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Topological Groupoid Quantales

Abstract. We associate a canonical unital involutive quantale to a topological groupoid. When the groupoid is also étale, this association is compatible with but independent from the theory of localic étale groupoids and their quantales [9] of P. Resende. As a motivating example, we describe the connection between the quantale and the C^* -algebra that both classify Penrose tilings, which was left as an open problem in [5].

Keywords: unital involutive quantale, regular frame, topological groupoid, étale groupoid, representation theorem, Penrose tilings.

Mathematics Subject Classification: 06D05, 06D22, 06D50, 06F07, 18B40, 20L05, 22A22, 54D10, 54D30, 54D80.

1. Introduction

Groupoids, i.e. small categories such that every morphism is an iso, have been first introduced by Brandt in 1926 as algebraic structures generalizing groups, by allowing the group operation to be partially defined. Groupoids can be usefully seen as the ‘categorification’ of equivalence relations: indeed, since every morphism is an iso, any two objects joined by at least one arrow are equivalent in “as many ways” as there are arrows between them. Moreover, equivalence relations can often be meaningfully represented as the orbit equivalence relations of some nontrivial groupoids over their domains. This observation has led to important applications of groupoids in algebraic and non-commutative geometry: when an equivalence relation on a topological space induces a pathological quotient space, the equivalence relation itself can be studied as a groupoid, as was done for instance by Connes [2] with the classifying space K of Penrose tilings. The main role of groupoids in Connes’ noncommutative geometry, particularly when they are étale (see Definition 2.5 below) is their giving rise to C^* -algebras, a fact of which the space K of Penrose tilings is also an interesting example: indeed the C^* -algebra $A(K)$ associated with K seen as a groupoid classifies the Penrose tilings up to isomorphism [2] (see also [7] and [6]). Thanks to their connection with C^* -algebras, when endowed with suitable topological or localic structure, groupoids can also be regarded as noncommutative spaces.

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Finally, in algebraic logic, discrete groupoids have been used in Jónsson and Tarski's representation theorems for certain classes of relation algebras [3].

Quantales were introduced [4] as the noncommutative generalizations of locales (i.e. point-free topologies) and have been investigated in close connection with C^* -algebras, in the context of a research program aimed at developing noncommutative extensions of the Gelfand-Naimark duality (which establishes a dual equivalence of categories between unital *commutative* C^* -algebras and compact Hausdorff spaces).

In this context, the hope was that the functor Max , associating every C^* -algebra with the quantale of its closed linear subspaces, would provide the correspondence from C^* -algebras to unital involutive quantales generalizing the commutative Gelfand duality in a natural way. However, although Max is a faithful complete invariant of C^* -algebras, it does not preserve limits, so it does not have a left adjoint, and hence it is not viable for establishing a dual equivalence between C^* -algebras and quantales.

In the quest for an alternative way of establishing a duality between quantales and C^* -algebras, Penrose tilings provided again an interesting case study: Mulvey and Resende [5] associated the classifying space K of Penrose tilings with a unital involutive quantale **Pen**, defined as the Lindenbaum-Tarski algebra of a logic of finite observations on certain geometric properties of the tilings, and canonically interpreted as a certain relational quantale Pen , thanks to the fact that any relational representation of **Pen** factors through Pen . In particular this holds for any irreducible relational representation of **Pen**, which is used in [5] to show that these representations classify the Penrose tilings of the plane up to isomorphism, exactly like Connes' C^* -algebra $A(K)$ does. However the precise relation between Pen and $A(K)$ was left as an open problem in [5]. Since Pen is a different quantale than $MaxA(K)$, and both $A(K)$ and Pen arise from the same étale groupoid K , this case study suggested the possibility of an alternative correspondence between C^* -algebras and quantales using *groupoids* as intermediate structures. This line of investigation was further developed by Resende [9] who established an equivalence on objects between localic étale groupoids and *inverse quantale frames*.

The aim of our own contribution is extending Resende's correspondence to non étale topological groupoids: in this paper, any topological groupoid is associated with a unital involutive quantale (its *topological groupoid quantale*) in a way that is alternative to [9] but compatible with it¹ when

¹We will report about the comparison with the correspondence defined in [9] in a forthcoming paper.

the groupoids are étale. As a case study, we show that Pen is the topological groupoid quantale associated with the classifying space of Penrose tilings, from which fact we derive the relation between Pen and $A(K)^2$.

2. Basic definitions and examples

A *quantale* $\mathcal{Q}$ (see [4, 10]) is a complete join-semilattice endowed with an associative binary operation $\cdot$ that is completely distributive in each coordinate, i.e.

$$D1: c \cdot \bigvee I = \bigvee \{c \cdot q : q \in I\}$$

$$D2: \bigvee I \cdot c = \bigvee \{q \cdot c : q \in I\}$$

for every $c \in \mathcal{Q}$, $I \subseteq \mathcal{Q}$. Since it is a complete join-semilattice, $\mathcal{Q}$ is also a complete, hence bounded, lattice. Let $0, 1$ be the lattice bottom and top of $\mathcal{Q}$, respectively. Conditions $D1$ and $D2$ readily imply that $\cdot$ is order-preserving in both coordinates and, as $\bigvee \emptyset = 0$, that $c \cdot 0 = 0 = 0 \cdot c$ for every $c \in \mathcal{Q}$. $\mathcal{Q}$ is *unital* if there exists an element $e \in \mathcal{Q}$ for which

$$U: e \cdot c = c = c \cdot e \text{ for every } c \in \mathcal{Q},$$

and is *involutive* if it is endowed with a unary operation $*$ such that, for every $c, q \in \mathcal{Q}$ and every $I \subseteq \mathcal{Q}$,

$$I1: c^{**} = c.$$

$$I2: (c \cdot q)^* = q^* \cdot c^*.$$

$$I3: (\bigvee I)^* = \bigvee \{q^* : q \in I\}.$$

Relevant examples of unital involutive quantales are:

1. The quantale $\mathcal{P}(R)$ of subrelations of a given equivalence relation $R \subseteq X \times X$.
2. The quantale $\mathcal{P}(G)$, for every group G .
3. Any frame $\mathcal{Q}$, setting $\cdot := \wedge$, $*$:= id and $e := 1_{\mathcal{Q}}$.

DEFINITION 2.1. A *groupoid* is a tuple $\mathcal{G} = (G_0, G_1, m, d, r, u, ()^{-1})$, s.t.

G1. G_0 and G_1 are sets;

G2. $d, r : G_1 \rightarrow G_0$ and $u : G_0 \rightarrow G_1$ s.t. $d(u(p)) = p = r(u(p))$ for every $p \in G_0$;

G3. $m : (x, y) \mapsto xy$ is an associative map defined on $\{(x, y) \mid r(x) = d(y)\}$ and s.t. $d(xy) = d(x)$ and $r(xy) = r(y)$;

G4. $xu(r(x)) = x = u(d(x))x$ for every $x \in G_1$;

G5. $()^{-1} : G_1 \rightarrow G_1$ is an operator such that $xx^{-1} = u(d(x))$, $x^{-1}x = u(r(x))$, $d(x^{-1}) = r(x)$ and $r(x^{-1}) = d(x)$ for every $x \in G_1$.

²In private communication, P. Resende informed us that the relation between Pen and $A(K)$ was independently known to him and Mulvey, but was not officially communicated.

EXAMPLE 2.2.

1. For any equivalence relation $R \subseteq X \times X$, the tuple $(X, R, \circ, \pi_1, \pi_2, \Delta, ()^{-1})$ defines a groupoid. Later on we will discuss the topological version of this example.
2. For any group $(G, \cdot, e, ()^{-1})$, the tuple $(\{e\}, G, \cdot, d, r, u, ()^{-1})$ is a groupoid, and the equalities G4 and G5 just restate the group axioms.
3. The following example is a special but important case of the first one: every topological space X can be seen as a groupoid by setting $G_1 = G_0 = X$ and identity structure maps. In this case, $G_1 \times_{G_0} G_1 = \{(x, x) \mid x \in X\}$ and $xx = x$ for every $x \in X$.
4. A groupoid can be associated with any action³ $G \times X \rightarrow X$ of a group G on a set X , by setting $G_1 = G \times X$, $G_0 = X$, and for all $g, h \in G$ and $x, y \in X$, $d(g, x) = x$, $r(g, x) = gx$, $u(x) = (e, x)$ ($e \in G$ being the identity element), and $(g, x) \cdot (h, y) = (hg, x)$ iff $y = gx$.

LEMMA 2.3. For all $p \in G_0$, $x, y \in G_1$,

1. $u(p)^{-1} = u(p)$,
2. $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$,
3. if $xy^{-1}, x^{-1}y \in u[G_0]$ then $x = y$,
4. if $x = xyx$ and $xyx = y$, then $y = x^{-1}$,
5. $(x^{-1})^{-1} = x$,
6. $(xy)^{-1} = y^{-1}x^{-1}$.

PROOF. 1. G4, G5 and G2 imply that $u(p)^{-1} = u(p)^{-1}u(r(u(p)^{-1})) = u(p)^{-1}u(d(u(p))) = u(p)^{-1}u(p) = u(r(u(p))) = u(p)$.

2. By G5 and G4, $xx^{-1}x = u(d(x))x = x$. The second one is analogous.

3. Assume $xy^{-1} = u(p)$ and $x^{-1}y = u(q)$ for some $p, q \in G_0$. Then, by G2, G3 and G5, $p = d(u(p)) = d(xy^{-1}) = d(x)$ and $p = r(u(p)) = r(xy^{-1}) = r(y^{-1}) = d(y)$. Likewise, using the second part of the assumption, one shows that $r(x) = q = r(y)$. Hence, by G5, $x^{-1}x = u(r(x)) = u(q) = x^{-1}y = u(r(y)) = y^{-1}y$ and $xx^{-1} = u(d(x)) = u(p) = xy^{-1} = u(d(y)) = yy^{-1}$. By G4 and item 2 above, $x = u(d(x))xu(r(x)) = u(d(y))xx^{-1}y = u(d(y))yy^{-1}y = u(d(y))y = y$.

4. Since xyx is well defined by assumption, $r(x) = d(y)$ and $d(x) = r(y)$, so $x^{-1}x = x^{-1}xyx = u(r(x))yx = u(d(y))yx = yx = yxu(r(x)) = yxu(d(y)) = yxyy^{-1} = yy^{-1}$, and likewise $xx^{-1} = xy = y^{-1}y$, therefore $y = yy^{-1}y = x^{-1}xy = x^{-1}xx^{-1} = x^{-1}$.

5. Immediate consequence of items 2 and 4 above.

³For any group G , a (left) action of G on a set X is a function $\cdot : G \times X \rightarrow X$ s.t. for all $g, h \in G$ and $x \in X$, $(gh)x = g(hx)$ and $ex = x$ (e being the identity of G).

6. By (4), it is enough to show that $(y^{-1}x^{-1})(xy)(y^{-1}x^{-1}) = y^{-1}x^{-1}$ and $(xy)(y^{-1}x^{-1})(xy) = xy$. As $r(x) = d(y) = r(y^{-1})$, $y^{-1}x^{-1}x = y^{-1}u(r(x)) = y^{-1}u(r(y^{-1})) = y^{-1}$, so by (2), $(y^{-1}x^{-1})(xy)(y^{-1}x^{-1}) = y^{-1}yy^{-1}x^{-1} = y^{-1}x^{-1}$. ■

For every groupoid $\mathcal{G}$, $\mathcal{P}(G_1)$ can be given the structure of a unital involutive quantale (see also [9] 1.1 for a more detailed discussion): indeed, the product and involution on G_1 can be lifted to $\mathcal{P}(G_1)$ as follows:

$$S \cdot T = \{x \cdot y \mid x \in S, y \in T \text{ and } r(x) = d(y)\} \quad S^* = \{x^{-1} \mid x \in S\}.$$

Denoting by E the image of the structure map $u : G_0 \rightarrow G_1$, we get:

FACT 2.4. $\langle \mathcal{P}(G_1), \bigcup, \cdot, ()^*, E \rangle$ is a unital involutive quantale.

DEFINITION 2.5. A *topological groupoid* is a groupoid $\mathcal{G}$ such that G_0 and G_1 are topological spaces and the structure maps are continuous. A topological groupoid $\mathcal{G}$ is *étale* if $d : G_1 \rightarrow G_0$ is a local homeomorphism (d is étale).

Since $()^{-1}$ swaps the roles of d and r , if d is étale, then so is r .

3. Topological groupoid quantales

In this section we will define a general procedure that associates a unital involutive quantale with *any* topological groupoid $\mathcal{G}$. Our method is based on the following simple fact:

FACT 3.1. If $\mathcal{Q} = (Q, \bigvee, \cdot, *, e)$ is a unital involutive quantale and $S \subseteq Q$ contains e and is closed under $\cdot$ and $*$, then the sub-semilattice of $(Q, \bigvee)$ generated by S is a unital involutive subquantale of $\mathcal{Q}$.

So we will define the quantale $\mathcal{Q}(\mathcal{G})$ associated with a topological groupoid $\mathcal{G}$ as the sub join-semilattice of $\langle \mathcal{P}(G_1), \bigcup \rangle$ generated by a suitable subset $S \subseteq \mathcal{P}(G_1)$. The following definition will provide us with the building blocks for this subset:

DEFINITION 3.2. A *local section of d* is a continuous map $s : U \rightarrow G_1$ defined on some open subset U of G_0 , s.t. $d \circ s = id_U$. Such an s is a *local bisection* if $r \circ s : U \rightarrow V$ is a local homeomorphism for some open subset V of G_0 .

By G2, the structure map u is a local bisection. In the context of the topological version of Example 2.2.1, local bisections can be identified with the graphs $\Gamma_f = \{(x, f(x)) \mid x \in U\}$ of local homeomorphisms $f : U \rightarrow V$ such

that $\Gamma_f \subseteq R$ (see also the proof of Theorem 4.1.2 for an argument that applies more generally here as well). In the context of Example 2.2.2, local bisections can be identified with the elements of the group G . In Example 2.2.3, the local bisections are the identity maps $i : U \rightarrow U$ on open subsets $U \subseteq X$.

For every local bisection $s : U \rightarrow G_1$ and every $p \in U$, $d(s(p)) = p$. Therefore, s can be identified with its image $s[U]$. Images of local bisections are called *G-sets*. Let $\mathcal{S}(\mathcal{G}) \subseteq \mathcal{P}(G_1)$ be the collection of the G-sets of $\mathcal{G}$.

Besides containing $E = u[G_0]$, $\mathcal{S}(\mathcal{G})$ is also closed under product and involution on $\mathcal{P}(G_1)$. To see this, we first define composition and involution on local bisections as follows: If $s : U \rightarrow G_1$ and $t : U' \rightarrow G_1$ are local bisections, then the composition $s \cdot t$ (or just st) is defined by

$$(s \cdot t)(p) = s(p)t(r \circ s(p)) \quad (1)$$

on the open set $(r \circ s)^{-1}[U']$. Similarly, the involution s^* of a local bisection $s : U \rightarrow G_1$ is defined on the open set $V = (r \circ s)[U]$ by

$$s^*(r \circ s(p)) = s(p)^{-1}. \quad (2)$$

Again, it is easy to verify that in the context of Example 2.2.1 (resp. 2.2.2), compositions and involutions of local bisections respectively correspond to compositions⁴ (products) and inverses of the associated local homeomorphisms (elements of the group G).

LEMMA 3.3. *For all local bisections s and t ,*

1. *$s \cdot t$ is a local bisection of d .*
2. *$s \cdot s^*$ and $s^* \cdot s$ coincide with u wherever defined.*
3. *s^* is a local bisection of d .*

PROOF. 1. By (1) and G3, for every $p \in G_0$, $(r \circ (s \cdot t))(p) = r(s(p)t(r \circ s(p))) = r(t((r \circ s)(p))) = (r \circ t)((r \circ s)(p))$, hence $r \circ (s \cdot t)$ is a local homeomorphism. Moreover, by (1) and G3, $(d \circ (s \cdot t))(p) = d(s(p)t(r \circ s(p))) = d(s(p)) = p$, hence $d \circ (s \cdot t) = id$. This shows that $s \cdot t$ is a local bisection. 2. Let $U = \text{dom}(s)$. By (1), (2), G5 and $d \circ s = id_U$, $(s \cdot s^*)(p) = s(p)s^*(r \circ s(p)) = s(p)s(p)^{-1} = u(d(s(p))) = u(p)$. As for the second equality, for every q in the domain of s^* , $q = r(s(p))$ for some $p \in U$, hence $s^*(q) = s(p)^{-1}$,

⁴Notation is treacherous here: if s and t respectively correspond to the local homeomorphisms f and g , then the *algebraic* product $s \cdot t$ as defined in (1) set-theoretically corresponds to the relational composition of the graphs $\Gamma_f \circ \Gamma_g$ (and thus to the functional composition $g \circ f$).

and so, by (1), (2), G5 and $d \circ s = id_U$, $(s^* \cdot s)(q) = s^*(q)s(r(s^*(q))) = s(p)^{-1}s(r(s(p)^{-1})) = s(p)^{-1}s(d(s(p))) = s(p)^{-1}s(p) = u(r(s(p))) = u(q)$.

3. Let $p \in G_0$. By G2, G3 and item 2 above, $(r \circ s^*)[(r \circ s)(p)] = (r \circ (s^*s))(p) = r(u(p)) = p$ (The first equality is shown like in the proof of item 1). Likewise, $(r \circ s)[(r \circ s^*)(p)] = p$. This shows that $(r \circ s^*)$ is the inverse of the local homeomorphism $(r \circ s)$ and so is itself a local homeomorphism. Moreover, by (2) and G5, for every $p = r(s(q))$ in the domain of s^* , $d(s^*(p)) = d(s^*(r(s(q)))) = d(s(q)^{-1}) = r(s(q)) = p$, i.e. $d \circ s^* = id$. ■

COROLLARY 3.4. *For every local bisection $s : U \rightarrow G_1$ and $t : U' \rightarrow G_1$, let $S = s[U]$, $T = t[U']$, V be the domain of st and $W = (r \circ s)[U]$ be the domain of s^* . Then:*

1. $(st)[V] = S \cdot T = \{xy \mid x \in S, y \in T \text{ and } r(x) = d(y)\}$.
2. $s^*[W] = S^* = \{x^{-1} \mid x \in S\}$.

PROOF. 1. By (1), if $x \in (st)[V]$ then $x = s(p)t(r(s(p)))$ for some $p \in V$, hence $x \in S \cdot T$. Conversely, if $x = s(p)t(q)$ for some $p \in U$, $q \in U'$, then $r(s(p)) = d(t(q)) = q$, and so $x = s(p)t(r(s(p))) \in (st)[V]$.

2. If $x \in s^*[W]$, then $x = s^*(r(s(p))) = s(p)^{-1}$ for some $p \in U$, hence $x \in S^*$. Conversely, if $p \in U$, then $s(p)^{-1} = s^*(r(s(p)))$, which proves that $S^* \subseteq s^*[W]$. ■

The following facts (cf. [7] chapter I, Definition 2.6, Lemma 2.7 and Proposition 2.8) will be used later on:

- FACT 3.5.** 1. If G_0 is locally compact and $\mathcal{G}$ is étale, then $\mathcal{S}(\mathcal{G})$ forms a base for the topology of G_1 . Then in particular $u[G_0]$ is open.
2. If, for a topological groupoid $\mathcal{G}$, G_0 is locally compact and there exists a base of G -sets for the topology of G_1 , then $\mathcal{G}$ is étale.
3. Under the assumptions of item 2 above, every G -set is open in G_1 .

We are ready to introduce our main definition, i.e. the unital involutive quantale that we will associate with any topological groupoid $\mathcal{G}$:

DEFINITION 3.6. The *topological groupoid quantale* $\mathcal{Q}(\mathcal{G})$ associated with $\mathcal{G}$ is the sub $\bigcup$ -semilattice of $\mathcal{P}(G_1)$ generated by the collection $\mathcal{S}(\mathcal{G})$ of the G -sets. Composition and involution in $\mathcal{Q}(\mathcal{G})$ are defined as the lifted operations from G_1 . The unit $e_{\mathcal{Q}(\mathcal{G})}$ is E .

By Fact 3.1 and Lemmas 3.3 and 3.4, $\mathcal{Q}(\mathcal{G})$ is indeed a unital involutive subquantale of $\mathcal{P}(G_1)$. The three basic examples of unital involutive quantales given above can be retrieved as instances of topological groupoid quantales:

If X is a discrete space and R is an equivalence relation on X , then singletons $\{(x, y)\} \subset R$ are local bisections and so $\mathcal{Q}(\mathcal{G}) = \mathcal{P}(R)$. Similarly, if G is a group, then $\mathcal{Q}(\mathcal{G}) = \mathcal{P}(G)$. As we remarked early on, for topological spaces X seen as groupoids (cf. Example 2.2.3), local bisections are the identity maps $i : U \rightarrow U$ on open subsets. So $\mathcal{Q}(\mathcal{G})$ is the frame $\Omega(X)$.

EXAMPLE 3.7. 1. Let $(X, X \times G)$ be as in Example 2.2(4) s.t. moreover X is a locally connected topological space and G a group with the discrete topology. Then the local bisections are the locally constant maps $U \rightarrow G$ s.t. $U \subseteq X$ is an open set. Hence $\mathcal{Q}(\mathcal{G})$ coincides with the product topology on $G_1 = G \times X$ and it is obviously étale.

2. On the other hand, let $R \subseteq X \times X$ be the equivalence relation induced by the action of G , i.e. xRy iff $y = gx$ for some $g \in G$. If R is endowed with the quotient topology induced by the map $(d, r) : G \times X \rightarrow R$, defined by $(g, x) \mapsto (x, gx)$, then the first projection $\pi_1 : R \rightarrow X$ is not necessarily étale. For example, let $X = \mathbb{C}$ and $G = \{z \in \mathbb{C} \mid z^n = 1\}$ be the group of the n th roots of the unity, for $n \geq 2$. Consider the action of G on X given by the product $(z, x) \mapsto zx$. Its induced equivalence relation is $R = \{(x, y) \mid y = zx, z \in G\}$. For any $z \neq w \in G$ consider the two local bisections of the groupoid R defined respectively by $x \mapsto (x, zx)$ and $x \mapsto (x, wx)$. Their images intersect only at $(0, 0) \in R$, so $d : R \rightarrow X$ is not étale (which implies [9] that the topological groupoid quantale $\mathcal{Q}(R)$ is not a frame).

In the following section we introduce the quantale of Penrose tilings, our motivating case study.

4. The quantale associated with Penrose tilings

Let $X \subseteq 2^{\mathbb{N}_0}$ be the set of *Penrose sequences* [2, 5], i.e. the sequences $x = (x_k)_{k \in \mathbb{N}}$ such that $x_k = 1$ implies $x_{k+1} = 0$. X is a closed subset of $2^{\mathbb{N}_0}$ w.r.t. the Tychonoff topology, so it is homeomorphic to the Cantor space $2^{\mathbb{N}_0}$. For any $n \in \mathbb{N}$ and $\varepsilon \in \{0, 1\}^n$, we denote

$$X_\varepsilon = \{x \in X \mid x_k = \varepsilon_k, k = 1, \dots, n\}.$$

Note that the sets X_ε form a basis for the Tychonoff topology of X . Consider the equivalence relation R on X defined by xRy iff there exists some $n \in \mathbb{N}$ such that $x_k = y_k$ for every $k > n$. Let R_n be the subrelation of R defined by $xR_n y$ if $x_k = y_k$ for any $k > n$. Then clearly $R = \bigcup_n R_n$. The equivalence classes of R classify the isomorphism classes of the Penrose

tilings of the plane. In [5], Mulvey and Resende defined a quantale **Pen** by generators and relations and proved that its algebraically irreducible relational representations are in one-to-one correspondence with the equivalence classes of R . This quantale admits a concrete representation as a quantale Pen of subrelations of R , which is canonical, in the sense that every relational representation of **Pen** factors through Pen . Hence Pen classifies the isomorphism classes of Penrose tilings too. Pen is defined in [5] as the subquantale of $\mathcal{P}(R)$ generated by the following relations: for every $n \in \mathbb{N}$,

$$\begin{aligned} l_n &= \{(x, y) \in R \mid y_n = 0 \text{ and } x_k = y_k \text{ for any } k > n\} \\ s_n &= \{(x, y) \in R \mid y_n = 1 \text{ and } x_k = y_k \text{ for any } k > n\} \end{aligned}$$

and their inverses l_n^{-1} and s_n^{-1} . In order to study the quantale Pen and show that it fits with the general framework of the previous sections, we introduce the following special subsets of R . Let Γ_f be the graph of a local homeomorphism f of the form

$$f(\varepsilon, x_{n+1}, x_{n+2}, \dots) = (\eta, x_{n+1}, x_{n+2}, \dots), \quad (3)$$

with $\varepsilon, \eta \in \{0, 1\}^n$. Clearly, each such an f is uniquely determined by a pair of $\varepsilon, \eta \in \{0, 1\}^n$ and is associated with a unique (least) $n \in \mathbb{N}$. Sometimes we will write $f_{(n)}$ to indicate this, but most of the times we will keep the notation minimal. The identity id_X on X can be accounted for in this notation as the only $f_{(0)}$.

The following theorem is the central result of this section, and its proof does not follow straightforwardly from the theory developed in [5].

THEOREM 4.1. *The quantale Pen has the following properties:*

1. Pen is $\bigcup$ -generated by the graphs Γ_f .
2. Pen is a topology on R which makes (X, R) an étale groupoid.
3. Pen is the final topology on R generated by the chain of open subsets $R_0 \subset R_1 \subset \dots \subset R_n \subset \dots$, each of which endowed with the inherited product topology $R_n \subseteq X \times X$.
4. Pen is finer than the inherited product topology $R \subset X \times X$.
5. Pen is the topological groupoid quantale associated with (X, R) .

PROOF. 1. For a given $n \geq 1$, fix $\varepsilon, \eta \in \{0, 1\}^n$ and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\eta = (\eta_1, \dots, \eta_n)$. Let f be their associated homeomorphism, defined as in (3), and let:

$$X_i = \begin{cases} l_i^{-1} & \text{if } \varepsilon_i = 0 \\ s_i^{-1} & \text{if } \varepsilon_i = 1 \end{cases} \quad Y_i = \begin{cases} l_i & \text{if } \eta_i = 0 \\ s_i & \text{if } \eta_i = 1. \end{cases}$$

Claim. $\Gamma_f = X_1 \circ \dots \circ X_n \circ Y_n \circ \dots \circ Y_1$. In particular, $\Gamma_f \in Pen$.

The proof of this claim is by induction on n . If $n = 1$, an easy calculation from the definitions shows that

$$\begin{aligned} (x, y) \in X_1 \circ Y_1 & \text{ iff } x_1 = \varepsilon_1, y_1 = \eta_1 \text{ and } x_k = y_k \text{ for any } k > 1 \\ & \text{ iff } (x, y) \in \Gamma_f. \end{aligned}$$

If $n > 1$, then note that $\theta = X_1 \circ (X_2 \circ \dots \circ X_n \circ Y_n \circ \dots \circ Y_2) \circ Y_1$ is contained in the set of tuples (x, y) such that $x_1 = \varepsilon_1$ and $y_1 = \eta_1$: this is because $\text{dom}(\theta) \subseteq \text{dom}(X_1)$ and $\text{range}(\theta) \subseteq \text{range}(Y_1)$. Suppressing these first components, we can assume that $(x, y) \in X \times X$ and X_1 and Y_1 are the graphs of the identity map. Then we can calculate the other components $x_2, x_3, \dots, y_2, y_3, \dots$ by just considering $X_2 \circ \dots \circ X_n \circ Y_n \circ \dots \circ Y_2$, and the result follows by induction. This shows that Pen contains the quantale generated by $\bigcup \{ \Gamma_{f(n)} \mid \varepsilon, \eta \in \{0, 1\}^n \mid n \in \mathbb{N} \}$.

Conversely, s_n, l_n and their inverses are finite unions of graphs of homeomorphisms of the form (3): for example s_n is the union of the graphs $\Gamma_{f(n)}$ with $f(n)$ defined as in (3), $\varepsilon, \eta \in \{0, 1\}^n$ and $\eta_n = 1$. Hence, Pen is contained in the quantale generated by $\bigcup \{ \Gamma_{f(n)} \mid \varepsilon, \eta \in \{0, 1\}^n \mid n \in \mathbb{N} \}$.

2. In the previous item we showed that Pen is join-generated by the sets of type Γ_f . So, in order to show that Pen is a topology on R , it is sufficient to show that the sets Γ_f form the basis of a topology. Firstly, we need to show that if f and g are defined as in (3) for some ε, η and some ε', η' respectively, then $\Gamma_f \cap \Gamma_g$ is the union of sets of the same shape, i.e. of form Γ_h . Indeed, let $f(\varepsilon, x_{n+1}, x_{n+2}, \dots) = (\eta, x_{n+1}, x_{n+2}, \dots)$ and $g(\varepsilon', x_{m+1}, x_{m+2}, \dots) = (\eta', x_{m+1}, x_{m+2}, \dots)$ and let us assume $n \leq m$. Then either $\Gamma_f \cap \Gamma_g = \emptyset$, or $\varepsilon_k = \varepsilon'_k$ and $\eta_k = \eta'_k$ for any $k = 1, \dots, n$ and $\varepsilon'_k = \eta'_k$ for $k = n+1, \dots, m$. In this latter case g is a restriction of f , i.e. $\Gamma_g \subseteq \Gamma_f$. Secondly, the Γ_f cover R : let $(x, y) \in R$; if $x = y$ then $(x, y) \in \Gamma_{id_X}$, otherwise by definition, there exists some $n \geq 1$ such that $x_k = y_k$ for all $k \geq n+1$. Consider $\varepsilon = (x_1, \dots, x_n)$, $\eta = (y_1, \dots, y_n)$. Then clearly $(x, y) \in \Gamma_f$, where f is the local homeomorphism associated with ε, η . This completes the proof that Pen is a topology. Finally, the sets Γ_f are G-sets for the topological groupoid R with Pen as a topology: indeed, for $f : X_\varepsilon \rightarrow X_\eta$ defined as in (3), consider $s : X_\varepsilon \rightarrow R$ defined as $s(x) = (x, f(x))$; clearly, $d \circ s = \pi_1 \circ s = id_{X_\varepsilon}$ and $r \circ s = \pi_2 \circ s = f$ is a local homeomorphism of X ; in order to conclude the proof that s is a local bisection of R we only need to see that $s^{-1}[\Gamma_g] = \{x \mid f(x) = g(x)\}$ is an open set for any g : early on in item 2we

showed that either $\Gamma_f \cap \Gamma_g$ is empty or (w.l.o.g.) $\Gamma_g \subseteq \Gamma_f$. Thus $s^{-1}[\Gamma_g]$, which is the domain of $\Gamma_f \cap \Gamma_g$, is either empty or coincides with the domain of g , which is an open. This shows that the sets Γ_f are G -sets. By item 1. above, we know that Pen is join-generated by these G -sets. From Fact 3.5 it then follows that Pen is a topology on R which makes R an étale groupoid over X .

3. $R_0 \subset R_1 \subset \cdots R_n \subset \cdots$ is a chain of clopen inclusions: indeed, $R_n = R_{n+1} \cap \{(x, y) \mid x_{n+1} = y_{n+1}\}$ and $\{(x, y) \mid x_{n+1} = y_{n+1}\}$, which is a basic open of the Tychonoff topology on X , is also a closed set, since it is the intersection of the graphs of continuous functions on a Hausdorff space. Hence, R_n is clopen in any R_m with $m > n$.

The *final topology* is defined as the collection of subsets $S \subseteq R$ such that, for any n , $S \cap R_n$ is open in R_n (endowed with the inherited product topology). The final topology is characterized as the unique topology Ω on $R = \bigcup R_n$ such that, for every n , R_n , endowed with the inherited product topology, is an open subspace of (R, Ω) : indeed, if τ is a topology on R with the property above and $S \in \tau$, then $S \cap R_n$ is an open set of R_n for every n , hence $S \in \Omega$. Conversely, if $S \in \Omega$, then, for every n , $S \cap R_n$ is an open set of R_n , hence of τ , and so $S = S \cap R = S \cap \bigcup_{n \in \mathbb{N}} R_n = \bigcup_{n \in \mathbb{N}} (S \cap R_n) \in \tau$. In the final topology, the subspaces $R_n \subset R$ are all clopen: this follows from the definition of final topology and the fact, shown above, that R_n is clopen in any R_m with $m > n$. By definition, for every $f = f_{(n)}$ defined as in (3), $\Gamma_{f_{(n)}}$ is contained in R_n . Moreover, the collection $\{\Gamma_{f_{(n)}} \mid \varepsilon, \eta \in \{0, 1\}^n\}$ is a base of the inherited product topology of $R_n \subseteq X \times X$: indeed, for any $\varepsilon, \eta \in \{0, 1\}^n$, their associated f is s.t. $\Gamma_f = R_n \cap X_\varepsilon \times X_\eta$. Then, since $\bigcup \{\Gamma_{f_{(n)}} \mid \varepsilon, \eta \in \{0, 1\}^n\} \mid n \in \mathbb{N}\}$ is a base for the topology Pen , we get that for any n , the restriction of Pen to R_n coincides with the induced product topology $R_n \subset X \times X$. This proves that Pen coincides with the final topology on R .

4. The sets $B_{\varepsilon, \eta} = (X_\varepsilon \times X_\eta) \cap R$, for every $\varepsilon, \eta \in 2^n$ form a basis for the product topology for R . Moreover, for any n , the set $B_{\varepsilon, \eta} \cap R_n$ is open in R_n , since this latter space has the induced topology of $X \times X$. Hence by definition of the final topology, $B_{\varepsilon, \eta}$ belongs to Pen , which implies that Pen is finer than the product topology on R . It is strictly finer, since, for example, no basic open set of the form (3), being the graph of a function cannot contain any open set $B_{\varepsilon, \eta}$: hence it cannot be open in the topology induced from $X \times X$.

5. Assume that $s : U \rightarrow R$ is a local bisection, with U open in X and R with the topology given by Pen . Then its image $S \subset R$ is a G -set, and by Fact 3 in section 3, S is generated by the given basis of G -sets as in (3), that is,

$S \in \text{Pen}$. Hence Pen is the topological groupoid quantale associated with $K = (X, R)$. ■

A question left open in [5] was to characterize the relation between **Pen** and the C^* -algebra $A(K)$ that Connes associates with the space $K = (X, R)$ of Penrose tilings. We can now answer this question by saying that $\text{Pen} = \lim R_n$, i.e. Pen is the limit topology on R that Connes used to construct $A(K)$ as the completion of a space of *continuous* functions $g : R \rightarrow \mathbb{C}$. Intuitively, this means that Pen encodes the purely topological content of $A(K)$. For sake of completeness, let us now remind the reader how $A(K)$ is defined in [2]. Using the limit topology $\lim R_n$, the ring $\mathcal{C}_c(R)$ of complex-valued continuous functions with compact support is introduced, the continuity of the functions in $\mathcal{C}_c(R)$ being assumed with respect to this limit topology. Since the subspace R_n is compact for every n , then $\mathcal{C}_c(R_n) = \bigcup_n \mathcal{C}(R_n)$ and by the universal properties of the final topology of R (cf. [1], I.2.4), it is not difficult to see that $\mathcal{C}_c(R) = \bigcup_n \mathcal{C}(R_n)$. The convolution product of $f, g \in \mathcal{C}_c(R)$ is defined as

$$(f * g)(x, z) = \sum_{xRy} f(x, y)g(y, z),$$

the sum containing only finitely many non-zero summands by the hypothesis of compact support. Involution is defined as $f^*(x, y) = \overline{f(y, x)}$. With these operations $\mathcal{C}_c(R)$ becomes a C^* -algebra. The algebra $\mathcal{C}_c(R)$ admits canonical irreducible representations in the Hilbert spaces $l^2(R[x])$, $R[x]$ being the equivalence class of $x \in X$. If $v = (v_y)_{y \in R[x]} \in l^2(R[x])$ and $f \in \mathcal{C}_c(R)$, then $fv = ((fv)_z)_{z \in R[x]}$ is defined as $(fv)_z = \sum f(z, y)v_y$. Then $\mathcal{C}_c(R)$ inherits an operator norm from the given representation on $l^2(R[x])$. Then the *reduced norm* on $\mathcal{C}_c(R)$ is defined by taking the supremum of all the operator norms arising in this way from the distinct equivalence classes $R[x]$ (see [2], chapter II, section 3 and [6] pp. 105-109). The C^* -algebra $A(K)$ associated with the Penrose tilings is the norm-closure of $\mathcal{C}_c(R)$ with respect to this reduced norm. The fact that this C^* -algebra is limit of the $\mathcal{C}(R_n)$ was used in [2] to compute the group K_0 of the C^* -algebra of Penrose tilings.

References

- [1] BOURBAKI, N., *General Topology, part I*, Hermann, 1966.
- [2] CONNES, A., *Noncommutative Geometry*, Academic Press, 1994.
- [3] JÓNSSON, B., and A. TARSKI, 'Boolean Algebras with Operators, Part II', *Amer. J. of Math.* 74, 2:127–162, 1952.
- [4] MULVEY, C. J., '&', *Rend. Circ. Mat. Palermo* 12:99–104, 1986.

- [5] MULVEY, C. J., and P. RESENDE, ‘A noncommutative theory of Penrose tilings’, *Int. J. Theoret. Physics* 44:655–689, 2005.
- [6] PATERSON, A. L. T., *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progress in mathematics 170, Birkhäuser (Boston), 1999.
- [7] RENAULT, J., *A Groupoid Approach to C^* -Algebras*, Lecture Notes in Mathematics 793, Springer, 1980.
- [8] RESENDE, P., *Lectures on étale groupoids, inverse semigroups and quantales*, Lecture Notes for the GAMAP IP Meeting, Antwerp, 2006.
- [9] RESENDE, P., ‘Étale groupoids and their quantales’, *Adv. Math.* 208:147–209, 2007.
- [10] ROSENTHAL, K., *Quantales an Their Applications*, Pitman Research Notes in Math. 234, Longman, 1990.

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