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Energy Flux Positivity and Unitarity in Conformal Field Theories

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We show that in most conformal field theories the condition of the energy flux positivity, proposed by Hofman and Maldacena, is equivalent to the absence of ghosts. At finite temperature and large energy and momenta, the two-point functions of the stress energy tensor develop lightlike poles. The residues of the poles can be computed, as long as the only spin-two conserved current, which appears in the stress energy tensor operator-product expansion and acquires a nonvanishing expectation value at finite temperature, is the stress energy tensor. The condition for the residues to stay positive and the theory to remain ghost-free is equivalent to the condition of positivity of energy flux.

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Introduction.—Recently, considerable discussion was devoted to the AdS/CFT correspondence for gravitational theories with higher derivative interactions. In particular, it has been observed that conformal field theories (CFTs) dual to some of these models share an interesting property. Namely, the requirement of causal propagation of high energy modes at finite temperature [1] is equivalent to requiring the positivity of energy flux [2]. (Positivity of energy flux and its equivalence to causality has also been studied in Refs. [3–10]. See also [11] for earlier work in this direction and [12] for a check of energy flux positivity in a number of interacting superconformal theories.) The causality constraints follow from the dispersion relation in the regime where the frequency w and momentum q are much larger than the temperature T. The positivity of energy constraints involve the two- and three-point functions of the stress energy tensor, which can be determined from the singular terms in the operator-product expansion (OPE) of the stress energy tensor with itself. Hence, both sets of constraints follow from the high energy (UV) properties of the CFT, and it is natural to ask whether there is a general argument which relates them.

In this Letter, we investigate this question from the field theoretic point of view (earlier arguments for the positivity of energy flux in CFTs can be found in an interesting paper [4]). We start by considering a CFT where the only operator which takes an expectation value at finite temperature is the stress energy tensor. This condition seems natural for consistent CFTs defined by pure gravitational theories where the only degree of freedom is a massless graviton in the bulk dual to the boundary stress energy tensor. We then compute the first nontrivial finite temperature correction to the two-point function of the stress energy tensor in the regime of small temperatures $q/T, w/T \gg 1$. We show that whenever the flux positivity conditions of Ref. [2] are violated, the residue of the lightlike pole acquires a negative sign, signifying the appearance of ghosts in the spectrum and violation of unitarity.

More precisely, the positivity of energy flux constraints in a four-dimensional CFT are given by Eq. (2.38) of Ref. [2]:

$$\begin{aligned} \mathcal{C}_{\text{tensor}} &= \left(1 - \frac{t_2}{3} - \frac{t_4}{15}\right) \geq 0, \\ \mathcal{C}_{\text{vector}} &= \left(1 - \frac{t_2}{3} - \frac{t_4}{15}\right) + \frac{t_2}{2} \geq 0, \\ \mathcal{C}_{\text{scalar}} &= \left(1 - \frac{t_2}{3} - \frac{t_4}{15}\right) + \frac{2t_2}{3} + \frac{2t_4}{3} \geq 0, \end{aligned} \tag{1.1}$$

where the subscript in $\mathcal{C}$ corresponds to the properties of the state with respect to rotation around the unit vector which specifies the point on S^2 where the energy flux is measured [2]. The variables t_2 and t_4 can be expressed as functions of the parameters a, b, and c that determine the two- and three-point functions of the stress energy tensor. They are given by Eq. (C.12) in Ref. [2] [these expressions, as well as (1.1), have been generalized to six [5] and arbitrary [6,7] spacetime dimensions] and upon substitution into (1.1) give rise to

$$\begin{aligned} \mathcal{C}_{\text{tensor}} &= -\frac{5(7a + 2b - c)}{(14a - 2b - 5c)} \geq 0, \\ \mathcal{C}_{\text{vector}} &= \frac{10(16a + 5b - 4c)}{(14a - 2b - 5c)} \geq 0, \\ \mathcal{C}_{\text{scalar}} &= -\frac{45(4a + 2b - c)}{(14a - 2b - 5c)} \geq 0. \end{aligned} \tag{1.2}$$

In the next section, we compute the two-point function of stress energy tensor in the regime $w/T, q/T \gg 1$. There are three independent propagators which correspond to scalar, shear, and sound modes. We show that the residues of the lightlike poles in scalar, shear, and sound channels are equal (up to a positive numerical coefficient) to $\mathcal{C}_{\text{tensor}}$, $\mathcal{C}_{\text{vector}}$, and $\mathcal{C}_{\text{scalar}}$, respectively. This means that the energy flux positivity is related to the absence of ghosts in the spectrum. In the discussion section, we discuss our results and outline some open questions. We explain that the only case where our results can be modified involves a conserved spin-two current, other than the stress energy tensor, which appears in the stress energy tensor OPE and takes an expectation value at finite temperature. The only such case we are aware of involves a sum of decoupled CFTs.

Flux positivity and ghosts.—In this section we are going to compute the leading finite temperature contribution to the short-distance behavior of the two-point function of the stress energy tensor. We start by considering the OPE of the stress energy tensor $T_{\mu\nu}$ with itself, which takes the form [13]

$$T_{\mu\nu}(x)T_{\sigma\rho}(0) \sim \frac{C_T I_{\mu\nu,\sigma\rho}(x)}{x^{2d}} + \hat{A}_{\mu\nu\sigma\rho\alpha\beta}(x)T_{\alpha\beta}(0) + \cdots, \tag{2.1}$$

where $I_{\mu\nu,\sigma\rho}(x)$ and $\hat{A}_{\mu\nu\sigma\rho\alpha\beta}(x)$ are known functions of x which can be found in [13]. In (2.1) we keep only the composites of the stress energy tensor in the right-hand side. In principle, there can be other contributing terms, including relevant and marginal operators, but we postpone the discussion of those until the discussion section. At this stage, we would just like to comment that consistent CFTs dual to pure gravitational theories, whose operator content in this limit consists only of the stress energy tensor and its composites, should be described by (2.1). This is natural from the point of view of gravity where the sole degree of freedom is a massless graviton. Interestingly, studies of the superconformal algebra of the $\mathcal{N} = 4$ super Yang-Mills theory in the strong coupling and large N limit have also provided evidence in favor of the OPE (2.1) [14].

To compute the two-point function, one takes the expectation value of both sides in (2.1). Upon the Fourier transform to the momentum space, the first term in the right-hand side of (2.1) produces the usual Lorentz invariant (zero-temperature) two-point function

$$G_0(k) \sim k^d \log k^2, \tag{2.2}$$

where k is the energy-momentum d vector and the suppressed index structure is uniquely determined by Lorentz invariance and conformality. We will be interested in the finite temperature correction to (2.2), obtained by subtracting the zero-temperature term

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle_T = \langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle - \langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle|_{T=0}, \tag{2.3}$$

where the expectation value $\langle \ldots \rangle$ is taken at finite temperature T; consequently, $\langle \ldots \rangle_T$ denotes the finite temperature expectation value with the zero-temperature contribution subtracted. The leading contribution to $\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle_T$ in the limit $k/T \gg 1$ ($xT \ll 1$) comes from the second term in (2.1). Contributions from less singular terms [denoted by the dots in (2.1)] are suppressed by powers of T/k. Since we are interested in the small temperature limit, it is sufficient to compute (2.3) in the Euclidean signature, use the integral Fourier transform, and then Wick rotate to Minkowski space. One can presumably recover interesting subleading finite temperature effects by performing this calculation in the real time formalism.

In the following, we will restrict our attention to the $d = 4$ case, although the generalization to arbitrary d should be straightforward. We also choose the coordinates so that the spatial momentum is oriented along the x_3 direction: $k = (w, 0, 0, q)$. We start by considering the case of transversely polarized $T_{\mu\nu}$:

$$G_{12,12}(w, q)_T = \int d^4x \langle T_{12}(x)T_{12}(0)\rangle_T e^{-iwx_0 - iqx_3}. \tag{2.4}$$

The leading contribution to $\langle T_{12}(x)T_{12}(0)\rangle_T$ can be written as

$$\langle T_{12}(x)T_{12}(0)\rangle_T = \mathcal{C}T^4\left(-3\hat{A}_{121200}(x) + \sum_{i=1}^{3}\hat{A}_{1212ii}(x)\right) + \cdots, \tag{2.5}$$

where $\mathcal{C}$ is a positive real constant and the dots denote terms suppressed by powers of T/w and T/q. In deriving (2.5) we took the expectation value of (2.1) at finite temperature, subtracted the zero-temperature term, and used the nonzero expectation values of $T_{\mu\nu}$:

$$\langle T_{00}\rangle = -3\mathcal{C}T^4, \qquad \langle T_{ii}\rangle = \mathcal{C}T^4, \qquad i = 1, \ldots, 3, \tag{2.6}$$

where the familiar Lorentzian result $T_{\mu\nu} = \mathcal{C}\mathrm{diag}[3p, p, p, p]$, with p denoting the pressure, has been Wick rotated to the Euclidean signature. Hence, to compute $G_{12,12}(w, q)_T$ we need only to know $\hat{A}_{1212\alpha\beta}(x)$. Fortunately, it has already been computed: It is given by Eq. (6.38) in Ref. [13].

The first step in performing the Fourier transform (2.4) on the powerlike term in (2.5) involves integrating over the transverse coordinates x_1 and x_2. The only contributions that survive this integration come from $\hat{A}_{121200}(x)$ and $\hat{A}_{121233}(x)$. The expression for $G_T(w, q)$ takes a simple form:

$$G_{12,12}(w, q)_T = \mathcal{C}_{\text{tensor}} \int dx_0 dx_3 e^{-iwx_0 - iqx_3} \frac{x_3^2 - x_0^2}{(x_0^2 + x_3^2)^2}, \tag{2.7}$$

where we neglected an overall positive numerical factor and $\mathcal{C}_{\text{tensor}}$ is defined in the first line of (1.2). Remarkably, the correlator is proportional to the combination of parameters required to be positive by the condition of energy flux positivity. The integral in (2.7) can be evaluated by using the standard techniques and then Wick rotated to Lorentzian signature to yield

$$G_{12,12}(w,q)T = \mathcal{C}_{\text{tensor}} \frac{w^2+q^2}{w^2-q^2}. \tag{2.8}$$

There is a lightlike pole whose residue changes when $\mathcal{C}_{\text{tensor}}$ changes sign. We see that the first line in (1.1) is equivalent to the absence of ghosts in the scalar channel.

The situation is similar for the other polarizations of the stress energy tensor. Recall that generally there are three independent components in the two-point function of the stress energy tensor: scalar, shear, and sound. The corresponding correlators take the form (see, e.g., [15])

$$\begin{aligned} G_{12,12}(k) &= \frac{1}{2} G_3(w,q), \\ G_{10,13}(k) &= -\frac{1}{2}\frac{wq}{w^2-q^2} G_1(w,q), \\ G_{00,00}(k) &= \frac{2q^4}{3(w^2-q^2)^2} G_2(w,q). \end{aligned} \tag{2.9}$$

The correlator computed in (2.8) corresponds to the scalar mode $G_{12,12}(k)$. The computation in the shear channel involves

$$\langle T_{10}(x)T_{13}(0)\rangle_T = \mathcal{C}T^4\Big(-3\hat{A}_{101300}(x) + \sum_{i=1}^{3}\hat{A}_{1013ii}(x)\Big). \tag{2.10}$$

Now all $\hat{A}_{1013ii}(x)$ contribute, and after integration over x_1 and x_2, we have

$$G_{10,13}(k)_T = \int dx_0 dx_3 e^{-iwx_0 - iqx_3} \times \frac{(64a+14b-19c)x_0^3x_3 - (64a+26b-13c)x_0x_3^3}{(14a-2b-5c)(x_0^2+x_3^2)^2}, \tag{2.11}$$

where we omitted an overall positive numerical factor. We can perform the integration to obtain

$$\int dx_0 dx_3 e^{-iwx_0-iqx_3} \frac{x_0^3x_3}{(x_0^2+x_3^2)^3} = \frac{wq(13w^2-3q^2)}{(q^2+w^2)^3}. \tag{2.12}$$

Substituting (2.12), and an expression obtained from (2.12) by the simultaneous interchange ($x_0 \leftrightarrow x_3$, $w \leftrightarrow q$), into (2.11) and (2.9) and performing a Wick rotation, we obtain

$$G_1(w,q) = -\frac{(512a+130b-143c)w^2 + (512a+190b-113c)q^2}{(14a-2b-5c)(w^2-q^2)}. \tag{2.13}$$

The residue of the pole at $w^2 = q^2$ in the limit $q, w \to \infty$ is proportional to $\mathcal{C}_{\text{vector}}$, up to a positive number. The condition for the absence of ghosts in the shear channel is therefore equivalent to the second line in (1.1). Finally, we expect the last line in (1.1) to correspond to the absence of ghosts in the sound channel. This can be verified through the computation of $G_{00,00}(k)$. The part that contains the pole can be computed by using the same technique as above. This integral can be computed by using the same techniques as above. The result, up to a positive real numerical factor, is

$$G_2(w,q) = \frac{(-68a+32b+23c)q^6 + 3(52a+16b-15c)q^4w^2 + (116a-32b-35c)q^2w^4 - (76a-16b-25c)w^6}{-(14a-2b-5c)(w^2-q^2)}, \tag{2.14}$$

which implies that the residue in $G_2(w,q)$ at $w^2 = q^2$ is proportional to C_{scalar} in the limit $w, q \to \infty$ and the absence of ghosts is equivalent to the third line in (1.1).

Discussion.—We showed that the positivity of energy flux in a CFT, described by (1.1), is equivalent to the absence of ghosts. (The positivity of the residues of the poles can be shown to be equivalent to the absence of ghosts in a number of ways. For example, the corresponding Wightman functions must be positive. This statement translates into reflection positivity of the Euclidean theory.) Their positivity at finite temperature, provided the only operator with a nonvanishing expectation value which appears in the right-hand side of (2.1), is the stress energy tensor itself. (It would be interesting to understand how our work is related to the recent discussion of unitarity in the context of AdS/CFT [16].) An irrelevant operator with a nonvanishing expectation value would not alter the discussion, since its contribution would be further suppressed by positive powers of T/q and T/w. However, a marginal operator with a nonvanishing finite temperature expectation value would contribute to $G(w,q)_T$ at the same order, while the corresponding contribution from a relevant operator would dominate in the low temperature limit. We need only to consider primary operators, since the descendants involve ∂_μ (something), and their expectation values vanish in the CFT at finite temperature. First, consider a scalar operator Φ, other than the identity, with conformal dimension Δ between one and four and a nonvanishing expectation value. The Lorentz invariance of the OPE, together with the fact that $\langle\Phi\rangle \sim T^\Delta$ does not break Lorentz invariance, implies that it does not contribute to the poles in $G_i(w,q)$. (This scaling follows from T being the only dimensionful parameter in the theory.) This is because the contribution from such a scalar to $G_i(w,q)$

would be proportional to $T^{\Delta}(k^2)^{2-(\Delta/2)}$, which is nonsingular for $\Delta \leq 4$. We have also explicitly checked this by using the $TT \sim \Phi$ OPE. As evident from (2.2) there can be corrections to the scaling which go as $\log k^2$, but these do not affect the pole structure.

The only relevant operator whose expectation value could break Lorentz invariance is a vector. Rotational invariance implies that only the time component of a vector could take an expectation value. One can then use a rotation by $\theta = \pi$ in the x^0-x^1 plane, which is still a symmetry of the finite temperature theory, to deduce that the time component of a vector has to vanish as well, as long as this residual symmetry is not spontaneously broken. Hence, the only operator which can spoil the correspondence between the positivity of energy flux and the absence of ghosts is a traceless symmetric spin-two conserved current, which is not proportional to $T_{\mu\nu}$. (The trace part does not violate Lorentz invariance, while a nonconserved spin-two operator is necessarily irrelevant [17]; see also, e.g., [18] for a recent discussion.) (The arguments in this paragraph are due to J. Maldacena.)

Such an operator, which we denote by $X_{\mu\nu}$ below, generates a copy of the conformal algebra and can in principle appear in the $T_{\mu\nu}(x)T_{\alpha\beta}(0)$ OPE. The simplest (and the only known to us) example is the case of two decoupled CFTs, whose stress energy tensors are denoted by $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$. (The case of more than two decoupled CFTs is a simple generalization of this.) In this case the structure of the OPE implies that, in addition to $T_{\mu\nu} = T^{(1)}_{\mu\nu} + T^{(2)}_{\mu\nu}$, there is another linear combination of $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$ which appears in the right-hand side of the OPE (2.1). Of course, in this case it is possible to diagonalize the OPE, so that $T^{(1)}_{\mu\nu}T^{(1)}_{\mu\nu} \sim T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}T^{(2)}_{\mu\nu} \sim T^{(2)}_{\mu\nu}$.

One may wonder whether there exists some rotationally and translationally invariant state, where the only spin-two operator with a nonvanishing expectation value is $T_{\mu\nu}$, and the value of the pressure, given by $\langle T_{ii}\rangle$, is positive. In this case the results of the previous section would go through, because these were the only assumptions involved. (Of course, a finite temperature state described by the density matrix $\rho \sim e^{-H/T}$ satisfies these assumptions.) It would be interesting to see if one can find such a state, perhaps subject to the constraints discussed in Ref. [11].

In CFTs dual to pure gravitational theories, the composites of the stress energy tensor should be the only operators appearing in (2.1). It is therefore not surprising that the correspondence between the energy flux positivity and the violation of causality has been first observed in CFTs dual to Gauss-Bonnet [1–7] and Lovelock theories of gravity [8,9] with a negative cosmological constant. In these models, once the parameters of the theory are taken outside of the flux positivity region, a set of tachyonic quasinormal modes appears at finite temperature. The corresponding states in the dual CFT are stable in the limit $w/T, q/T \to \infty$ and have velocities which vary from unity to a finite number $c_g > 1$. These states form a continuum in the $w/T, q/T \to \infty$ limit. Presumably, our CFT calculation observes the lower edge of this continuum of states and predicts that these states are also ghosts, in addition to being tachyonic.

It would be interesting to verify this picture directly. Exploring consequences of our results in the theories at finite size is an interesting direction of research. The generalization of the results of this Letter to the supersymmetric case should be straightforward. One would need to consider the OPE of the R-current operator with itself. The results should be consistent with vanishing t_4 in the supersymmetric case [19].

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