely often, then from some moment on we have $e_i \equiv e_{i+1}$. But then, from that moment on, we have $w_i < w_{i+1}$, which is an infinite ascending chain in $W$, contradicting the converse well-foundedness of $\leq$. Then, the infinite chain in $W \times E$ cannot exists, and hence $\leq'$ is conversely well-founded.

This completes the proof.