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### Short-circuit logic

Bergstra, J.A.; Ponse, A.

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# Short-Circuit Logic

Jan A. Bergstra\*  
Alban Ponse

Section Theory of Computer Science, Informatics Institute, Faculty of Science  
University of Amsterdam  
Url: [www.science.uva.nl/~{janb,alban}](http://www.science.uva.nl/~{janb,alban})

## Abstract

Short-circuit evaluation denotes the semantics of propositional connectives in which the second argument is only evaluated if the first argument does not suffice to determine the value of the expression. In programming, short-circuit evaluation is widely used.

A short-circuit logic is a variant of propositional logic (PL) that can be defined by short-circuit evaluation and implies the set of consequences defined by a module SCL. The module SCL is defined using Hoare's conditional, a ternary connective comparable to if-then-else, and implies all identities that follow from four basic axioms for the conditional and can be expressed using the classical connectives for PL (e.g., axioms for associativity of conjunction and double negation shift). In the absence of side effects, short-circuit evaluation characterizes PL. However, short-circuit logic admits the possibility to model side effects. We use sequential conjunction as a primitive connective because it immediately relates to short-circuit evaluation. Sequential conjunction gives rise to many different short-circuit logics. The first extreme case is FSCL (*free short-circuit logic*), which characterizes the setting in which evaluation of each propositional variable can yield a side effect. The other extreme case is MSCL (*memorizing short-circuit logic*), the strongest (most identifying) variant we distinguish below PL. In MSCL, only static side effects can be modelled, while sequential conjunction is non-commutative. We provide sets of equations for FSCL and MSCL, and for MSCL we have a non-trivial completeness result.

Extending MSCL with one simple axiom yields SSCL (static short-circuit logic, or sequential PL), for which we also provide a completeness result. We briefly discuss two variants in between FSCL and MSCL, among which a logic that admits the contraction of propositional variables and of their negations.

*Key Words and Phrases:* Non-commutative conjunction, conditional composition, reactive valuation, sequential connective, short-circuit evaluation, side effect.

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## 1 Introduction

Propositional statements are built up from propositional variables and connectives, and possibly constants for *true* and/or *false*, for which we use the notation  $T$  and  $F$ , respectively. For brevity, a propositional variable is further called an *atom*.

Our starting point concerns *short-circuit evaluation* and *side effects*, and can be captured by the following question: Given some programming language, what is the logic that implies the equivalence of conditions, notably in if-then-else and while-do constructs and the like? In the (hypothetical) case that a programming language prohibits the use of conditions that yield side effects this underlying logic is just propositional logic (PL), but in most cases it certainly is not PL. In this paper we consider a few sequential variants of PL in which conjunction is not commutative and we shall use a fresh notation for sequential conjunction. We specify these logics in an equational style (cf. an equational basis of PL comprising  $x \vee \neg x = T$  as an axiom) because otherwise we would have to mix sequential connectives with classical connectives such as  $\leftrightarrow$ , which would obscure the presentation.

Consider the following example, taken from the MatLab<sup>1</sup> documentation about *short-circuit evaluation*<sup>2</sup> of conjunction that explains a certain form of sequential conjunction.

**Example 1** (Short-Circuit Operators). The following operators perform AND and OR operations on logical expressions containing scalar values. They are short-circuit operators in that they evaluate their second operand only when the result is not fully determined by the first operand.

Operator	Description
&&	Returns logical 1 (true) if both inputs evaluate to true, and logical 0 (false) if they do not.
	Returns logical 1 (true) if either input, or both, evaluate to true, and logical 0 (false) if they do not.

The statement shown here performs an AND of two logical terms, A and B:

A && B

If A equals zero, then the entire expression will evaluate to logical 0 (false), regardless of the value of B. Under these circumstances, there is no need to evaluate B because the result is already known. In this case, MATLAB short-circuits the statement by evaluating only the first term.

A similar case is when you OR two terms and the first term is true. Again, regardless of the value of B, the statement will evaluate to true. There is no need to evaluate the second term, and MATLAB does not do so.

Advantage of Short-Circuiting. You can use the short-circuit operators to evaluate an expression only when certain conditions are satisfied. For example, [...] this statement avoids divide-by-zero errors when b equals zero:

x = (b ~= 0) && (a/b > 18.5)

*End example.*

Clearly, conjunction as discussed in this example is not commutative: the results of a short-circuit evaluation of A && B and of B && A can be different. Furthermore, the explanation in this example (here only partly quoted) suggests that negation evaluates in the expected way. This raises the following questions: What exactly is the underlying variant of propositional logic in a setting with short-circuit evaluation? Which (equational) laws are valid, in particular if we restrict to *T* and *F* as the only possible truth values? Here is a list of more specific questions, where we write  $\neg A$  for the negation of A:

- Is the double negation shift ( $\neg\neg A = A$ ) valid?
- Are && and || associative?
- Are De Morgan's laws ( $\neg(A \ \&\& \ B) = \neg A \ || \ \neg B$  and its dual) valid?
- Is the law of the excluded middle valid, that is, does  $A \ || \ \neg A = T$  hold?
- Does  $A \ \&\& \ F = F$  hold?
- Are && and || idempotent?
- Are the sequential versions of laws for distributivity of && and || valid?
- Do && and || satisfy sequential versions of absorption, e.g.,  $(A \ || \ B) \ \&\& \ A = A$ ?

<sup>1</sup> <http://www.mathworks.com/access/helpdesk/help/techdoc/index.html>

<sup>2</sup> Other names used for short-circuit evaluation are *Minimal evaluation* and *McCarthy evaluation*.

The answer to the last question with respect to the mentioned version of absorption is NO if we allow erroneous statements: assume

$$A = ((b \sim = 0) \ \&\& \ (a/b > 18.5))$$

as in Example 1, and  $B = (a/b > 18.5)$ , and suppose  $(b \sim = 0)$  does not hold, thus  $A = F$ . Then  $B$  will be evaluated in  $(A \ || \ B) \ \&\& \ A$ , which will lead to an error, and thus  $(A \ || \ B) \ \&\& \ A \neq A$ . But also if each statement evaluates to either  $T$  or  $F$ , the answer to the last question should be NO because the evaluation of  $B$  can yield a side effect that changes the second evaluation of  $A$ . For the same reason, right-distributivity of  $\&\&$ , thus

$$(A \ || \ B) \ \&\& \ C = (A \ \&\& \ C) \ || \ (B \ \&\& \ C)$$

is not valid, and neither are the law of the excluded middle and the identity  $A \ \&\& \ F = F$ .

A *side effect* takes place in the evaluation of a propositional statement  $A$  if the evaluation of one of its atoms changes the evaluation value of one or more atoms to be subsequently evaluated in  $A$ . For example, a valuation (interpretation function)  $f$  can be such that  $f(A) = f(B) = T$ , while  $f(A \ \&\& \ B) = F$  because the side effect of  $f(A)$  is that  $f(B)$  has become  $F$ . This immediately implies that side effects and symmetric connectives do not go together. Side effects can be modeled by valuations that are defined on strings of atoms instead of only on a set of atoms. We call a side effect *static* if within the evaluation of a propositional statement the value of each atom remains fixed after its first evaluation. So, a valuation that admits static side effects is determined by its definition on the set of strings of atoms in which each atom occurs at most once. In the setting of static side effects, sequential conjunction is not commutative if at least two atoms are involved.

In order to analyze short-circuit evaluation in a systematic way, we use so-called *left-sequential* conjunction

$$x \ \underset{\circ}{\wedge} \ y$$

where the particular asymmetric notation is taken from [2]: the small circle indicates which argument must be evaluated first. As a consequence, other connectives such as *left-sequential* disjunction  $\underset{\circ}{\vee}$  and *right-sequential* conjunction  $\underset{\circ}{\wedge}$  have a straightforward notation. Thus, conjunction that is subject to short-circuit evaluation is further called left-sequential conjunction, and in  $x \ \underset{\circ}{\wedge} \ y$  it is required that first  $x$  is evaluated and if this yields  $F$  then the conjunction yields  $F$ ; if the evaluation of  $x$  yields  $T$ , then  $y$  is evaluated and determines the overall evaluation result. So  $x \ \underset{\circ}{\wedge} \ y$  satisfies the “equation”

$$x \ \underset{\circ}{\wedge} \ y = \text{if } x \text{ then } y \text{ else } F,$$

which characterizes the short-circuit evaluation of left-sequential conjunction. We use Hoare’s notation  $x \triangleleft y \triangleright z$  for *if y then x else z* and four of his equational laws that express basic identities between conditional expressions, and we define an (indirect) axiomatization SCL (abbreviating short-circuit logic) that implies all consequences that follow from these four axioms and that can be expressed using only  $T$ ,  $\neg$  and  $\underset{\circ}{\wedge}$ . As an example, axioms expressing the associativity of conjunction and the double negation shift are consequences of SCL. We formally define SCL with help of *module algebra* [3].

We define short-circuit logic in a generic way: a *short-circuit logic* is any logic that implies all consequences of SCL. We note that in each short-circuit logic the constant  $F$  and left-sequential disjunction  $\underset{\circ}{\vee}$  are definable in the expected way, and SCL implies De Morgan’s

laws for left-sequential connectives. So, the answers to the first three questions listed above are YES. We use the name *free short-circuit logic*, or briefly FSCL, for the most basic one among these logics: FSCL implies no more consequences than SCL (thus identifies “as least as possible” propositional statements) and is a reasonable short-circuit logic in the sense that realistic examples and counter-examples can be found. We do not know whether the logic FSCL has a straightforward and simple equational basis, but we come up with a list of axioms that is sound and that provides the answers to all remaining questions: all these answers are NO.

Next we propose *Memorizing short-circuit logic*, or briefly MSCL, as the variant of SCL that identifies “as much as possible” below PL. We claim that MSCL is a reasonable candidate for modeling short-circuit evaluation in a context where only static side effects occur. MSCL has a straightforward and relatively simple equational basis. We will show that in MSCL the answers to the “remaining questions” mentioned are NO ( $x \overset{\circ}{\vee} \neg x = T$ ), NO ( $x \overset{\circ}{\wedge} F = F$ ), YES (idempotency of  $\overset{\circ}{\wedge}$  and  $\overset{\circ}{\vee}$ ), SOME (distributivity laws), SOME (absorption laws).

The further contents of the paper can be summarized as follows: Hoare’s conditionals were the starting point for our work on *proposition algebra*, and both are briefly discussed in Section 2 because they are used to define SCL. In Section 3 we provide a generic definition of a short-circuit logic and we define FSCL as the least identifying short-circuit logic we distinguish. In Section 4 we define MSCL, the most identifying short-circuit logic below PL that we distinguish. In Section 5 we consider some other variants of short-circuit logic. We end the paper with some conclusions in Section 6. The three appendices A, B and C contain an example program in Perl, some detailed proofs, and a simple axiomatization of PL, respectively.

## 2 Short-circuit evaluation and proposition algebra

In this section we briefly discuss *proposition algebra* [6] as the basis we use to define short-circuit logic.

### 2.1 Hoare’s conditional and proposition algebra

In 1985, Hoare introduced in the paper [9] the ternary connective

$$x \triangleleft y \triangleright z,$$

and called this connective the *conditional*.<sup>3</sup> A more common expression for the conditional  $x \triangleleft y \triangleright z$  is

$$\text{if } y \text{ then } x \text{ else } z$$

with  $x$ ,  $y$  and  $z$  ranging over propositional statements. However, in order to reason systematically with conditionals, a notation such as  $x \triangleleft y \triangleright z$  seems indispensable.

In [9], Hoare proves that PL can be equationally characterized over the signature  $\Sigma_{\text{CP}} = \{T, F, \_ \triangleleft \_ \triangleright \_ \}$  and provides a set of elegant axioms to this end, including those in Table 1.

---

<sup>3</sup>Not to be confused with Hoare’s *conditional* introduced in his 1985 book on CSP [8] and in his well-known 1987 paper *Laws of Programming* [7] for expressions  $P \triangleleft b \triangleright Q$  with  $P$  and  $Q$  programs and  $b$  a Boolean expression; these sources do not refer to [9] that appeared in 1985.

---

$x \triangleleft T \triangleright y = x$	(CP1)
$x \triangleleft F \triangleright y = y$	(CP2)
$T \triangleleft x \triangleright F = x$	(CP3)
$x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v)$	(CP4)

---

Table 1: The set CP of axioms for proposition algebra

In our paper [6] we introduce *proposition algebra* and we define varieties of so-called *valuation algebra's*. These varieties serve the interpretation of a logic over  $\Sigma_{\text{CP}}$  by means of short-circuit evaluation: in the evaluation of  $t_1 \triangleleft t_2 \triangleright t_3$ , first  $t_2$  is evaluated, and the result of this evaluation determines further evaluation; upon  $T$ ,  $t_1$  is evaluated and determines the final evaluation result ( $t_3$  is not evaluated); upon  $F$ ,  $t_3$  is evaluated and determines the final evaluation result ( $t_1$  is not evaluated).

All varieties discussed in [6] satisfy the axioms in Table 1, while the most distinguishing variety is axiomatized by exactly these four axioms. We write CP for this set of axioms (where CP abbreviates conditional propositions) and  $=_{fr}$  (free valuation congruence) for the associated valuation congruence. Thus for each pair of closed terms  $t, t'$  over  $\Sigma_{\text{CP}}$ , i.e., terms that do not contain variables, but that of course may contain atoms (propositional variables),

$$\text{CP} \vdash t = t' \iff t =_{fr} t'.$$

In [11] it is shown that the axioms of CP are independent, and also that they are  $\omega$ -complete if the set of atoms involved contains at least two elements.

With the conditional as a primitive connective, negation can be defined by

$$\neg x = F \triangleleft x \triangleright T,$$

and the following consequences are easily derived from the extension of CP with negation:

$$\begin{aligned} F &= \neg T, \\ \neg \neg x &= x, \\ \neg(x \triangleleft y \triangleright z) &= \neg x \triangleleft y \triangleright \neg z, \\ x \triangleleft \neg y \triangleright z &= z \triangleleft y \triangleright x. \end{aligned}$$

Furthermore, left-sequential conjunction  $x \triangleleft y$  can be defined in CP by

$$x \triangleleft y = y \triangleleft x \triangleright F,$$

and left-sequential disjunction  $x \triangleright y$  can be defined by

$$\text{either } x \triangleright y = \neg(\neg x \triangleleft \neg y) \quad \text{or} \quad x \triangleright y = T \triangleleft x \triangleright y.$$

A proof of the latter inter-definability is as follows:

$$\begin{aligned}
\neg(\neg x \wp \neg y) &= F \triangleleft ((F \triangleleft y \triangleright T) \triangleleft (F \triangleleft x \triangleright T) \triangleright F) \triangleright T \\
&= F \triangleleft (F \triangleleft x \triangleright (F \triangleleft y \triangleright T)) \triangleright T \\
&= T \triangleleft x \triangleright (F \triangleleft (F \triangleleft y \triangleright T) \triangleright T) \\
&= T \triangleleft x \triangleright (T \triangleleft y \triangleright F) \\
&= T \triangleleft x \triangleright y.
\end{aligned}$$

The connectives  $\wp$  and  $\wp$  are associative and the dual of each other. For  $\wp$  a proof of this is as follows:

$$\begin{aligned}
(x \wp y) \wp z &= z \triangleleft (y \triangleleft x \triangleright F) \triangleright F \\
&= (z \triangleleft y \triangleright F) \triangleleft x \triangleright (z \triangleleft F \triangleright F) \\
&= (z \triangleleft y \triangleright F) \triangleleft x \triangleright F \\
&= x \wp (y \wp z),
\end{aligned}$$

and duality (a sequential version of De Morgan's laws) immediately follows from the first definition of  $\wp$ . Finally, observe that from CP extended with defining equations for  $\wp$  and  $\wp$  the following equations (and their duals) are derivable:

$$T \wp x = x, \quad x \wp T = x, \quad T \wp x = T,$$

in contrast to  $x \wp T = T$  (and  $x \wp F = F$ ).

A typical inequality is  $a \wp a \neq_{fr} a$  where  $a$  is an atom: an evaluation can be such that  $a \wp a$  yields  $F$ , while  $a$  yields  $T$ .

## 2.2 Memorizing CP

CP can be strengthened in various ways, among which its extension to  $CP_{mem}$  defined by adding this axiom to CP:

$$x \triangleleft y \triangleright (z \triangleleft u \triangleright (v \triangleleft y \triangleright w)) = x \triangleleft y \triangleright (z \triangleleft u \triangleright w). \quad (\text{CPmem})$$

The axiom (CPmem) expresses that the first evaluation value of  $y$  is memorized. Note that with  $u = F$  we find the *contraction law*

$$x \triangleleft y \triangleright (v \triangleleft y \triangleright w) = x \triangleleft y \triangleright w,$$

and replacing  $y$  by  $\neg y$  yields with  $x \triangleleft \neg y \triangleright z = z \triangleleft y \triangleright x$  the symmetric contraction law  $(w \triangleleft y \triangleright v) \triangleleft y \triangleright x = w \triangleleft y \triangleright x$ . A simple consequence of contraction is  $x \wp x = x$ :

$$\begin{aligned}
x \wp x &= x \triangleleft x \triangleright F \\
&= (T \triangleleft x \triangleright F) \triangleleft x \triangleright F \\
&= T \triangleleft x \triangleright F \\
&= x.
\end{aligned}$$



We use the name “memorizing CP” for the signature and axioms of  $\text{CP}_{mem}$ . In memorizing CP extended with defining equations for  $\neg$  and  $\triangleleft$ , the conditional is definable:  $\circlearrowleft$  is expressible, and

$$\begin{aligned}
(y \triangleleft x) \circlearrowleft (\neg y \triangleleft z) &= T \triangleleft (x \triangleleft y \triangleright F) \triangleright (z \triangleleft (F \triangleleft y \triangleright T) \triangleright F) \\
&= T \triangleleft (x \triangleleft y \triangleright F) \triangleright (F \triangleleft y \triangleright z) \\
&= (T \triangleleft x \triangleright (F \triangleleft y \triangleright z)) \triangleleft y \triangleright (F \triangleleft y \triangleright z) \\
&= (T \triangleleft x \triangleright F) \triangleleft y \triangleright (F \triangleleft y \triangleright z) \\
&= x \triangleleft y \triangleright z.
\end{aligned}$$

Another (equivalent) definition is  $x \triangleleft y \triangleright z = (y \circlearrowleft z) \triangleleft (\neg y \circlearrowleft x)$ . Furthermore, we will use the fact that replacing in axiom (CPmem) the variables  $y$  and/or  $u$  by their negation yields various equivalent versions of this axiom, e.g.,

$$(x \triangleleft y \triangleright (z \triangleleft u \triangleright v)) \triangleleft u \triangleright w = (x \triangleleft y \triangleright z) \triangleleft u \triangleright w. \quad (\text{CPmem}')$$

In Section 4 we provide axioms over the signature  $\{T, \neg, \triangleleft\}$  that define  $F$  and  $\circlearrowleft$ , and that constitute an equational basis for  $\text{CP}_{mem}$ .

We write  $=_{mem}$  (memorizing valuation congruence) for the valuation congruence axiomatized by  $\text{CP}_{mem}$ : in [6] we prove that for each pair of closed terms  $t, t'$  over  $\Sigma_{\text{CP}}$ ,

$$\text{CP}_{mem} \vdash t = t' \iff t =_{mem} t'.$$

From the above-mentioned consequence  $x \triangleleft x = x$  it follows that  $=_{mem}$  identifies more than  $=_{fr}$ , and a typical inequality is  $a \triangleleft b \not\equiv_{mem} b \triangleleft a$  for atoms  $a$  and  $b$ : an evaluation can be such that  $a \triangleleft b$  yields  $T$  and  $b \triangleleft a$  yields  $F$ . (In the deprecated case that propositional statements may contain at most one atom,  $\triangleleft$  is commutative in  $\text{CP}_{mem}$ , see [6, 11] for more details.)

### 2.3 Static CP

The most identifying extension to CP can be obtained by adding the axiom

$$(x \triangleleft y \triangleright z) \triangleleft u \triangleright v = (x \triangleleft u \triangleright v) \triangleleft y \triangleright (z \triangleleft u \triangleright v) \quad (\text{CPstat})$$

to  $\text{CP}_{mem}$ . The axiom (CPstat) expresses how the order of evaluation of  $u$  and  $y$  can be swapped. In [6] the set  $\text{CP}_{stat}$  of axioms is defined as the extension of CP with (CPstat) and with the contraction axiom

$$(x \triangleleft y \triangleright z) \triangleleft y \triangleright u = x \triangleleft y \triangleright u. \quad (\text{CPcontr})$$

We use the name “static CP” for the signature and axioms of  $\text{CP}_{stat}$ .

A simple consequence of  $\text{CP}_{stat}$  is  $v = v \triangleleft y \triangleright v$  (take  $u = F$  in axiom (CPstat)), and with this equation one easily derives  $x \triangleleft y = y \triangleleft x$ :

$$\begin{aligned}
x \triangleleft y &= y \triangleleft x \triangleright F \\
&= (T \triangleleft y \triangleright F) \triangleleft x \triangleright F \\
&= (T \triangleleft x \triangleright F) \triangleleft y \triangleright (F \triangleleft x \triangleright F) \\
&= x \triangleleft y \triangleright F \\
&= y \triangleleft x.
\end{aligned}$$

We write  $=_{stat}$  (static valuation congruence) for the associated valuation congruence, thus for any pair of closed terms  $t, t'$  over  $\Sigma_{CP}$ ,

$$CP_{stat} \vdash t = t' \iff t =_{stat} t'.$$

We finally note that the extension of  $CP_{stat}$  with defining equations for  $\neg$  and  $\triangleleft$ , say  $CP_{stat}^+$ , coincides with PL in the following sense:

Any propositional statement  $\phi$  expressed using  $T, \neg$ , and  $\wedge$  is derivable in PL if, and only if,  $CP_{stat}^+ \vdash \bar{\phi} = T$ , where  $\bar{\phi}$  is obtained from  $\phi$  by replacing  $\wedge$  by  $\triangleleft$ .

This immediately implies that  $=_{stat}$  coincides with the usual interpretation of PL, and thus that short-circuit evaluation in PL coincides with the usual interpretation in PL. Stated differently: in PL it is not possible to define propositions with side effects.

### 3 Free short-circuit logic: FSCL

In this section we provide a generic definition of a short-circuit logic and a definition of FSCL, the least identifying short-circuit logic we consider. In Section 3.2 we define a set of equations that is sound in FSCL and raise the question of its completeness.

#### 3.1 A generic definition of short-circuit logics

We define short-circuit logics using *Module algebra* [3]. Intuitively, a short-circuit logic is a logic that implies<sup>4</sup> all consequences that can be expressed in the signature  $\{T, \neg, \triangleleft\}$  of some CP-axiomatization. The definition below uses the export-operator  $\square$  of module algebra to define this in a precise manner, where it is assumed that CP satisfies the format of a module specification. In module algebra,  $S \square X$  is the operation that exports the signature  $S$  from module  $X$  while declaring other signature elements hidden. In this case it declares conditional composition to be an auxiliary operator.

**Definition 1.** A *short-circuit logic* is a logic that implies the consequences of the module expression

$$\begin{aligned} \text{SCL} = \{T, \neg, \triangleleft\} \square (\text{CP} \\ + \langle \neg x = F \triangleleft x \triangleright T \rangle \\ + \langle x \triangleleft y = y \triangleleft x \triangleright F \rangle). \end{aligned}$$

As an example,  $\text{SCL} \vdash \neg\neg x = x$  can be proved as follows:

$$\begin{aligned} \neg\neg x &= F \triangleleft (F \triangleleft x \triangleright T) \triangleright T \\ &= (F \triangleleft F \triangleright T) \triangleleft x \triangleright (F \triangleleft T \triangleright T) \\ &= T \triangleleft x \triangleright F \\ &= x. \end{aligned}$$

Following Definition 1, the most basic (least identifying) short-circuit logic we distinguish is the following (but see our remark on leaving out  $T$  in the exported signature in Section 6).

---

<sup>4</sup>Or, if one prefers the semantical point of view, “satisfies”.

---

$F = \neg T$	(SCL1)
$x \overset{\circ}{\vee} y = \neg(\neg x \wedge \neg y)$	(SCL2)
$\neg\neg x = x$	(SCL3)
$T \wedge x = x$	(SCL4)
$x \wedge T = x$	(SCL5)
$F \wedge x = F$	(SCL6)
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	(SCL7)
$(x \overset{\circ}{\vee} (y \wedge F)) \wedge (z \wedge F) = (\neg x \overset{\circ}{\vee} (z \wedge F)) \wedge (y \wedge F)$	(SCL8)
$(x \overset{\circ}{\vee} y) \wedge (z \overset{\circ}{\vee} T) = (x \wedge (z \overset{\circ}{\vee} T)) \overset{\circ}{\vee} (y \wedge (z \overset{\circ}{\vee} T))$	(SCL9)
$((x \wedge F) \overset{\circ}{\vee} y) \wedge z = (x \wedge F) \overset{\circ}{\vee} (y \wedge z)$	(SCL10)

---

Table 2: EqFSCL, a set of equations for FSCL

**Definition 2. FSCL (free short-circuit logic)** is the short-circuit logic that implies no other consequences than those of the module expression SCL.

### 3.2 Equations for FSCL

Although the constant  $F$  does not occur in the exported signature of SCL, we discuss FSCL using this constant to enhance readability. This is not problematic because

$$\text{CP} + \langle \neg x = F \triangleleft x \triangleright T \rangle \vdash F = \neg T,$$

so  $F$  can be used as a shorthand for  $\neg T$  in FSCL.

In Table 2 we provide equations for FSCL: equation (SCL1) defines  $F$ , and equations (SCL2) – (SCL7) are derivable in FSCL (cf. the similar equations in Section 2.1). Also, equations (SCL8) – (SCL10) are derivable in FSCL (see Proposition 1). Some comments:

- Equation (SCL8) defines a property of the mix of negation and the sequential connectives: its left-hand side states that if evaluation of  $x$  yields  $F$ , then  $y \wedge F$  is subsequently evaluated and leads to  $F$  (and  $z$  is not evaluated), which also is the case for its right-hand side; if evaluation of  $x$  yields  $T$ , then  $z \wedge F$  is evaluated next (while  $y$  is not) and leads to  $F$ , which also is the case for its right-hand side.
- Equation (SCL9) defines a restricted form of right-distributivity of  $\wedge$ , and so does equation (SCL10) (observe that  $(x \wedge F) \wedge z = x \wedge F$ ).

We use the name EqFSCL for this set of equations and we note that equations (SCL2) and (SCL3) imply sequential versions of De Morgan’s laws,<sup>5</sup> which allows us to use the duality principle.

Observe that EqFSCL does not contain the equation

$$x \wedge F = F.$$

---

<sup>5</sup>These are  $\neg(x \wedge y) = \neg x \overset{\circ}{\vee} \neg y$  and  $\neg(x \overset{\circ}{\vee} y) = \neg x \wedge \neg y$ .

This illustrates a typical property of a logic that models side effects: although it is the case that for any closed term  $t$  evaluation of  $t \triangleleft F$  yields  $F$ , the evaluation of  $t$  might also yield a side effect. However, the same side effect and evaluation result are obtained upon evaluation of  $\neg t \triangleleft F$ , and this explains a simple consequence of equation (SCL8) (take  $y = z = F$ ):

$$x \triangleleft F = \neg x \triangleleft F. \quad (\text{SCL8}^*)$$

This consequence implies another useful consequence that can be derived as follows:

$$\begin{aligned} (x \circlearrowleft y) \triangleleft F &= \neg(x \circlearrowright y) \triangleleft F \\ &= (\neg x \triangleleft \neg y) \triangleleft F \\ &= \neg x \triangleleft (\neg y \triangleleft F) \\ &= \neg x \triangleleft (y \triangleleft F). \end{aligned}$$

With  $y = T$  we find  $(x \circlearrowleft T) \triangleleft F = \neg x \triangleleft F = x \triangleleft F$ , and with the dual  $(x \triangleleft F) \circlearrowright T = x \circlearrowright T$  we derive

$$\begin{aligned} (x \circlearrowleft T) \triangleleft y &= ((x \triangleleft F) \circlearrowright T) \triangleleft y \\ &= (x \triangleleft F) \circlearrowright (T \triangleleft y) && \text{by (SCL10)} \\ &= (x \triangleleft F) \circlearrowright y. \end{aligned}$$

**Proposition 1** (Soundness). *The equations in EqFSCL (see Table 2) are derivable in FSCL.*

*Proof.* Trivial. As an example we prove equation (SCL10), where we use that  $\circlearrowleft$  can be defined in FSCL in exactly the same way as is done in Table 2 and that  $x \circlearrowleft y = \neg(\neg x \triangleleft \neg y) = T \triangleleft x \triangleright y$ , where the latter identity follows in CP extended with defining equations for  $\neg$  and  $\triangleleft$  (cf. Section 2.1):

$$\begin{aligned} ((x \triangleleft F) \circlearrowleft y) \triangleleft z &= z \triangleleft (T \triangleleft (F \triangleleft x \triangleright F) \triangleright y) \triangleright F \\ &= z \triangleleft (y \triangleleft x \triangleright y) \triangleright F \\ &= (z \triangleleft y \triangleright F) \triangleleft x \triangleright (z \triangleleft y \triangleright F) \\ &= T \triangleleft (F \triangleleft x \triangleright F) \triangleright (z \triangleleft y \triangleright F) \\ &= (x \triangleleft F) \circlearrowleft (y \triangleleft z). \end{aligned}$$

□

The programming language **Perl** [10] can be used to illustrate our claim that FSCL defines a reasonable logic because the language definition is rather liberal with respect to conditionals and satisfies all consequences of FSCL. In Perl it is allowed that conditionals (if-then-else statements) have conditions that yield side effects. In particular, assignments may occur in a condition and these always evaluate to  $T$ . Of course, assignments are *meant* to have a side effect and should perhaps not be used as propositional statements in “proper programming practice”. However, just because Perl admits assignments as propositional statements, it can be used to demonstrate that certain laws that perhaps seem reasonable, should *not* be added to FSCL, as for example these:

$$\begin{aligned} x \triangleleft x &= x, \\ x \triangleleft \neg x &= \neg x \triangleleft x. \end{aligned}$$

---

```
perl Not.pl

$x=0 (assignment)
"(($x=$x+1) && not($x=$x+1)) || $x==2" is true

$x=0 (assignment)
"(not($x=$x+1) && ($x=$x+1)) || $x==2" is false
```

---

Figure 1: A run of a Perl program `Not.pl` exemplifying  $A \ \&\& \ \text{not}(A) \neq \text{not}(A) \ \&\& \ A$

It is not hard to write Perl programs that demonstrate the non-validity of these identities. As an example, consider the run depicted in Figure 1 that shows that  $A \ \&\& \ \text{not}(A) = \text{not}(A) \ \&\& \ A$  does not hold in Perl, not even if  $A$  is an “atom”, as in the code of `Not.pl` (see Appendix A for this code).

While not having found any equations that are derivable in FSCL and not from EqFSCL, we failed to prove completeness of EqFSCL in the following sense:

$$\text{For all SCL-terms } t \text{ and } t', \quad \text{EqFSCL} \vdash t = t' \iff \text{FSCL} \vdash t = t'.$$

(Of course,  $\implies$  follows from Proposition 1.)

## 4 Memorizing short-circuit logic: MSCL

In this section we define a “most liberal” short-circuit logic in which (only static) side effects can occur and in which  $\wp$  is not commutative.

**Definition 3.** *MSCL (memorizing short-circuit logic) is the short-circuit logic that implies no other consequences than those of the module expression*

$$\begin{aligned} \{T, \neg, \wp\} \sqcap (\text{CP}_{mem} \\ + \langle \neg x = F \triangleleft x \triangleright T \rangle \\ + \langle x \wp y = y \triangleleft x \triangleright F \rangle). \end{aligned}$$

According to Definition 1, MSCL is a short-circuit logic because  $\text{CP}_{mem}$  is an axiomatic extension of CP ( $\text{CP}_{mem} = \text{CP} + \langle (\text{CP}_{mem}) \rangle$ , see Section 2.2). In Section 4.1 we provide axioms for MSCL and in Section 4.2 we prove their completeness.

### 4.1 Axioms for MSCL

In Table 3 we present a set of axioms for MSCL and we call this set EqMSCL.

Axioms (SCL1) – (SCL7) occur in EqFSCL (see Table 2) and thus need no further comment, and neither does axiom (SCL8\*). The EqFSCL-equations (SCL8) – (SCL10) are derivable from EqMSCL (see Appendix B). Some comments on the new axioms (MSCL1) – (MSCL4) (each of which is not a consequence of EqFSCL), where we use the notation  $(n)'$  for the dual version of axiom  $(n)$ :

---

$F = \neg T$	(SCL1)
$x \overset{\circ}{\vee} y = \neg(\neg x \triangleleft y)$	(SCL2)
$\neg\neg x = x$	(SCL3)
$T \triangleleft x = x$	(SCL4)
$x \triangleleft T = x$	(SCL5)
$F \triangleleft x = F$	(SCL6)
$(x \triangleleft y) \triangleleft z = x \triangleleft (y \triangleleft z)$	(SCL7)
$x \triangleleft F = \neg x \triangleleft F$	(SCL8*)
$x \triangleleft (x \overset{\circ}{\vee} y) = x$	(MSCL1)
$x \triangleleft (y \overset{\circ}{\vee} z) = (x \triangleleft y) \overset{\circ}{\vee} (x \triangleleft z)$	(MSCL2)
$(x \overset{\circ}{\vee} y) \triangleleft (\neg x \overset{\circ}{\vee} z) = (\neg x \overset{\circ}{\vee} z) \triangleleft (x \overset{\circ}{\vee} y)$	(MSCL3)
$((x \triangleleft y) \overset{\circ}{\vee} (\neg x \triangleleft z)) \triangleleft u = (x \overset{\circ}{\vee} (z \triangleleft u)) \triangleleft (\neg x \overset{\circ}{\vee} (y \triangleleft u))$	(MSCL4)

---

Table 3: EqMSCL, a set of axioms for MSCL.

- Axiom (MSCL1) defines a sequential form of absorption that implies the idempotence of  $\triangleleft$  (with (SCL5)' and  $y = F$ ) and  $\overset{\circ}{\vee}$ .
- Axiom (MSCL2) defines the left-distributivity of  $\triangleleft$ , and that of  $\overset{\circ}{\vee}$  follows by duality.
- Axiom (MSCL3) and its dual define a restricted form of commutativity of  $\triangleleft$  and  $\overset{\circ}{\vee}$ , reminiscent of the identity  $y \triangleleft x \triangleright z = z \triangleleft \neg x \triangleright y$  (cf. the identity  $y \triangleleft x \triangleright z = (x \triangleleft y) \overset{\circ}{\vee} (\neg x \triangleleft z)$  discussed in Section 2.2). We will sometimes use this identity with  $y$  and/or  $z$  equal to  $F$ , as in  $x \triangleleft (\neg x \overset{\circ}{\vee} z) = (\neg x \overset{\circ}{\vee} z) \triangleleft x$ .
- Axiom (MSCL4) is a combination of two more comprehensible equations: first, with  $u = T$  it yields

$$(x \triangleleft y) \overset{\circ}{\vee} (\neg x \triangleleft z) = (x \overset{\circ}{\vee} z) \triangleleft (\neg x \overset{\circ}{\vee} y), \quad (1)$$

which introduces another identity for  $y \triangleleft x \triangleright z$  (cf. Section 2.2). Application of (1) to the right-hand side of equation (MSCL4) reveals a restricted form of right-distributivity of  $\triangleleft$ :

$$((x \triangleleft y) \overset{\circ}{\vee} (\neg x \triangleleft z)) \triangleleft u = (x \triangleleft (y \triangleleft u)) \overset{\circ}{\vee} (\neg x \triangleleft (z \triangleleft u)), \quad (2)$$

and with  $y = x$  and  $z = \neg x$  this yields

$$(x \overset{\circ}{\vee} \neg x) \triangleleft u = (x \triangleleft u) \overset{\circ}{\vee} (\neg x \triangleleft u). \quad (3)$$

Right-distributivity is restricted in (2) in the sense that the “guards”  $x \triangleleft ..$  and  $\neg x \triangleleft ..$  must be present; if this were not the case we obtain e.g.  $(T \overset{\circ}{\vee} z) \triangleleft u = T \triangleleft u = u$ , while  $(T \triangleleft u) \overset{\circ}{\vee} (z \triangleleft u) = u \overset{\circ}{\vee} (z \triangleleft u)$  and the equation  $u = u \overset{\circ}{\vee} (z \triangleleft u)$  defines a form of absorption that is not valid (we explain this below). Conversely, axiom (MSCL4) follows from equations (1) + (2) in MSCL and can be replaced by these two equations if one prefers elegance to conciseness.

Observe that  $x \triangleleft F = F$  is not present in EqMSCL. In fact, the addition of this equation to EqMSCL yields an equational characterization of PL (in sequential notation), as we prove in Section 5.1. As discussed in the introduction, MSCL does not imply commutativity of  $x \triangleleft y$  (apart from the case of axiom (MSCL3)), but otherwise has some familiar consequences. However, the absorption law (MSCL1) and its dual are the *only* variants of absorption that occur in MSCL. The other variants are  $x \triangleleft (y \circlearrowleft x) = x$  and  $(y \circlearrowleft x) \triangleleft x = x$  and  $(x \circlearrowleft y) \triangleleft x = x$  (and their duals). The first two immediately imply  $y \circlearrowleft T = T$ , and the third one with  $x = F$  implies  $y \triangleleft F = F$ .

We end this section with a result on the soundness of MSCL.

**Proposition 2** (Soundness). *The axioms of EqMSCL (see Table 3) are derivable in MSCL.*

*Proof.* We use that  $\circlearrowleft$  can be defined in exactly the same way in MSCL as in EqMSCL (cf. the proof of Proposition 1). With respect to the soundness of EqMSCL, axiom (MSCL4), i.e.,

$$((x \triangleleft y) \circlearrowleft (\neg x \triangleleft z)) \triangleleft u = (x \circlearrowleft (z \triangleleft u)) \triangleleft (\neg x \circlearrowleft (y \triangleleft u)),$$

is the only non-trivial case. Write  $L = R$  for axiom (MSCL4), then

$$\begin{aligned} L &= u \triangleleft (T \triangleleft (y \triangleleft x \triangleright F) \triangleright (F \triangleleft x \triangleright z)) \triangleright F \\ &= u \triangleleft ((T \triangleleft y \triangleright (F \triangleleft x \triangleright z)) \triangleleft x \triangleright (F \triangleleft x \triangleright z)) \triangleright F \\ &= (u \triangleleft (T \triangleleft y \triangleright (F \triangleleft x \triangleright z)) \triangleright F) \triangleleft x \triangleright (u \triangleleft (F \triangleleft x \triangleright z) \triangleright F) \\ &= (u \triangleleft y \triangleright (u \triangleleft (F \triangleleft x \triangleright z) \triangleright F)) \triangleleft x \triangleright (F \triangleleft x \triangleright (u \triangleleft z \triangleright F)) \\ &= u \triangleleft y \triangleright (F \triangleleft x \triangleright (u \triangleleft z \triangleright F)) \triangleleft x \triangleright (u \triangleleft z \triangleright F) \\ &= (u \triangleleft y \triangleright F) \triangleleft x \triangleright (u \triangleleft z \triangleright F), \end{aligned}$$

and

$$\begin{aligned} R &= ((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft (T \triangleleft x \triangleright (u \triangleleft z \triangleright F)) \triangleright F \\ &= ((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft x \triangleright (((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft (u \triangleleft z \triangleright F) \triangleright F) \\ &= (u \triangleleft y \triangleright F) \triangleleft x \triangleright (((u \triangleleft y \triangleright F) \triangleleft x \triangleright T) \triangleleft u \triangleright F) \triangleleft z \triangleright F \\ &= (u \triangleleft y \triangleright F) \triangleleft x \triangleright (((y \triangleleft x \triangleright T) \triangleleft u \triangleright F) \triangleleft z \triangleright F) \\ &= (u \triangleleft y \triangleright F) \triangleleft x \triangleright (u \triangleleft z \triangleright F). \end{aligned}$$

□

## 4.2 A complete axiomatization of MSCL

In this section we prove that EqMSCL is a complete axiomatization of MSCL. We first derive a number of useful consequences that follow from EqMSCL, notably:

$$x \triangleleft \neg x = x \triangleleft F, \tag{4}$$

$$x \triangleleft y = x \triangleleft (\neg x \circlearrowleft y), \tag{5}$$

$$(x \triangleleft y) \circlearrowleft z = (x \triangleleft (y \circlearrowleft z)) \circlearrowleft (\neg x \triangleleft z), \tag{6}$$

$$x \triangleleft (y \circlearrowleft ((x \triangleleft z) \circlearrowleft (\neg x \triangleleft u))) = x \triangleleft (y \circlearrowleft z), \tag{7}$$

$$x \triangleleft (y \triangleleft x) = x \triangleleft y. \tag{8}$$

- Equation (4) can be derived as follows:

$$\begin{aligned}
x \triangleleft \neg x &= (x \triangleleft F) \triangleleft (\neg x \triangleleft F) \\
&= (x \triangleleft F) \triangleleft (\neg x \triangleleft F) && \text{by (1)} \\
&= (x \triangleleft F) \triangleleft (x \triangleleft F) && \text{by (SCL8*)} \\
&= x \triangleleft F,
\end{aligned}$$

and hence,  $\neg x \triangleleft x = \neg x \triangleleft \neg \neg x = \neg x \triangleleft F = x \triangleleft \neg x$ . Note that the dual of (4), thus

$$x \triangleleft \neg x = x \triangleleft T,$$

can be seen as a weak version of the law of the excluded middle.

- Equation (5), i.e.,  $x \triangleleft y = x \triangleleft (\neg x \triangleleft y)$ , can be derived as follows:

$$\begin{aligned}
x \triangleleft y &= x \triangleleft (F \triangleleft y) && \text{by (SCL4)'} \\
&= (x \triangleleft F) \triangleleft (x \triangleleft y) && \text{by (MSCL2)} \\
&= (x \triangleleft \neg x) \triangleleft (x \triangleleft y) && \text{by (4)} \\
&= x \triangleleft (\neg x \triangleleft y). && \text{by (MSCL2)}
\end{aligned}$$

Two immediate consequences of this identity are

$$x = x \triangleleft (\neg x \triangleleft T) \quad \text{and its dual} \quad x = x \triangleleft (\neg x \triangleleft F). \quad (9)$$

- Equation (6), i.e.,  $(x \triangleleft y) \triangleleft z = (x \triangleleft (y \triangleleft z)) \triangleleft (\neg x \triangleleft z)$ , can be derived as follows:

$$\begin{aligned}
(x \triangleleft y) \triangleleft z &= (x \triangleleft (\neg x \triangleleft y)) \triangleleft z && \text{by (5)} \\
&= ((x \triangleleft F) \triangleleft (\neg x \triangleleft y)) \triangleleft z \\
&= (x \triangleleft (y \triangleleft z)) \triangleleft (\neg x \triangleleft (F \triangleleft z)) && \text{by (MSCL4)'} \\
&= (x \triangleleft (y \triangleleft z)) \triangleleft (\neg x \triangleleft z).
\end{aligned}$$

- Equation (7), i.e.,  $x \triangleleft (y \triangleleft ((x \triangleleft z) \triangleleft (\neg x \triangleleft u))) = x \triangleleft (y \triangleleft z)$ , can be proved as follows:

$$\begin{aligned}
x \triangleleft (y \triangleleft ((x \triangleleft z) \triangleleft (\neg x \triangleleft u))) &= (x \triangleleft y) \triangleleft ((x \triangleleft z) \triangleleft (x \triangleleft (\neg x \triangleleft u))) \\
&= (x \triangleleft y) \triangleleft ((x \triangleleft z) \triangleleft ((x \triangleleft \neg x) \triangleleft u)) \\
&= (x \triangleleft y) \triangleleft ((x \triangleleft z) \triangleleft ((\neg x \triangleleft x) \triangleleft u)) \\
&= (x \triangleleft y) \triangleleft ((x \triangleleft z) \triangleleft (\neg x \triangleleft (x \triangleleft u))) \\
&= (x \triangleleft y) \triangleleft ((x \triangleleft z) \triangleleft (\neg x \triangleleft (x \triangleleft u))) && \text{by (1)} \\
&= (x \triangleleft y) \triangleleft (x \triangleleft (\neg x \triangleleft z)) && \text{by (MSCL1)'} \\
&= (x \triangleleft y) \triangleleft (x \triangleleft z) && \text{by (5)} \\
&= x \triangleleft (y \triangleleft z). && \text{by (MSCL2)}
\end{aligned}$$



- Equation (8), i.e.,  $x \triangleleft (y \triangleleft x) = x \triangleleft y$ , expresses a property of MSCL that one may call *memory* (like its dual  $x \triangleright (y \triangleright x) = x \triangleright y$ ). We employ (7) to prove equation (8):

$$\begin{aligned}
x \triangleleft (y \triangleleft x) &= x \triangleleft (y \triangleleft (\neg y \triangleright x)) \\
&= x \triangleleft ((\neg y \triangleright x) \triangleleft y) \\
&= (x \triangleleft (\neg y \triangleright x)) \triangleleft y \\
&= (x \triangleleft (\neg y \triangleright (x \triangleright (\neg x \triangleleft F)))) \triangleleft y && \text{by (9)} \\
&= (x \triangleleft (\neg y \triangleright ((x \triangleleft T) \triangleright (\neg x \triangleleft F)))) \triangleleft y \\
&= (x \triangleleft (\neg y \triangleright T)) \triangleleft y && \text{by (7)} \\
&= x \triangleleft ((\neg y \triangleright T) \triangleleft y) \\
&= x \triangleleft (y \triangleleft (\neg y \triangleright T)) \\
&= x \triangleleft y. && \text{by (9)}
\end{aligned}$$

Note that with  $y = T$  we find  $x \triangleleft x = x$ .

**Theorem 1** (Completeness). *For all SCL-terms  $t$  and  $t'$ ,*

$$\text{EqMSCL} \vdash t = t' \iff \text{MSCL} \vdash t = t'.$$

*Proof.* According to Proposition 2 it suffices to prove that the axioms of  $\text{CP}_{mem}$  are derivable from EqMSCL. In this proof we use  $F$  and  $\triangleright$  in the familiar way and also the definability of the conditional by  $x \triangleleft y \triangleright z = (y \triangleleft x) \triangleright (\neg y \triangleleft z)$  (see Section 2.2), and equation (1), i.e.,

$$(y \triangleleft x) \triangleright (\neg y \triangleleft z) = (y \triangleright z) \triangleleft (\neg y \triangleright x).$$

(CP1):  $x \triangleleft T \triangleright y = (T \triangleleft x) \triangleright (F \triangleleft y) = x \triangleright F = x$ .

(CP2):  $x \triangleleft F \triangleright y = (F \triangleleft x) \triangleright (T \triangleleft y) = F \triangleright y = y$ .

(CP3):  $T \triangleleft x \triangleright F = (x \triangleleft T) \triangleright (\neg x \triangleleft F) = x \triangleright (\neg x \triangleleft F) = x$  by (9).

(CP4): To derive (CP4), i.e.,  $x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v)$ , to which we further refer to by  $L = R$ , we use the identity

$$(x \triangleright (y \triangleleft z)) \triangleleft (\neg x \triangleright (u \triangleleft z)) = ((x \triangleright y) \triangleleft (\neg x \triangleright u)) \triangleleft z, \quad (10)$$

which in MSCL is an equivalent version of (MSCL4). Now  $L = (X \triangleleft x) \triangleright (\neg X \triangleleft v)$  where

$$X = (z \triangleleft y) \triangleright (\neg z \triangleleft u) = (z \triangleright u) \triangleleft (\neg z \triangleright y)$$

and thus  $\neg X = (z \triangleright \neg u) \triangleleft (\neg z \triangleright \neg y)$ . We derive

$$\begin{aligned}
L &\stackrel{(10)}{=} [(z \triangleright (u \triangleleft x)) \triangleleft (\neg z \triangleright (y \triangleleft x))] \triangleright [(z \triangleright (\neg u \triangleleft v)) \triangleleft (\neg z \triangleright (\neg y \triangleleft v))] \\
&= (z \triangleleft (y \triangleleft x)) \triangleright (\neg z \triangleleft (u \triangleleft x)) \triangleright (z \triangleleft (\neg y \triangleleft v)) \triangleright (\neg z \triangleleft (\neg u \triangleleft v)) \\
&= (z \triangleleft (y \triangleleft x)) \triangleright (z \triangleleft (\neg y \triangleleft v)) \triangleright (\neg z \triangleleft (u \triangleleft x)) \triangleright (\neg z \triangleleft (\neg u \triangleleft v)) \\
&= (z \triangleleft ((y \triangleleft x) \triangleright (\neg y \triangleleft v))) \triangleright (\neg z \triangleleft ((u \triangleleft x) \triangleright (\neg u \triangleleft v))) \\
&= R.
\end{aligned}$$

(CPmem): As argued in Section 2.2 it is sufficient to derive the axiom (CPmem'), i.e.,

$$(w \triangleleft y \triangleright (z \triangleleft x \triangleright u)) \triangleleft x \triangleright v = (w \triangleleft y \triangleright z) \triangleleft x \triangleright v,$$

say  $L = R$ . We will use equation (7), which we repeat here:

$$x \wp (y \wp ((x \wp z) \wp (\neg x \wp u))) = x \wp (y \wp z).$$

We derive

$$\begin{aligned} L &= (x \wp (w \wp y \wp (z \wp x \wp u))) \wp (\neg x \wp v) \\ &= (x \wp (y \wp (z \wp x \wp u)) \wp (\neg y \wp w)) \wp (\neg x \wp v) \\ &= (x \wp (y \wp (x \wp z) \wp (\neg x \wp u)) \wp (\neg y \wp w)) \wp (\neg x \wp v) \\ &= (x \wp (y \wp z) \wp (\neg y \wp w)) \wp (\neg x \wp v) && \text{by (7)} \\ &= R. \end{aligned}$$

□

## 5 Other short-circuit logics

In this section we consider some other variants of short-circuit logic. First we discuss a sequential variant of PL, and then two more variants that stem from axiomatizations of proposition algebra that are in between CP and  $CP_{mem}$ .

### 5.1 Static short-circuit logic: SSCL

In this section we prove that the equation  $x \wp F = F$  marks the distinguishing feature between MSCL and PL: adding this axiom to EqMSCL yields an equational characterization of PL (be it in sequential notation and defined with short-circuit evaluation).<sup>6</sup>

We write SSCL (static short-circuit logic) for the extension of MSCL obtained by adding the axiom  $F \wp x \wp F = F$  to  $CP_{mem}$ , and we write EqSSCL for the extension of EqMSCL with axiom  $x \wp F = F$ . Combining identity (4) (i.e.,  $x \wp \neg x = x \wp F$ ) with this axiom yields

$$x \wp \neg x = F.$$

First we prove that  $\text{EqSSCL} \vdash x \wp (y \wp \neg x) = F$ . The derivation is similar to the one we used to prove the memory property (8):

$$\begin{aligned} x \wp (y \wp \neg x) &= x \wp (y \wp (\neg y \wp \neg x)) \\ &= x \wp ((\neg y \wp \neg x) \wp y) \\ &= (x \wp (\neg y \wp \neg x)) \wp y \\ &= (x \wp (\neg y \wp (\neg x \wp (x \wp F)))) \wp y && \text{by (9)} \\ &= (x \wp (\neg y \wp ((\neg x \wp T) \wp (x \wp F)))) \wp y \\ &= (x \wp (\neg y \wp ((x \wp F) \wp (\neg x \wp T)))) \wp y \\ &= (x \wp (\neg y \wp F)) \wp y && \text{by (7)} \\ &= x \wp (\neg y \wp y) \\ &= F. \end{aligned}$$

---

<sup>6</sup>Formally, the axiom  $F \wp x \wp F = F$  is added to  $CP_{mem}$ .

From this equation and the memory property, commutativity of  $\triangleleft$  can be easily derived:

$$\begin{aligned}
x \triangleleft y &= (y \circlearrowleft \neg y) \triangleleft (x \triangleleft y) \\
&= (y \triangleleft (x \triangleleft y)) \circlearrowleft (\neg y \triangleleft (x \triangleleft y)) && \text{by (3)} \\
&= y \triangleleft x.
\end{aligned}$$

By duality it follows that full distributivity holds in EqSSCL, and it is not difficult to see that EqSSCL defines the mentioned variant of PL: this follows for example immediately from [12] in which equational bases for Boolean algebra are provided, and each of these bases can be easily derived from EqSSCL (we return to this point in Section 6).

## 5.2 Repetition-proof and Contractive short-circuit logic

We briefly mention two other variants of short-circuit logics which both involve explicit reference to a set  $A$  of atoms (propositional variables). Both these variants can be located in between SCL and MSCL.

In [6] we introduced  $\text{CP}_{rp}$  (repetition-proof CP) for the axioms in CP extended with these axiom schemes ( $a \in A$ ):

$$(x \triangleleft a \triangleright y) \triangleleft a \triangleright z = (x \triangleleft a \triangleright x) \triangleleft a \triangleright z, \quad (\text{CPrp1})$$

$$x \triangleleft a \triangleright (y \triangleleft a \triangleright z) = x \triangleleft a \triangleright (z \triangleleft a \triangleright z). \quad (\text{CPrp2})$$

We write  $\text{Eq}_{rp}(A)$  to denote the set of these axioms in the format of module algebra [3].

**Definition 4. RPSCL (*repetition-proof short-circuit logic*)** is the short-circuit logic that implies no other consequences than those of the module expression

$$\begin{aligned}
\{T, \neg, \triangleleft, a \mid a \in A\} \square (\text{CP} + \text{Eq}_{rp}(A) \\
+ \langle \neg x = F \triangleleft x \triangleright T \rangle \\
+ \langle x \triangleleft y = y \triangleleft x \triangleright F \rangle).
\end{aligned}$$

The equations defined by RPSCL include those that are derivable from EqFSCL (see Table 2) and

$$\begin{aligned}
a \triangleleft (a \circlearrowleft x) &= a \triangleleft (a \circlearrowleft T), \\
a \circlearrowleft (a \triangleleft x) &= a \circlearrowleft (a \triangleleft F).
\end{aligned}$$

We failed to find a convincing example for this type of short-circuit logic, and we leave it as an open question whether the extension of EqFSCL with these two equations axiomatizes RPSCL. Note that

$$\begin{aligned}
\neg a \triangleleft (\neg a \circlearrowleft x) &= \neg a \triangleleft (\neg a \circlearrowleft T), \\
\neg a \circlearrowleft (\neg a \triangleleft x) &= \neg a \circlearrowleft (\neg a \triangleleft F)
\end{aligned}$$

are derivable in RPSCL.

In [6] we also introduced  $\text{CP}_{cr}$  (contractive CP) which is defined by extending CP with these axiom schemes ( $a \in A$ ):

$$(x \triangleleft a \triangleright y) \triangleleft a \triangleright z = x \triangleleft a \triangleright z, \quad (\text{CPcr1})$$

$$x \triangleleft a \triangleright (y \triangleleft a \triangleright z) = x \triangleleft a \triangleright z. \quad (\text{CPcr2})$$

These schemes contract for each atom  $a$  respectively the  $T$ -case and the  $F$ -case, and it is easily seen that the axiom schemes (CPrp1) and (CPrp2) are derivable in  $\text{CP}_{cr}$  (so  $\text{CP}_{cr}$  is also an extension of  $\text{CP}_{rp}$ ). We write  $\text{Eq}_{cr}(A)$  to denote the set of these axioms in the format of module algebra.

**Definition 5. CSCL (*contractive short-circuit logic*)** is the short-circuit logic that implies no other consequences than those of the module expression

$$\begin{aligned} \{T, \neg, \delta, \wedge, a \mid a \in A\} \square (\text{CP} + \text{Eq}_{cr}(A) \\ + \langle \neg x = F \triangleleft x \triangleright T \rangle \\ + \langle x \delta y = y \triangleleft x \triangleright F \rangle). \end{aligned}$$

The equations defined by CSCL include those that are derivable from EqFSCL and

$$\begin{aligned} a \delta (a \vee x) &= a, \\ a \vee (a \delta x) &= a. \end{aligned}$$

It is an open question whether the extension of EqFSCL with these two equations yields an axiomatization of CSCL. Observe that the following equations are consequences in CSCL:

$$\begin{aligned} a \delta a &= a, & a \vee a &= a, \\ \neg a \delta (\neg a \vee x) &= \neg a, & \neg a \vee (\neg a \delta x) &= \neg a, \\ \neg a \delta \neg a &= \neg a, & \neg a \vee \neg a &= \neg a. \end{aligned}$$

An example that illustrates the use of CSCL concerns atoms that define manipulation of Boolean registers can be described as follows:

- Consider atoms  $\text{set}:i:j$  and  $\text{eq}:i:j$  with  $i \in \{1, \dots, n\}$  (the number of registers) and  $j \in \{T, F\}$  (the value of registers).
- An atom  $\text{set}:i:j$  can have a side effect (it sets register  $i$  to value  $j$ ) and yields upon evaluation always  $T$ .
- An atom  $\text{eq}:i:j$  has no side effect but yields upon evaluation only  $T$  if register  $i$  has value  $j$ .

Clearly, the consequences mentioned above are derivable in CSCL, but  $x \delta x = x$  is not: assume register 1 has value  $F$  and let  $t = \text{eq}:1:F \delta \text{set}:1:T$ . Then  $t$  yields  $T$  upon evaluation in this state, while  $t \delta t$  yields  $F$ .

## 6 Conclusions and future work

In our paper [6] we introduced proposition algebra using Hoare's conditional  $x \triangleleft y \triangleright z$  and the constants  $T$  and  $F$ . We distinguished various valuation congruences that are defined by means of short-circuit evaluation, and provided axiomatizations of these congruences: CP (four axioms) characterizes the least identifying valuation congruence we consider, and the extension  $\text{CP}_{mem}$  (one extra axiom) characterizes the most identifying valuation congruence below propositional logic. Various other valuation congruences in between these two and

axiomatizations thereof are also described in [6]. This paper arose by an attempt to answer the question whether the extension of  $CP_{mem}$  with  $\neg$  and  $\triangleleft$  characterizes a reasonable logic if one restricts to axioms defined over the signature  $\{T, \neg, \triangleleft, \triangle\}$  (and with  $F$  and  $\triangleleft$  being definable). After having found an axiomatization of MSCL (memorizing short-circuit logic), we distinguished FSCL (free short-circuit logic) as the most basic (least identifying) short-circuit logic, where we took CP as a point of departure. We used the module expression

$$SCL = \{T, \neg, \triangleleft\} \square (CP + \langle \neg x = F \triangleleft x \triangleright T \rangle + \langle x \triangleleft y = y \triangleleft x \triangleright F \rangle)$$

in our generic definition of short-circuit logics (Definition 1) and presented the set EqFSCL of equations that are derivable in FSCL. However, we did not prove independence and completeness of EqFSCL, although we have not found FSCL-identities that are not derivable from EqFSCL.

We used a Perl program to illustrate that FSCL is a reasonable logic. A next question is to provide a natural example that supports MSCL. A first idea is to consider only programs (say, in a Perl-like language) that allow in conditions only (comparative) tests on scalar variables (no side effects), but also special conditions that test whether a program-variable has been evaluated (initialized) before, say `eval$x` for scalar variable `x`. This combines well with the consequences of MSCL, but refutes identities that are typically not in MSCL, such as for example  $(\$x==2 \triangleleft \text{eval}\$x) = (\text{eval}\$x \triangleleft \$x==2)$  where the left-hand side always yields `T`, while the right-hand side can yield `F`.

The extension of  $CP_{mem}$  with the axiom  $F \triangleleft x \triangleright F = x$  that defines SSCL (static short-circuit logic) yields an axiomatization of static CP (a sequential variant of PL) that is more elegant than the one provided in Section 2.3 and in [6]: it is a simple exercise to derive the axiom (CPstat) using the expressibility of conditional composition and commutativity of  $\triangleleft$  and  $\triangleleft$  (and hence full distributivity). We say a little more about this matter in Appendix C.

In [6], more variants in between CP and  $CP_{mem}$  are distinguished, and thus more short-circuit logics can be defined. In this paper we defined RPSCL (repetition-proof short-circuit logic) and CSCL (contractive short-circuit logic), and listed some obvious equations for the associated extensions of EqFSCL, but leave their completeness open, and we provided a small example on the use of CSCL.

The focus on left-sequential conjunction that is typical for this paper leads to the following considerations. First, the valuation congruences that we consider can be axiomatized in an incremental way: the axiom systems  $CP_{rp}$  up to and including  $CP_{stat}$  all share the axioms of CP and each next system can be defined by the addition of either one or two axioms, in most cases making previously added axiom(s) redundant (the last inter-derivability  $\dashv\vdash$  is explained in Section 6):

$$\begin{aligned} CP_{rp} &= CP + (CPrp1) + (CPrp2), \\ CP_{cr} &\dashv\vdash CP_{rp} + (CPcr1) + (CPcr2), \\ CP_{mem} &\dashv\vdash CP_{cr} + (CPmem), \\ CP_{stat} &\dashv\vdash CP_{mem} + \langle F \triangleleft x \triangleright F = F \rangle. \end{aligned}$$

(In [6] we also define *weakly memorizing valuation congruence* that has an axiomatization with the same property in between  $CP_{cr}$  and  $CP_{mem}$ .) Secondly, the focus on left-sequential conjunction led us to a new, elegant axiomatization of  $CP_{stat}$  (see Appendix C) consisting of five axioms that are simpler than those of  $CP_{stat}$  (which stem from the eleven axioms in

Hoare’s paper [9]). Note that “simple” is not an easy qualification in this case: having (CP1) and (CP2) available, axioms  $L_1 = R_1$  and  $L_2 = R_2$  can be combined using a fresh variable, say  $u$ , to a single axiom  $L_1 \triangleleft u \triangleright L_2 = R_1 \triangleleft u \triangleright R_2$ . So,  $CP_{stat}$  could have been defined as the extension of CP with a single (and ugly) axiom that combines (CPstat) and (CPcontr).

We note that a perhaps interesting variant of SCL is obtained by leaving out the constant  $T$  in the exported signature (and thus also leaving out  $F$  as a definable constant). This weakened variant of SCL can be motivated by the fact that these constants are usually absent in conditions in imperative programming (although they may be always used; also  $T$  can be mimicked by a void equality test like  $(1 == 1)$  in Perl). Observe that under this restriction only the EqFSCL-equations (SCL2), (SCL3) and (SCL7) remain (expressing duality, double negation shift, and associativity of  $\triangleleft$ ).

Future work: in [1] a connection is proposed between short-circuit logic and instruction sequences as studied in program algebra [4, 5]. In particular, the relation between non-atomic tests (i.e., test instructions involving sequential connectives) and their decomposition in atomic tests and jumps is considered, evolving into a discussion concerning the length of instruction sequences and their minimization. Also, the paper [1] contains a discussion about a classification of side effects, derived from a classification of atoms (thus partitioning  $A$ ), and a consideration about “real time” atoms, such as  $a \equiv (\text{height} > 3000 \text{ meter})$ .

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## A A Perl program

The Perl program `Not.pl`:

```

my $x = 0;
print "\n $x=$x (assignment)\n";

if ( (($x=$x+1) && not($x=$x+1)) || $x==2 )
  {print " \"(\($x=\$x+1) && not(\($x=\$x+1)) || \$x==2\" is true \n\n";}
else
  {print " \" \" is false \n\n";}

$x = 0;
print " \$x=$x (assignment)\n";

if ( (not($x=$x+1) && ($x=$x+1)) || $x==2 )
  {print " \" \" is true \n";}
else
  {print " \"(not(\($x=\$x+1) && (\($x=\$x+1)) || \$x==2\" is false \n";}

```

## B Some proofs

We prove that the EqFSCL-equations (SCL8)—(SCL10) can be derived from EqMSCL.

- Equation (SCL8) can be derived as follows, where we often use (SCL8\*):

$$\begin{aligned}
(x \overset{\circ}{\vee} (y \underset{\circ}{\wedge} F)) \underset{\circ}{\wedge} (z \underset{\circ}{\wedge} F) &= ((x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} (x \overset{\circ}{\vee} F)) \underset{\circ}{\wedge} (z \underset{\circ}{\wedge} F) \\
&= ((x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} x) \underset{\circ}{\wedge} (\neg z \underset{\circ}{\wedge} F) \\
&= (x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} (x \underset{\circ}{\wedge} (\neg z \underset{\circ}{\wedge} F)) \\
&= (x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} ((x \underset{\circ}{\wedge} \neg z) \underset{\circ}{\wedge} F) \\
&= (x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} ((\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} F) \\
&= ((x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} (\neg x \overset{\circ}{\vee} z)) \underset{\circ}{\wedge} F \\
&= ((\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} (x \overset{\circ}{\vee} y)) \underset{\circ}{\wedge} F \\
&= (\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} ((x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} F) \\
&= (\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} ((\neg x \underset{\circ}{\wedge} \neg y) \underset{\circ}{\wedge} F) \\
&= (\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} (\neg x \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} F)) \\
&= ((\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} \neg x) \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} F) \\
&= ((\neg x \overset{\circ}{\vee} z) \underset{\circ}{\wedge} (\neg x \overset{\circ}{\vee} F)) \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} F) \\
&= (\neg x \overset{\circ}{\vee} (z \underset{\circ}{\wedge} F)) \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} F).
\end{aligned}$$

- Equation (SCL9) can be derived as follows, where we use equation (6), i.e.,

$$(x \underset{\circ}{\wedge} y) \overset{\circ}{\vee} z = (x \underset{\circ}{\wedge} (y \overset{\circ}{\vee} z)) \overset{\circ}{\vee} (\neg x \underset{\circ}{\wedge} z)$$

and its dual, and also the identity  $\underline{(z \overset{\circ}{\vee} T)} = \underline{(z \overset{\circ}{\vee} T)} \overset{\circ}{\vee} u$ :

$$\begin{aligned}
(x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T) &= (x \overset{\circ}{\vee} (y \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T))) \underset{\circ}{\wedge} (\neg x \overset{\circ}{\vee} (z \overset{\circ}{\vee} T)) && \text{by (6)'} \\
&= (x \underset{\circ}{\wedge} \underline{(z \overset{\circ}{\vee} T)}) \overset{\circ}{\vee} (\neg x \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T))) \\
&= (x \underset{\circ}{\wedge} [\underline{(z \overset{\circ}{\vee} T)} \overset{\circ}{\vee} (y \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T))]) \overset{\circ}{\vee} (\neg x \underset{\circ}{\wedge} (y \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T))) \\
&= (x \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T)) \overset{\circ}{\vee} (y \underset{\circ}{\wedge} (z \overset{\circ}{\vee} T)). && \text{by (6)}
\end{aligned}$$

- Equation (SCL10) can be derived as follows:

$$\begin{aligned}
((x \underset{\circ}{\wedge} F) \overset{\circ}{\vee} y) \underset{\circ}{\wedge} z &= ((x \underset{\circ}{\wedge} \neg x) \overset{\circ}{\vee} y) \underset{\circ}{\wedge} z \\
&= ((x \overset{\circ}{\vee} y) \underset{\circ}{\wedge} (\neg x \overset{\circ}{\vee} y)) \underset{\circ}{\wedge} z \\
&= ((x \underset{\circ}{\wedge} y) \overset{\circ}{\vee} (\neg x \underset{\circ}{\wedge} y)) \underset{\circ}{\wedge} z \\
&= (x \overset{\circ}{\vee} (y \underset{\circ}{\wedge} z)) \underset{\circ}{\wedge} (\neg x \overset{\circ}{\vee} (y \underset{\circ}{\wedge} z)) && \text{by (MSCL4)} \\
&= (x \underset{\circ}{\wedge} \neg x) \overset{\circ}{\vee} (y \underset{\circ}{\wedge} z) \\
&= (x \underset{\circ}{\wedge} F) \overset{\circ}{\vee} (y \underset{\circ}{\wedge} z).
\end{aligned}$$



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$x \triangleleft T \triangleright y = x$	(CP1)
$x \triangleleft F \triangleright y = y$	(CP2)
$T \triangleleft x \triangleright y = T \triangleleft y \triangleright x$	(CP3*)
$x \triangleleft (y \triangleleft z \triangleright u) \triangleright v = (x \triangleleft y \triangleright v) \triangleleft z \triangleright (x \triangleleft u \triangleright v)$	(CP4)
$(x \triangleleft y \triangleright z) \triangleleft y \triangleright F = x \triangleleft y \triangleright F$	(CP5)

---

Table 4:  $\text{CP}_{stat}^*$ , a set of axioms for static CP

## C Simple axioms for static CP

A very simple axiomatization of static CP (see Section 2.3) is provided in Table 4 and we write  $\text{CP}_{stat}^*$  for this set of axioms.

Compared to CP, axiom (CP3\*) is a strengthening of axiom (CP3) (i.e.,  $T \triangleleft x \triangleright F = x$ ) that also implies  $x \overset{\circ}{\vee} y = y \overset{\circ}{\vee} x$  and thus  $x \overset{\circ}{\vee} T = T \overset{\circ}{\vee} x = T$ . Axiom (CP5) implies the absorption law  $x \triangleleft (x \overset{\circ}{\vee} y) = x$ , and also  $x \triangleleft \neg x = F \triangleleft x \triangleright F = x \triangleleft F$ .

Observe that the duality principle is derivable in  $\text{CP}_{stat}^*$  (because it is derivable in CP, cf. Section 2.1), so  $x \triangleleft y = y \triangleleft x$ , and thus  $x \triangleleft \neg x = x \triangleleft F = F \triangleleft x = F$ . Furthermore, distributivity is derivable in  $\text{CP}_{stat}^*$ :

$$\begin{aligned}
x \overset{\circ}{\vee} (y \triangleleft z) &= T \triangleleft x \triangleright (z \triangleleft y \triangleright F) \\
&= T \triangleleft (z \triangleleft y \triangleright F) \triangleright x && \text{by (CP3*)} \\
&= (T \triangleleft z \triangleright x) \triangleleft y \triangleright x && \text{by (CP4) and (CP2)} \\
&= (T \triangleleft x \triangleright z) \triangleleft y \triangleright (T \triangleleft x \triangleright F) && \text{by (CP3*)} \\
&= (T \triangleleft x \triangleright z) \triangleleft y \triangleright ((T \triangleleft x \triangleright z) \triangleleft x \triangleright F) && \text{by (CP5)} \\
&= (T \triangleleft x \triangleright z) \triangleleft (T \triangleleft y \triangleright x) \triangleright F && \text{by (CP4) and (CP1)} \\
&= (T \triangleleft x \triangleright z) \triangleleft (T \triangleleft x \triangleright y) \triangleright F && \text{by (CP3*)} \\
&= (x \overset{\circ}{\vee} y) \triangleleft (x \overset{\circ}{\vee} z),
\end{aligned}$$

and by duality, also  $x \triangleleft (y \overset{\circ}{\vee} z) = (x \triangleleft y) \overset{\circ}{\vee} (x \triangleleft z)$  is derivable in  $\text{CP}_{stat}^*$ .

Using the translation  $x \triangleleft y \triangleright z = (y \triangleleft x) \overset{\circ}{\vee} (\neg y \triangleleft z)$ , it is a simple exercise to derive the  $\text{CP}_{stat}$ -axioms (CPstat) and (CPcontr) in  $\text{CP}_{stat}^*$ :

$$\begin{aligned}
(x \triangleleft u \triangleright v) \triangleleft y \triangleright (z \triangleleft u \triangleright v) &= (y \triangleleft [(u \triangleleft x) \overset{\circ}{\vee} (\neg u \triangleleft v)]) \overset{\circ}{\vee} (\neg y \triangleleft [(u \triangleleft z) \overset{\circ}{\vee} (\neg u \triangleleft v)]) \\
&= (y \triangleleft u \triangleleft x) \overset{\circ}{\vee} (y \triangleleft \neg u \triangleleft v) \overset{\circ}{\vee} (\neg y \triangleleft u \triangleleft z) \overset{\circ}{\vee} (\neg y \triangleleft \neg u \triangleleft v) \\
&= (y \triangleleft u \triangleleft x) \overset{\circ}{\vee} ((y \overset{\circ}{\vee} \neg y) \triangleleft \neg u \triangleleft v) \overset{\circ}{\vee} (\neg y \triangleleft u \triangleleft z) \\
&= (u \triangleleft [(y \triangleleft x) \overset{\circ}{\vee} (\neg y \triangleleft z)]) \overset{\circ}{\vee} (\neg u \triangleleft v) \\
&= (x \triangleleft y \triangleright z) \triangleleft u \triangleright v,
\end{aligned}$$

and

$$\begin{aligned}
(x \triangleleft y \triangleright z) \triangleleft y \triangleright u &= (y \triangleleft [(y \wedge x) \wp \vee (\neg y \wedge z)]) \wp \vee (\neg y \wedge u) \\
&= (y \triangleleft y \wedge x) \wp \vee (y \triangleleft \neg y \wedge z) \wp \vee (\neg y \wedge u) \\
&= (y \triangleleft x) \wp \vee (\neg y \wedge u) \\
&= x \triangleleft y \triangleright u.
\end{aligned}$$

Hence,  $\text{CP}_{stat}^*$  is a complete axiomatization of static CP. In [11] it is proved that  $\text{CP}_{stat}$  is  $\omega$ -complete, so  $\text{CP}_{stat}^*$  is  $\omega$ -complete as well.

Next, we show that the axioms of  $\text{CP}_{stat}^*$  are independent. Inspired by [11] we provide the following independence-models, where  $P, Q$  and  $R$  range over closed terms, and where  $a$  and  $b$  are two atoms and  $\phi$  is the interpretation function. The domain of the first two independence-models is  $\{T, F\}$  and the interpretation refers to classical propositional logic.

1. The model defined by  $\phi(T) = F$ ,  $\phi(F) = \phi(a) = \phi(b) = T$  and  $\phi(P \triangleleft Q \triangleright R) = \phi(Q) \wedge \phi(R)$  satisfies all axioms but (CP1).
2. The model defined by  $\phi(T) = \phi(a) = \phi(b) = T$ ,  $\phi(F) = F$  and  $\phi(P \triangleleft Q \triangleright R) = \phi(P)$  satisfies all axioms but (CP2).
3. Any model for  $\text{CP}_{mem}$  that does not satisfy  $a \wp \vee b = b \wp \vee a$ , satisfies all axioms but (CP3\*). Such models exist and are discussed in [6].
4. The model with the natural numbers as its domain, and the interpretation defined by

$$\begin{aligned}
\phi(T) &= 0, \quad \phi(F) = 1, \quad \phi(a) = 2, \quad \phi(b) = 3, \\
\phi(P \triangleleft Q \triangleright R) &= \begin{cases} \phi(P) & \text{if } \phi(Q) = 0, \\ \phi(R) & \text{if } \phi(Q) = 1, \\ \phi(Q) \cdot \phi(R) & \text{otherwise,} \end{cases}
\end{aligned}$$

satisfies all axioms but (CP4):  $\phi(F \triangleleft a \triangleright T) = \phi(a) \cdot \phi(T) = 0$ , so

$$\phi(F \triangleleft (F \triangleleft a \triangleright T) \triangleright T) = \phi(F) = 1,$$

while  $\phi((F \triangleleft F \triangleright T) \triangleleft a \triangleright (F \triangleleft T \triangleright T)) = \phi(a) \cdot \phi(F) = 2$ .

5. The model with the integers numbers as its domain, and the interpretation defined by

$$\begin{aligned}
\phi(T) &= 0, \quad \phi(F) = 1, \quad \phi(a) = 2, \quad \phi(b) = 3, \\
\phi(P \triangleleft Q \triangleright R) &= (1 - \phi(Q)) \cdot \phi(P) + \phi(Q) \cdot \phi(R),
\end{aligned}$$

satisfies all axioms but (CP5):  $\phi(T \triangleleft a \triangleright F) = 2$ , while

$$\phi((T \triangleleft a \triangleright F) \triangleleft a \triangleright F) = (1 - 2) \cdot 2 + 2 \cdot 1 = 0.$$

A proof of the validity of axioms (CP3\*) and (CP4) is a simple arithmetical exercise.

Finally, we note that  $\text{CP}_{stat}^*$  has an elegant, symmetric nature: exchanging (CP3\*) with  $x \triangleleft y \triangleright F = y \triangleleft x \triangleright F$  and (CP5) with  $T \triangleleft x \triangleright (y \triangleleft x \triangleright z) = T \triangleleft x \triangleright z$  yields an equally strong axiomatization. Moreover, the resulting axioms are also independent (modifying the independence proofs given above is nothing more than a nice exercise).