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SPARSE TENSOR PRODUCT WAVELET APPROXIMATION OF SINGULAR FUNCTIONS*

MONIQUE DAUGE[†] AND ROB STEVENSON[‡]

Abstract. On product domains, sparse-grid approximation yields optimal, dimension-independent convergence rates when the function that is approximated has L^2 -bounded mixed derivatives of a sufficiently high order. We show that the solution of Poisson’s equation on the n -dimensional hypercube with Dirichlet boundary conditions and smooth right-hand side generally does not satisfy this condition. As suggested by P.-A. Nitsche in [*Constr. Approx.*, 21 (2005), pp. 63–81], the regularity conditions can be relaxed to corresponding ones in weighted L^2 spaces when the sparse-grid approach is combined with local refinement of the set of one-dimensional wavelet indices towards the end points. In this paper, we prove that for general smooth right-hand sides, the solution of Poisson’s problem satisfies these relaxed regularity conditions in any space dimension. Furthermore, since we remove log-factors from the energy-error estimates from Nitsche’s work, we show that in any space dimension, locally refined sparse-grid approximation yields the optimal, dimension-independent convergence rate.

Key words. wavelets, sparse grids, weighted anisotropic Sobolev spaces, regularity estimates, tensor products, local refinements

AMS subject classifications. 65T60, 35J25, 41A63, 65N99, 42C40

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1. Introduction. For standard, isotropic piecewise polynomial approximation of order d on an n -dimensional domain $\Omega \subset \mathbb{R}^n$, it is well known that the error, measured in $H^m(\Omega)$, in the best approximation of a sufficiently smooth function u is generally of order $N^{-\frac{d-m}{n}}$, where N denotes the dimension of the approximation space. The fact that the rate $\frac{d-m}{n}$ is inversely proportional to n is known as the *curse of dimensionality*. When working on a product domain, this curse can be overcome by considering sparse-grid approximation [Zen91, BG04], also known as hyperbolic wavelet approximation [DKT98]. With this type of approximation and $m \geq 0$, the error is of order $N^{-(d-m)}$, up to some log-factors, that for $m > 0$ can even be removed by considering optimized sparse grids [GK00].

At this point we note that, in this paper, with an error of order N^{-s} , we mean that the error can be bounded by a multiple of N^{-s} , where this multiple is independent of N but may depend on the space dimension n . Because of the latter, having shown that a convergence *rate* s is independent of n does not mean that the problem is *tractable*, that is, that the work for obtaining an error less than some tolerance ε may not grow exponentially in n . We refer to [NW08, Wer96] and the references cited there for results dealing with (in)tractability of approximations in high space dimensions.

A sparse grid is based on a tensorized one-dimensional multiscale decomposition. In the literature, hierarchical bases, also known as Faber bases, are often used. In this paper, we restrict ourselves to “true” one-dimensional wavelet decompositions, in

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[†]IRMAR, Institut Mathématique, Université de Rennes 1, 35042 Rennes, France (monique.dauge@univ-rennes1.fr).

[‡]Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands (R.P.Stevenson@uva.nl).

the sense that, properly scaled, they will generate Riesz bases for a range of Sobolev spaces including L^2 .

In order that the aforementioned error bounds are valid, certain mixed partial derivatives of u must be bounded. For the model problem of u being the solution of Poisson's problem on $\square = (0, 1)^n$ with homogeneous Dirichlet boundary conditions, we will show that generally these regularity conditions are *not* fulfilled, regardless of the smoothness of the right-hand side f . Instead, we show that for sufficiently smooth f , generally the error in $H^1(\square)$ is only of order $N^{-(\frac{1}{2} + \frac{1}{n})}$ (and also that the error is of order $N^{-(d-1)}$ when additionally f vanishes at the order $d - 1$ at the boundary).

Interior regularity theory shows that on any $\hat{\Omega} \subset\subset \square$ (as well as on the smooth parts of the boundary), any partial derivative of u is bounded, assuming a sufficient smoothness of f . So lacking smoothness of u is caused by boundary singularities, more accurately, by edge and corner singularities. This fact motivates Nitsche in the pioneering work [Nit05] to combine sparse-grid approximation with local refinement of the underlying one-dimensional wavelet collection towards $\{0, 1\}$. He showed that with this type of approximation, a function u can be approximated in the $H^1(\square)$ -norm with an error of order $(\log N)^t N^{-(d-1)}$ for some $t > 0$ depending on n and d , assuming u has mixed derivatives of sufficiently high order that are bounded in a *weighted* $L^2(\square)$ -norm, with the weight being a product of one-dimensional smooth weights that vanish at $\{0, 1\}$. Moreover, for the space dimension $n = 2$, he showed that this regularity assumption is fulfilled for the solution of Poisson's problem on \square with homogeneous Dirichlet boundary conditions when the right-hand side is sufficiently smooth (but does not necessarily vanish at the boundary).

In the present paper, we first provide a new regularity result of the solution of Poisson's problem on \square with homogeneous Dirichlet boundary conditions in certain anisotropic weighted Sobolev spaces. This result shows that the regularity assumptions made by Nitsche are fulfilled in *any* space dimension. At this point, we stress that evaluating regularity in *anisotropic* weighted spaces is mandatory as soon as the space dimension is larger than or equal to 3: The presence of edges generally prevents the sufficient regularity from holding in isotropic weighted spaces.

Second, we remove the log-factors from the error estimate in $H^1(\square)$, which arose from two different estimates in [Nit05]. That is, we demonstrate the best possible dimension-independent convergence rate $d - 1$ in $H^1(\square)$. The corresponding error estimate in $L^2(\square)$ will still contain some log-factors.

Third, in [Nit05], the approximation results were derived with wavelets that do not satisfy boundary conditions. Thinking of u being the solution of an (elliptic) boundary value problem with homogeneous Dirichlet boundary conditions, there is no obvious way to use such wavelets for actually *solving* such a boundary value problem. For approximating a function in $H_0^1(\square)$, we will use wavelets that are in that space. This will require some technicalities dealing with tensor products of weighted Sobolev spaces that incorporate essential boundary conditions.

Finally, we reduce the amount of local refinement needed to approximate the same type of boundary singularities, which may result in a quantitative improvement.

Note that the rates we demonstrate from the spans of wavelet sets constructed using a priori local refinements towards the boundaries are obviously also realized by *best N -term approximations* from the wavelet basis. The rate of best N -term approximation in turn can be realized, in linear complexity, by *adaptive* wavelet methods; cf. [DSS08].

This paper is organized as follows: Section 2 deals with wavelet approximation with local refinement of functions on the interval that are contained in certain weighted Sobolev spaces with a weight that vanishes at the end points.

In section 3, we discuss tensor products of Hilbert spaces. In particular, we show that (weighted) Sobolev spaces on product domains that incorporate essential boundary conditions can be written as intersections of tensor products of such spaces in the coordinate directions.

In section 4, we demonstrate optimal convergence rates with tensor product locally refined wavelet approximation of functions on \square that are contained in (intersections of) tensor products of weighted Sobolev spaces.

Assuming a sufficiently smooth right-hand side, in section 5 it is shown that the solution of Poisson’s problem on \square with homogeneous Dirichlet boundary conditions has arbitrarily high regularity in a suitable scale of anisotropic weighted Sobolev spaces.

Finally, in section 6, we show that the solution of this Poisson problem generally has limited smoothness in the scale of (unweighted) Sobolev spaces of dominating mixed derivatives that governs the rate of approximation of (optimized) sparse grids without local refinements.

In this paper, by $C \lesssim D$ we mean that C can be bounded by a *constant* multiple of D , whose constant is, unless stated otherwise, independent of parameters which C and D may depend on, possibly with *the exception of the space dimension n* . Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \approx D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Wavelet approximation of singular functions in one dimension using local refinement. For $k \in \mathbb{N}_0$, $\theta \geq 0$, and with

$$I := (0, 1),$$

let the *weighted Sobolev space* $H_\theta^k(I)$ be defined as the space of all measurable functions u for which the norm

$$\|u\|_{H_\theta^k(I)} := \left[\sum_{j=0}^k \int_I |x^\theta u^{(j)}(x)|^2 dx \right]^{\frac{1}{2}}$$

is finite. The seminorm $[\int_I |x^\theta u^{(k)}(x)|^2 dx]^{\frac{1}{2}}$ will be denoted as $|u|_{H_\theta^k(0,1)}$.

LEMMA 2.1 (see [Nit05, Lem. 2]). *For $q \in [1, \infty]$ and $\theta - (k - \frac{1}{2}) < \frac{1}{q}$, we have the continuous embedding*

$$H_\theta^k(I) \hookrightarrow L^q(I).$$

Remark 2.2. Since $H_{\theta_1}^k(I) \hookrightarrow H_{\theta_2}^k(I)$ when $\theta_1 \leq \theta_2$, clearly the additional condition $\theta > k - \frac{1}{2}$ imposed in [Nit05] can be omitted.

Remark 2.3. Lemma 2.1 is sharp in the sense that if $\theta - (k - \frac{1}{2}) > \frac{1}{q}$, then $H_\theta^k(I) \not\hookrightarrow L^q(I)$. Indeed, for $\alpha \in (k - \frac{1}{2} - \theta, -\frac{1}{q})$, $x \mapsto x^\alpha \in H_\theta^k(I)$, whereas this function is not in $L^q(I)$.

Remark 2.4. In the situation of Lemma 2.1 and for $j \in \mathbb{N}_0$, obviously also $H_\theta^{k+j}(I) \hookrightarrow W^{j,q}(I)$.

For $k \in \mathbb{N}$, $q \in [1, \infty]$, Ω being a domain in \mathbb{R}^n , and $\Gamma \subset \partial\Omega$, $W_{0,\Gamma}^{k,q}(\Omega)$ will denote the closure in $W^{k,q}(\Omega)$ of the space of smooth functions on Ω whose supports have

empty intersection with the Dirichlet boundary Γ . As usual, $W^{k,2}(\Omega)$ is denoted as $H^k(\Omega)$ and $W_{0,\Gamma}^{k,2}(\Omega)$ as $H_{0,\Gamma}^k(\Omega)$.

For some $m \in \mathbb{N}$ and $z \subseteq \partial I = \{0, 1\}$, we assume a *wavelet* collection

$$\{\psi_\lambda : \lambda \in \nabla\} \subset H_{0,z}^m(I)$$

such that

1. $\{\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $L^2(I)$;
2. $\{2^{-|\lambda|m}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $H_{0,z}^m(I)$,

where $|\lambda| \in \mathbb{N}_0$ denotes the *level* of ψ_λ or that of λ . Denoting the dual basis of $\{\psi_\lambda : \lambda \in \nabla\}$ for $L^2(I)$ as $\{\tilde{\psi}_\lambda : \lambda \in \nabla\}$, we furthermore assume that for some

$$\mathbb{N} \ni d > m,$$

for all $|\lambda| > 0$, for some $\tilde{\omega}_\lambda \subseteq \bar{I}$ with $\text{supp } \tilde{\psi}_\lambda \subseteq \tilde{\omega}_\lambda$,

3. $|\langle \tilde{\psi}_\lambda, u \rangle_{L^2(I)}| \lesssim 2^{-|\lambda|d} |u|_{H^d(\tilde{\omega}_\lambda)}$ ($u \in H^d(\tilde{\omega}_\lambda) \cap H_{0,\partial\tilde{\omega}_\lambda \cap z}^m(\tilde{\omega}_\lambda)$);
4. $\text{diam } \tilde{\omega}_\lambda \approx 2^{-|\lambda|}$;
5. $\sup_{\ell, j \in \mathbb{N}_0} \#\{|\lambda| = \ell : [j2^{-\ell}, (j+1)2^{-\ell}] \cap \tilde{\omega}_\lambda \neq \emptyset\} < \infty$.

The fourth and fifth conditions mean that the dual wavelets are locally supported and locally finite. From conditions 3, 5 and 1 or 2, one infers that

$$(2.1) \quad \left\| u - \sum_{\{\lambda \in \nabla : |\lambda| \leq \ell\}} \langle \tilde{\psi}_\lambda, u \rangle_{L^2(I)} \psi_\lambda \right\|_{L^2(I)} \lesssim 2^{-d\ell} |u|_{H^d(I)} \quad (u \in H^d(I) \cap H_{0,z}^m(I)),$$

$$(2.2) \quad \left\| u - \sum_{\{\lambda \in \nabla : |\lambda| \leq \ell\}} \langle \tilde{\psi}_\lambda, u \rangle_{L^2(I)} \psi_\lambda \right\|_{H^m(I)} \lesssim 2^{-(d-m)\ell} |u|_{H^d(I)} \quad (u \in H^d(I) \cap H_{0,z}^m(I)),$$

respectively, meaning that the primal wavelets satisfy a Jackson estimate of order d .

For any $m, d \in \mathbb{N}$, $d > m$, orthogonal or biorthogonal wavelets that satisfy the above conditions have been constructed in many papers. See, e.g., [CDV93] or [DKU99].

For $\beta \in [0, 1)$ and $\ell \in \mathbb{N}$, following [Nit05], we define

$$\nabla_\ell^\beta := \{\lambda \in \nabla : |\lambda| \leq \frac{\ell}{1-\beta}, \tilde{\omega}_\lambda \cap (0, 2^{\frac{\ell-|\lambda|}{1-\beta}}) \neq \emptyset\};$$

see Figure 1. Note that all λ 's with levels up to ℓ are in this set, and that, for $\beta > 0$, it contains also λ 's with higher levels, up to $\frac{\ell}{1-\beta}$, when the corresponding $\tilde{\omega}_\lambda$ are sufficiently close to zero. For $|\lambda|$ ranging from ℓ to $\frac{\ell}{1-\beta}$ (if the latter is an integer), $2^{\frac{\ell-|\lambda|}{1-\beta}}$ over the “mesh-size” $2^{-|\lambda|}$ ranges from 2^ℓ to 1.

On $L^2(I)$, we define the projector

$$P_\ell^\beta : u \mapsto \sum_{\lambda \in \nabla_\ell^\beta} \langle \tilde{\psi}_\lambda, u \rangle_{L^2(I)} \psi_\lambda.$$

PROPOSITION 2.5. *For any $\beta \in [0, 1)$, $\#\nabla_\ell^\beta \approx 2^\ell$.*

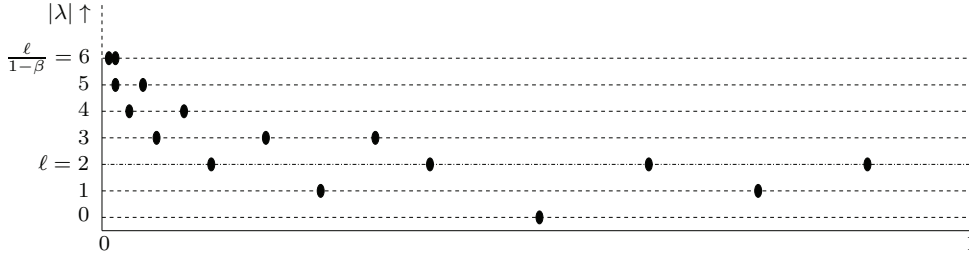


FIG. 1. ∇_ℓ^β for $\nabla = \cup_{j \in \mathbb{N}_0} 2^{-(j+1)}(2\mathbb{Z} + 1) \cap (0, 1)$, where $|\lambda| = j$, $\ell = 2$, $\beta = \frac{2}{3}$, $\tilde{\omega}_\lambda = [\lambda - 3 \cdot 2^{-|\lambda|}, \lambda + 3 \cdot 2^{-|\lambda|}] \cap [0, 1]$ (corresponds to, e.g., biorthogonal spline wavelets of order 2 and with 2 vanishing moments, and $z = \{0, 1\}$).

Proof. From the definition of ∇_ℓ^β and conditions 4 and 5, we have

$$\#\nabla_\ell^\beta \approx \sum_{j=0}^{\ell} 2^j + \sum_{j=\ell+1}^{\lfloor \frac{\ell}{1-\beta} \rfloor} 2^{\frac{\ell-j}{\beta}} 2^j \approx 2^\ell. \quad \square$$

Estimates for the approximation error for functions from weighted Sobolev spaces are derived in the next theorem and corollary. These estimates were inspired by [Nit05, Lemma 3]. Compared to that result, suboptimal log-factors are removed, and some of the arguments are corrected.

THEOREM 2.6. (a) Let $\theta \in [0, d)$, $\beta \in (\frac{\theta}{d}, 1)$, and $\beta \geq 1 - \frac{1}{2d}$. Then

$$\|u - P_\ell^\beta u\|_{L^2(\mathbb{I})} \lesssim 2^{-d\ell} \|u\|_{H_\theta^d(\mathbb{I})} \quad (u \in H_\theta^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})).$$

(b) Let $\theta \in [m, d)$, $\beta \in (\frac{\theta-m}{d-m}, 1)$, and $\beta \geq 1 - \frac{1}{2(d-m)}$. Then

$$\|u - P_\ell^\beta u\|_{H^m(\mathbb{I})} \lesssim 2^{-(d-m)\ell} \|u\|_{H_{\theta-m}^d(\mathbb{I})} \quad (u \in H_{\theta-m}^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})).$$

Proof. (a) We write $\nabla \setminus \nabla_\ell^\beta$ as the disjoint union of sets $\Sigma_\ell^{\beta,1}$ and $\Sigma_\ell^{\beta,2}$, where $\Sigma_\ell^{\beta,1}$ is the set of those $\lambda \in \nabla \setminus \nabla_\ell^\beta$ with $|\lambda| > \frac{\ell}{1-\beta}$ and $\tilde{\omega}_\lambda \cap (0, 2^{-\frac{\ell}{1-\beta}}) \neq \emptyset$. Consider $u \in H_\theta^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})$. By condition 1, we have

$$(2.3) \quad \|u - P_\ell^\beta u\|_{L^2(\mathbb{I})}^2 \approx \sum_{\lambda \in \Sigma_\ell^{\beta,1}} |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}|^2 + \sum_{\lambda \in \Sigma_\ell^{\beta,2}} |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}|^2.$$

By condition 4, the $\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}$ in the first sum depend only on u restricted to $(0, C2^{-\frac{\ell}{1-\beta}})$, where $C > 0$ is some absolute constant. By condition 1, this sum can therefore be bounded on some absolute multiple of $\|u\|_{L^2(0, C2^{-\frac{\ell}{1-\beta}})}^2$. Because of the conditions on the parameters d , θ , and β , there exists a $q \in (2, \infty]$ with $\theta - (d - \frac{1}{2}) < \frac{1}{q} \leq \frac{1}{2} + d(\beta - 1)$. We infer that

$$\|u\|_{L^2(0, C2^{-\frac{\ell}{1-\beta}})}^2 \leq (C2^{-\frac{\ell}{1-\beta}})^{1-2/q} \|u\|_{L^q(\mathbb{I})}^2 \lesssim 4^{-\ell d} \|u\|_{L^q(\mathbb{I})}^2 \lesssim 4^{-\ell d} \|u\|_{H_\theta^d(\mathbb{I})}^2,$$

where for the first, second, and third inequalities we used $\frac{1}{q} \in [0, \frac{1}{2}]$ in combination with Hölder's inequality, $\frac{1}{q} \leq \frac{1}{2} + d(\beta - 1)$, and $\theta - (d - \frac{1}{2}) < \frac{1}{q}$, respectively.

By definition of $\Sigma_\ell^{\beta,2}$, and by using conditions 3 and 5, the second sum on the right-hand side of (2.3) can be bounded on some absolute multiple of

$$\begin{aligned} & \sum_{j=\ell+1}^{\lfloor \frac{\ell}{1-\beta} \rfloor} 4^{-jd} |u|_{H^d(2^{-\frac{\ell-j}{\beta}}, 1)}^2 + \sum_{j > \frac{\ell}{1-\beta}} 4^{-jd} |u|_{H^d(2^{-\frac{\ell}{1-\beta}}, 1)}^2 \\ & \leq \sum_{i=1}^{\infty} \left(\sum_{j=i}^{\infty} 4^{-(\ell+j)d} \right) |u|_{H^d(2^{-\frac{i}{\beta}}, 2^{-\frac{i-1}{\beta}})}^2 \\ & \approx 4^{-\ell d} \sum_{i=1}^{\infty} 4^{-id} |u|_{H^d(2^{-\frac{i}{\beta}}, 2^{-\frac{i-1}{\beta}})}^2 \approx 4^{-\ell d} |u|_{H_{\beta d}^d(\mathbb{I})}^2 \lesssim 4^{-\ell d} |u|_{H_\theta^d(\mathbb{I})}^2, \end{aligned}$$

where we used that $4^{-id} \approx x^{2\beta d}$ on $(2^{-\frac{i}{\beta}}, 2^{-\frac{i-1}{\beta}})$, and finally that $\theta \leq \beta d$. The estimates for both sums on the right-hand side of (2.3) give the proof of (a).

(b) Consider $u \in H_{\theta-m}^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})$. Similar to (2.3), by condition 2 we have

$$(2.4) \quad \|u - P_\ell^\beta u\|_{H^m(\mathbb{I})}^2 \approx \sum_{\lambda \in \Sigma_\ell^{\beta,1}} 4^{|\lambda|m} |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}|^2 + \sum_{\lambda \in \Sigma_\ell^{\beta,2}} 4^{|\lambda|m} |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}|^2.$$

Because of condition 2, the first sum can be bounded on some absolute multiple of $\|u\|_{H^m(\mathbb{I})}^2$. By condition 4, we also know that this sum depends only on u restricted to $(0, C2^{-\frac{\ell}{1-\beta}})$ for some absolute constant $C > 0$, i.e., that for bounding this sum we may choose u outside this interval at our convenience, as long as it is in $H_{0,z}^m(\mathbb{I})$.

There exists a bounded extension operator $E : H^m(\mathbb{I}) \rightarrow H_{0,\{2\}}^m(0, 2)$. By a homogeneity argument, we infer that there exists an extension operator $E_h : H^m(0, h) \rightarrow H_{0,\{2h\}}^m(0, 2h) \subset H_{0,\{1\}}^m(\mathbb{I})$ with $\|E_h v\|_{H^m(\mathbb{I})}^2 \lesssim \sum_{k=0}^m h^{2(k-m)} |v|_{H^k(0,h)}^2$ ($v \in H^m(0, h)$). With $h = C2^{-\frac{\ell}{1-\beta}}$, we apply this extension operator to u .

If $0 \in z$, then $|u|_{H^k(0,h)}^2 \lesssim h^{2(m-k)} |u|_{H^m(0,h)}^2$, and so for the extended u we find $\|u\|_{H^m(\mathbb{I})}^2 \lesssim |u|_{H^m(0, C2^{-\frac{\ell}{1-\beta}})}^2$.

If $0 \notin z$, then by condition 3 the first sum on the right-hand side does not change if we subtract an arbitrary $p \in P_{m-1}$ from u . By choosing p suitably, for the resulting $\tilde{u} = u - p$, again we have $|\tilde{u}|_{H^k(0,h)}^2 \lesssim h^{2(m-k)} |\tilde{u}|_{H^m(0,h)}^2$. We conclude that also in this case the first sum on the right-hand side of (2.4) is bounded by some absolute multiple of $|\tilde{u}|_{H^m(0, C2^{-\frac{\ell}{1-\beta}})}^2 = |u|_{H^m(0, C2^{-\frac{\ell}{1-\beta}})}^2$.

The remainder of the proof is similar to that of (a). Because of the conditions on the parameters d, θ, β , and m , there exists a $q \in (2, \infty]$ with $\theta - m - (d - m - \frac{1}{2}) < \frac{1}{q} \leq \frac{1}{2} + (d - m)(\beta - 1)$. By applying Hölder's inequality and Lemma 2.1 we obtain

$$|u|_{H^m(0, C2^{-\frac{\ell}{1-\beta}})}^2 \leq (C2^{-\frac{\ell}{1-\beta}})^{1-2/q} \|u^{(m)}\|_{L^q(\mathbb{I})}^2 \lesssim 4^{-\ell(d-m)} \|u^{(m)}\|_{H_{\theta-m}^{d-m}(\mathbb{I})}^2.$$

Finally, by conditions 3 and 5, the second sum on the right-hand side of (2.4) can be bounded on some absolute multiple of

$$\sum_{j=\ell+1}^{\lfloor \frac{\ell}{1-\beta} \rfloor} 4^{-j(d-m)} |u|_{H^d(2^{-\frac{\ell-j}{\beta}}, 1)}^2 + \sum_{j > \frac{\ell}{1-\beta}} 4^{-j(d-m)} |u|_{H^d(2^{-\frac{\ell}{1-\beta}}, 1)}^2 \lesssim 4^{-\ell(d-m)} |u|_{H_{\theta-m}^d(\mathbb{I})}^2$$

because of $\theta - m \leq \beta(d - m)$. \square

Remark 2.7. So far we considered weighted Sobolev spaces with a vanishing weight at $x = 0$ appropriate for functions that have a singularity at $x = 0$. For functions that may have singularities at both $x = 0$ or $x = 1$, a suitable adaptation of the $H_\theta^k(\mathbb{I})$ -norm reads as

$$\|u\|_{H_\theta^k(\mathbb{I})} := \left[\sum_{i=0}^k \int_{\mathbb{I}} |x^\theta(1-x)^\theta u^{(i)}(x)|^2 dx \right]^{\frac{1}{2}}.$$

By redefining ∇_ℓ^β as

$$\nabla_\ell^\beta := \left\{ \lambda \in \nabla : |\lambda| \leq \frac{\ell}{1-\beta}, \tilde{\omega}_\lambda \cap \left((0, 2^{\frac{\ell-|\lambda|}{\beta}}) \cup (1 - 2^{\frac{\ell-|\lambda|}{\beta}}, 1) \right) \neq \emptyset \right\},$$

it is easily seen that Proposition 2.5 and Theorem 2.6 remain valid, where Theorem 2.6(a) now even reads as

$$\|u - P_\ell^\beta u\|_{L^2(\mathbb{I})} \lesssim 2^{-d\ell} \|u\|_{H_\theta^d(\mathbb{I})} \quad (u \in H_\theta^d(\mathbb{I})),$$

assuming that $\theta \in [0, d)$, $\beta \in (\frac{\theta}{d}, 1)$, and $\beta \geq 1 - \frac{1}{2d}$.

As follows from (2.1), (2.2), and the uniform $L^2(\mathbb{I})$ -boundedness of the P_ℓ^β , if in Theorem 2.6(a) $\theta = 0$, or in (b) $\theta = m$, then the corresponding statements are valid for $\beta = 0$, i.e., without local refinement, at least when in part (a) additionally $u \in H_{0,z}^m(\mathbb{I})$ is assumed. In view of this, the additional condition $\beta \geq 1 - \frac{1}{2d}$ or $\beta \geq 1 - \frac{1}{2(d-m)}$, besides $\beta > \frac{\theta}{d}$, seems unnatural. Below, for completeness, we will show that indeed they can be omitted for part (a) at the expense of adding some boundary conditions when $\beta \in (\frac{\theta}{d}, 1 - \frac{1}{2d})$. We consider only part (a), since the arguments for part (b) are similar.

Let $\frac{\theta}{d} < 1 - \frac{1}{2d}$, i.e., $\theta < d - \frac{1}{2}$, and let $\beta \in (\frac{\theta}{d}, 1 - \frac{1}{2d})$. The condition $\beta \geq 1 - \frac{1}{2d}$ was used only for bounding the first sum on the right-hand side of (2.3). For doing so, only the localness of the dual wavelets was used as well as the fact that the primal wavelets form a Riesz basis for $L^2(\mathbb{I})$. Alternatively, here we will use the approximation properties of the wavelets given by condition 3. When homogeneous Dirichlet boundary conditions are incorporated into the wavelet construction, then the function that is approximated must also satisfy certain homogeneous Dirichlet boundary conditions.

From conditions 3 and 4 and $\|\tilde{\psi}_\lambda\|_{L^2(\mathbb{I})} \lesssim 1$, we are going to infer that for $j \in \{1, \dots, d\}$

$$(2.5) \quad |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}| \lesssim 2^{-|\lambda|(j-\frac{1}{2})} |u|_{W^{j,1}(\tilde{\omega}_\lambda)}$$

for any sufficiently smooth u that vanishes at order $\min(m, j)$ at $\tilde{\omega}_\lambda \cap z$. Indeed, by condition 4 and a homogeneity argument, it is sufficient to show that $|\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}| \lesssim \|u^{(j)}\|_{L^1(\mathbb{I})}$. When $\tilde{\omega}_\lambda \cap z = \emptyset$, this follows from $\|\tilde{\psi}_\lambda\|_{L^2(\mathbb{I})} \lesssim 1$, $W^{j,1} \hookrightarrow L^2$, condition 3, which implies $\tilde{\psi}_\lambda \perp_{L^2(\mathbb{I})} P_{j-1}$, and the Bramble–Hilbert lemma. When, say $\tilde{\omega}_\lambda \cap z = \{0\}$, the combination of $\|\tilde{\psi}_\lambda\|_{L^2(\mathbb{I})} \lesssim 1$, the embedding $W^{\min(j,m),1} \hookrightarrow L^2$, and the Poincaré–Friedrichs’ inequality show that $|\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}| \lesssim \|u^{(\min(j,m))}\|_{L^1(\mathbb{I})}$. Now for $j > m$, let $p \in P_{j-1}$ be such that $p^{(m)}$ is the Taylor polynomial of $u^{(m)}$ of degree $j - 1 - m$ at 0 with $p(0) = \dots = p^{(m-1)}(0) = 0$. Then, by condition 3, $|\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}| = |\langle \tilde{\psi}_\lambda, u - p \rangle_{L^2(\mathbb{I})}| \lesssim \|u^{(m)} - p^{(m)}\|_{L^1(\mathbb{I})} \lesssim \|u^{(j)}\|_{L^1(\mathbb{I})}$, as required.

With $\gamma_v := u \mapsto u(v)$ denoting the trace operator, from $W^{1,1}(\mathbb{I}) \hookrightarrow L^\infty(\mathbb{I})$ we infer that the closure of the set of smooth functions that vanish at order $\min(m, j)$ at $\tilde{\omega}_\lambda \cap z$ in $W^{j,1}(\tilde{\omega}_\lambda)$ is

$$\{u \in W^{j,1}(\tilde{\omega}_\lambda) : \gamma_v D^r u = 0, 0 \leq r \leq \min(m, j) - 1, v \in \tilde{\omega}_\lambda \cap z\},$$

and thus that (2.5) also holds for all u in this space.

Now we take $j := \lceil d - \theta - \frac{1}{2} \rceil \in \{1, \dots, d\}$. Then, recalling that $\beta \in (\frac{\theta}{d}, 1 - \frac{1}{2d})$, there exists a $q \in (1, \infty)$ with $\theta - (d - j - \frac{1}{2}) < \frac{1}{q} \leq j + \frac{1}{2} - (1 - \beta)d$, where the first inequality implies that $H_\theta^d(\mathbb{I}) \hookrightarrow W^{j,q}(\mathbb{I})$ (see Remark 2.4) ($\hookrightarrow W^{j,1}(\mathbb{I})$), and the second one that $\frac{j + \frac{1}{2} - \frac{1}{q}}{1 - \beta} \geq d$. For u in

$$(2.6) \quad \tilde{H}_\theta^{d,m}(\mathbb{I}) := \{u \in H_\theta^d(\mathbb{I}) : \gamma_v D^r u = 0, 0 \leq r \leq \min(m, \lceil d - \theta - \frac{1}{2} \rceil) - 1, v \in z\},$$

from (2.5) and Hölder’s inequality we obtain that

$$\begin{aligned} \sum_{\lambda \in \Sigma_\ell^{\beta,1}} |\langle \tilde{\psi}_\lambda, u \rangle_{L^2(\mathbb{I})}|^2 &\lesssim 4^{-\frac{\ell}{1-\beta}(j-\frac{1}{2})} |u|_{W^{j,1}(0, C2^{-\frac{\ell}{1-\beta}})}^2 \\ &\leq 4^{-\frac{\ell}{1-\beta}(j+\frac{1}{2}-\frac{1}{q})} |u|_{W^{j,q}(\mathbb{I})}^2 \lesssim 4^{-\ell d} \|u\|_{H_\theta^d(\mathbb{I})}^2, \end{aligned}$$

which is the upper bound that was required.

Noting that for $\theta \geq d - \frac{1}{2}$ the set of additional conditions incorporated into the definition of $\tilde{H}_\theta^{d,m}(\mathbb{I})$ is empty, and thus that $\tilde{H}_\theta^{d,m}(\mathbb{I}) = H_\theta^d(\mathbb{I})$, we have shown the first statement of the following corollary. As noted before, the proof of the second part is similar.

COROLLARY 2.8. (a) *Let $\theta \in [0, d]$ and $\beta \in (\frac{\theta}{d}, 1)$. Then*

$$\|u - P_\ell^\beta u\|_{L^2(\mathbb{I})} \lesssim 2^{-d\ell} \|u\|_{H_\theta^d(\mathbb{I})} \quad (u \in \tilde{H}_\theta^{d,m}(\mathbb{I})).$$

(b) *Let $\theta \in [m, d]$ and $\beta \in (\frac{\theta-m}{d-m}, 1)$. Then*

$$\|u - P_\ell^\beta u\|_{H^m(\mathbb{I})} \lesssim 2^{-(d-m)\ell} \|u\|_{H_{\theta-m}^d(\mathbb{I})} \quad (u \in H_{\theta-m}^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})).$$

To indicate the dependence on the prescribed Dirichlet boundary $z \subseteq \partial\mathbb{I}$ of the wavelets ψ_λ , the dual wavelets $\tilde{\psi}_\lambda$, the sets $\tilde{\omega}_\lambda \supseteq \text{supp } \tilde{\psi}_\lambda$, the index sets ∇ and ∇_ℓ^β , the projector P_ℓ^β , and the space $\tilde{H}_\theta^{d,m}(\mathbb{I})$, from now on we will denote them as

$$\psi_\lambda^{(z)}, \tilde{\psi}_\lambda^{(z)}, \tilde{\omega}_\lambda^{(z)}, \nabla^{(z)}, \nabla_\ell^{\beta,z}, P_\ell^{\beta,z}, \tilde{H}_{\theta,z}^{d,m}(\mathbb{I}),$$

respectively.

3. Tensor products of Hilbert spaces. We recall some facts and give some new results about tensor products of Hilbert spaces, in particular (weighted) Sobolev spaces. Although for notational convenience, we start with formulating some results for tensor products of two spaces, these results obviously generalize to multiple products.

PROPOSITION 3.1. *For $i \in \{1, 2\}$, let Ω_i be domains in \mathbb{R}^{n_i} that satisfy the uniform cone property (cf. [Ada75, Thm. 4.32]). Then for $k \in \mathbb{N}_0$,*

$$H^k(\Omega_1 \times \Omega_2) = H^k(\Omega_1) \otimes L^2(\Omega_2) \cap L^2(\Omega_1) \otimes H^k(\Omega_2).$$

Proof. For $i \in \{1, 2\}$, let Σ_i be a Riesz basis for $L^2(\Omega_i)$, formally viewed as a column vector, such that for some invertible diagonal matrix \mathbf{D}_i and $0 \leq j \leq k$, $\mathbf{D}_i^{-j/k} \Sigma_i$ is a Riesz basis for $H^j(\Omega_i)$, meaning that $\sum_{|\alpha| \leq j} \langle \partial^\alpha \Sigma_i, \partial^\alpha \Sigma_i \rangle_{L^2(\Omega_i)} \approx \mathbf{D}_i^{2j/k}$. Later we will show that such bases exist.

Using that $L^2(\Omega_1 \times \Omega_2) = L^2(\Omega_1) \otimes L^2(\Omega_2)$, and that the tensor product of Riesz bases is a Riesz basis for the tensor product, each $u \in L^2(\Omega_1 \times \Omega_2)$ has a unique expansion $u = \mathbf{c}^\top \Sigma_1 \otimes \Sigma_2$, where $\mathbf{c} = (c_{\sigma_1 \sigma_2})_{\sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2} \in \ell^2(\Sigma_1 \times \Sigma_2)$. We have that

$$\begin{aligned} & \sum_{|\alpha_1|+|\alpha_2| \leq k} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} u\|_{L^2(\Omega_1 \times \Omega_2)}^2 \\ &= \sum_{|\alpha_1|+|\alpha_2| \leq k} \mathbf{c}^\top \langle \partial^{\alpha_1} \Sigma_1, \partial^{\alpha_1} \Sigma_1 \rangle_{L^2(\Omega_1)} \otimes \langle \partial^{\alpha_2} \Sigma_2, \partial^{\alpha_2} \Sigma_2 \rangle_{L^2(\Omega_2)} \mathbf{c} \\ &\approx \mathbf{c}^\top (\mathbf{D}_1^2 \otimes \text{Id} + \text{Id} \otimes \mathbf{D}_2^2) \mathbf{c} \approx \|u\|_{H^k(\Omega_1) \otimes L^2(\Omega_2)}^2 + \|u\|_{L^2(\Omega_1) \otimes H^k(\Omega_2)}^2, \end{aligned}$$

where for the one but last “ \approx ” we used that for $0 < j < k$, $\mathbf{D}_1^{2j/k} \otimes \mathbf{D}_2^{2(k-j)/k} \leq \mathbf{D}_1^2 \otimes \text{Id} + \text{Id} \otimes \mathbf{D}_2^2$.

The existence of the aforementioned bases Σ_i has still to be shown. Let $V_0 \subset V_1 \subset \dots \subset L_2(\Omega_i)$ be such that for some $d, \gamma > 0$, $\inf_{v_\ell \in V_\ell} \|u - v_\ell\|_{L^2(\Omega_i)} \lesssim 2^{-\ell d} \|u\|_{H^d(\Omega_i)}$ ($u \in H^d(\Omega_i)$) (*Jackson estimate*) and for all $s < \gamma$, $\|\cdot\|_{H^s(\Omega_i)} \lesssim 2^{\ell s} \|\cdot\|_{L^2(\Omega_i)}$ on V_ℓ (*Bernstein estimate*). With $V_{-1} := \{0\}$, let Ψ_ℓ be an $L_2(\Omega_i)$ -orthonormal basis for $V_\ell \cap V_{\ell-1}^\perp$. Then for $|s| < \min(\gamma, d)$, $\cup_{\ell \in \mathbb{N}_0} 2^{-s\ell} \Psi_\ell$ is a Riesz basis for $H^s(\Omega_i)$ (cf., e.g., [Dah96]), and so when $\min(\gamma, d) > k$, $\Sigma_i := \cup_{\ell \in \mathbb{N}_0} \Psi_\ell$ satisfies the assumptions.

When $\Omega_i = \mathbb{R}^{n_i}$, for any $d \in \mathbb{N}$ and with $\gamma := d - \frac{1}{2}$ such a ladder of spaces, denoted by $(V_\ell^{\mathbb{R}^{n_i}})_\ell$, can be constructed as a sequence of spline spaces with respect to nested dyadic partitions. Then, since the restriction of a function on \mathbb{R}^{n_i} to Ω_i is a bounded operator from $H^s(\mathbb{R}^{n_i}) \rightarrow H^s(\Omega_i)$, the sequence $(V_\ell)_\ell$ defined by $V_\ell := V_\ell^{\mathbb{R}^{n_i}}|_{\Omega_i}$ satisfies the corresponding Bernstein inequality on Ω_i . Thanks to the uniform cone condition, there exists a bounded extension operator $E : H^d(\Omega_i) \rightarrow H^d(\mathbb{R}^{n_i})$ (see [CZ52]). For $u \in H^d(\Omega_i)$, we have

$$\inf_{v_j \in V_\ell^{\mathbb{R}^{n_i}}} \|u - v_j|_{\Omega_i}\|_{L^2(\Omega_i)} \leq \inf_{v_j \in V_\ell^{\mathbb{R}^{n_i}}} \|Eu - v_j\|_{L^2(\mathbb{R}^{n_i})} \lesssim 2^{-\ell d} \|Eu\|_{H^d(\mathbb{R}^{n_i})} \lesssim \|u\|_{H^d(\Omega_i)},$$

i.e., the corresponding Jackson estimate is valid on Ω_i , with which the existence of Σ_i is demonstrated. \square

THEOREM 3.2. *Let H, K, Z be separable Hilbert spaces, and let $G : H \rightarrow Z$ be a linear map that is bounded and onto. Then*

$$\{u \in H \otimes K : (G \otimes \text{Id})u = 0\} = \{v \in H : Gv = 0\} \otimes K.$$

Proof. Since G is bounded, $\tilde{H} := \{v \in H : Gv = 0\}$ is closed, and so H is the direct sum of the Hilbert spaces \tilde{H} and \tilde{H}^\perp .

Let Ξ, Σ , and Υ be orthonormal bases for $\tilde{H}, \tilde{H}^\perp$, and K , respectively. Then $\Xi \otimes \Upsilon, \Sigma \otimes \Upsilon$, and $(\Xi \cup \Sigma) \otimes \Upsilon$ are orthonormal bases for $\tilde{H} \otimes K, \tilde{H}^\perp \otimes K$, and $H \otimes K$, respectively. Any $u \in H \otimes K$ has a unique expansion $u = \mathbf{c}^\top \Xi \otimes \Upsilon + \mathbf{d}^\top \Sigma \otimes \Upsilon$, where $\mathbf{c} \in \ell^2(\Xi \times \Upsilon), \mathbf{d} \in \ell^2(\Sigma \times \Upsilon)$. The condition $(G \otimes \text{Id})u = 0$ means that $\mathbf{d}^\top G(\Sigma) \otimes \Upsilon = 0$.

The linear mapping $G|_{\tilde{H}^\perp} : \tilde{H}^\perp \rightarrow Z$ is onto, bounded, and injective. Since \tilde{H}^\perp and Z are Hilbert spaces, the open mapping theorem shows that $G|_{\tilde{H}^\perp}$ is boundedly

invertible. We conclude that $G(\Sigma)$ is a Riesz basis for Z and thus that $G(\Sigma) \otimes \Upsilon$ is a Riesz basis for $Z \otimes K$. So $\mathbf{d}^\top G(\Sigma) \otimes \Upsilon = 0$ implies $\mathbf{d} = 0$, which shows that $\Xi \otimes \Upsilon$ is an (orthonormal) basis for $\{u \in H \otimes K : (G \otimes \text{Id})u = 0\}$, and thus completes the proof. \square

COROLLARY 3.3. *Let $k \in \mathbb{N}$, and for $i \in \{1, 2\}$, let Ω_i be a domain in \mathbb{R}^{n_i} with a $C^{k-1,1}$ boundary. Then*

$$H_0^k(\Omega_1 \times \Omega_2) = H_0^k(\Omega_1) \otimes L^2(\Omega_2) \cap L^2(\Omega_1) \otimes H_0^k(\Omega_2).$$

Proof. With γ_i denoting the trace operator on $\partial\Omega_i$, the mapping $G_i : H^k(\Omega_i) \rightarrow \prod_{j=0}^{k-1} H^{k-j-\frac{1}{2}}(\partial\Omega_i) : u \mapsto (\gamma_i u, \gamma_i \frac{\partial u}{\partial n}, \dots, \gamma_i \frac{\partial^{k-1} u}{\partial n^{k-1}})$ is bounded and onto, the latter because of $\partial\Omega_i \in C^{k-1,1}$ (see, e.g., [Gri85, Thm. 1.5.1.1]). Using

$$\begin{aligned} H_0^k(\Omega_1 \times \Omega_2) &= \{u \in H^k(\Omega_1 \times \Omega_2) : (G_1 \otimes \text{Id})u = 0, (\text{Id} \otimes G_2)u = 0\} \\ &= \{u \in H^k(\Omega_1) \otimes L^2(\Omega_2) : (G_1 \otimes \text{Id})u = 0\} \cap \{u \in L^2(\Omega_1) \otimes H^k(\Omega_2) : (\text{Id} \otimes G_2)u = 0\} \end{aligned}$$

by Proposition 3.1, and $H_0^k(\Omega_i) = \{u \in H^k(\Omega_i) : G_i u = 0\}$, an application of Theorem 3.2 completes the proof. \square

Results related to Corollary 3.3 can be found in [Dau88, (AC.7)], [GO95], [Hoc01]. The technique employed here allows us also to deal with weighted Sobolev spaces that incorporate boundary conditions.

Applications of Theorem 3.2 specific for this work involve multiple tensor products of (weighted) Sobolev spaces on \mathbb{I} . With, for $n \in \mathbb{N}$,

$$\square := (0, 1)^n,$$

let Γ be a union of $(n - 1)$ -dimensional faces of $\bar{\square}$, i.e., $\Gamma = \cup_{i=1}^n \bar{\mathbb{I}}^{i-1} \times z_i \times \bar{\mathbb{I}}^{n-i}$ for some $z_i \subseteq \partial\mathbb{I}$; see Figure 2 for an illustration.

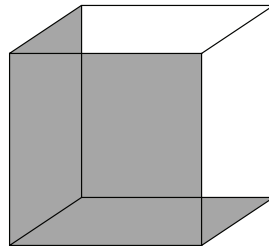


FIG. 2. Γ for $n = 3$ and $z_i = \{0\}$ for $1 \leq i \leq n$.

As a special case of Proposition 3.1 (for multiple tensor products), we have

$$(3.1) \quad H^m(\square) = H^m(\mathbb{I}) \otimes L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I}) \cap \dots \cap L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I}) \otimes H^m(\mathbb{I}).$$

Furthermore, we have

$$(3.2) \quad H_{0,\Gamma}^m(\square) = H_{0,z_1}^m(\mathbb{I}) \otimes L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I}) \cap \dots \cap L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I}) \otimes H_{0,z_n}^m(\mathbb{I}).$$

To see the latter, for $k \in \mathbb{N}$ and $1 \leq i \leq n$, let us put

$$G_{i,k} := u \mapsto \prod_{v \in z_i} \prod_{r=0}^{k-1} \gamma_v D^r u.$$

Then, since $G_{i,m} : H^m(\mathbb{I}) \rightarrow \mathbb{R}^{m \cdot \#z_i}$ is bounded and onto, and

$$\begin{aligned} H_{0,\Gamma}^m(\square) &= \{u \in H^m(\square) : (\text{Id} \otimes \cdots \otimes G_{i,m} \otimes \cdots \otimes \text{Id})u = 0, 1 \leq i \leq n\} \\ &= \cap_{i=1}^n \left\{ u \in L^2(\mathbb{I}) \otimes \cdots \otimes \underbrace{H_\theta^d(\mathbb{I})}_{i\text{th position}} \otimes \cdots \otimes L^2(\mathbb{I}) : (\text{Id} \otimes \cdots \otimes G_{i,m} \otimes \cdots \otimes \text{Id})u = 0 \right\}, \end{aligned}$$

the result follows from an application of Theorem 3.2.

Similarly, we have that

$$(3.3) \quad \otimes_{i=1}^n \tilde{H}_{\theta,z_i}^{d,m}(\mathbb{I}) = \{u \in \otimes_{i=1}^n H_\theta^d(\mathbb{I}) : \partial_n^r u = 0 \text{ on } \Gamma, 0 \leq r \leq \min(m, \lceil d - \theta - \frac{1}{2} \rceil) - 1\}.$$

Indeed, for $\theta < d - \frac{1}{2}$ putting $k := \min(m, \lceil d - \theta - \frac{1}{2} \rceil)$, $G_{i,k} : H_\theta^d(\mathbb{I}) \rightarrow \mathbb{R}^{k \cdot \#z_i}$ is bounded (cf. arguments that led to Corollary 2.8) and onto, and the result follows from Theorem 3.2 using the space on the right-hand side of (3.3) being equal to $\{u \in \otimes_{i=1}^n H_\theta^d(\mathbb{I}) : (\text{Id} \otimes \cdots \otimes G_{i,k} \otimes \cdots \otimes \text{Id})u = 0, 1 \leq i \leq n\}$ and the definition of $\tilde{H}_{\theta,z_i}^{d,m}(\mathbb{I})$ in (2.6).

Finally, we note that for $\theta \in [0, d)$,

$$(3.4a) \quad \cap_{p=1}^n \otimes_{i=1}^n H_{\theta - \delta_{ip} \min(m,\theta)}^d(\mathbb{I}) \cap H_{0,\Gamma}^m(\square) = \cap_{p=1}^n \otimes_{i=1}^n \mathcal{W}_{ip}(\mathbb{I}),$$

where

$$(3.4b) \quad \mathcal{W}_{pp} = H_{\max(0,\theta-m)}^d(\mathbb{I}) \cap H_{0,z_p}^m(\mathbb{I}),$$

and for $i \neq p$,

$$(3.4c) \quad \mathcal{W}_{ip} = H_\theta^d(\mathbb{I}) \quad \text{or} \quad \mathcal{W}_{ip} = \tilde{H}_{\theta,z_i}^{d,m}(\mathbb{I}).$$

Indeed, with the first option $\mathcal{W}_{ip} = H_\theta^d(\mathbb{I})$ for $i \neq p$, the result follows from Theorem 3.2 by writing the space on the left-hand side of (3.4a) as

$$\cap_{p=1}^n \{u \in \otimes_{i=1}^n H_{\theta - \delta_{ip} \min(m,\theta)}^d(\mathbb{I}) : (\text{Id} \otimes \cdots \otimes G_{p,m} \otimes \cdots \otimes \text{Id})u = 0\}.$$

Here we used that $\cap_{p=1}^n \otimes_{i=1}^n H_{\theta - \delta_{ip} \min(m,\theta)}^d(\mathbb{I}) \hookrightarrow H^m(\square)$, which follows from (3.1) and $H_\theta^d(\mathbb{I}) \hookrightarrow L^2(\mathbb{I})$ and $H_{\max(0,\theta-m)}^d(\mathbb{I}) \hookrightarrow H^m(\mathbb{I})$ by $\theta < d$ (cf. Lemma 2.1 and Remark 2.4).

With the second option $\mathcal{W}_{ip} = \tilde{H}_{\theta,z_i}^{d,m}(\mathbb{I})$ for $i \neq p$, the result follows by ‘‘copying’’ certain boundary conditions, i.e., with $k := \min(m, \lceil d - \theta - \frac{1}{2} \rceil)$, by writing the space on the left-hand side of (3.4a) as

$$\begin{aligned} \cap_{p=1}^n \{u \in \otimes_{i=1}^n H_{\theta - \delta_{ip} \min(m,\theta)}^d(\mathbb{I}) : (\text{Id} \otimes \cdots \otimes G_{p,m} \otimes \cdots \otimes \text{Id})u = 0, \\ (\text{Id} \otimes \cdots \otimes G_{i,k} \otimes \cdots \otimes \text{Id})u = 0 \ (1 \leq i \leq n, i \neq p)\}. \end{aligned}$$

4. Optimal wavelet approximation of singular functions using sparse products of locally refined index sets. As shown in [GO95], from $L^2(\square) = \otimes_{i=1}^n L^2(\mathbb{I})$, (3.2), and conditions 1 and 2, we have that

$$\left\{ \psi_\lambda := \psi_{\lambda_1}^{(z_1)} \otimes \cdots \otimes \psi_{\lambda_n}^{(z_n)} : \lambda \in \nabla := \prod_{i=1}^n \nabla^{(z_i)} \right\}, \quad \left\{ \left(\sum_{i=1}^n 4^{|\lambda_i|} \right)^{-m/2} \psi_\lambda : \lambda \in \nabla \right\}$$

are Riesz bases for $L^2(\square)$ and $H_{0,\Gamma}^m(\square)$, respectively.

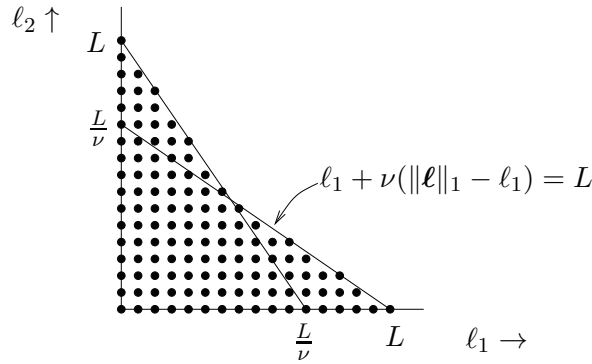


FIG. 3. $\{\ell \in \mathbb{N}_0^n : \nu \|\ell\|_1 + (1 - \nu)\|\ell\|_\infty \leq L\}$ for $n = 2$ and $\nu = \frac{16}{11}$.

For $\beta \in [0, 1)$, $\nu \geq 1$, and $L \in \mathbb{N}_0$, and with $\nabla_{-1}^{\beta, z_i} := \emptyset$, we define

$$\begin{aligned} \nabla_{L, \nu}^\beta &:= \left\{ \prod_{i=1}^n \nabla_{\ell_i}^{\beta, z_i} \setminus \nabla_{\ell_{i-1}}^{\beta, z_i} : \ell \in \mathbb{N}_0^n, \nu \|\ell\|_1 + (1 - \nu)\|\ell\|_\infty \leq L \right\} \\ &= \bigcup_{p=1}^n \left\{ \prod_{i=1}^n \nabla_{\ell_i}^{\beta, z_i} \setminus \nabla_{\ell_{i-1}}^{\beta, z_i} : \ell \in \mathbb{N}_0^n, \ell_p + \nu(\|\ell\|_1 - \ell_p) \leq L \right\}; \end{aligned}$$

see Figure 3. The index set $\nabla_{L, 1}^0$ underlies the well-known sparse grid, also called hyperbolic wavelet approximation; see, e.g., [Zen91, BG04, DKT98]. By considering $\nabla_{L, 1}^\beta$ for $\beta > 0$, in [Nit05] this type of approximation was combined with (product-type) local refinements towards $\partial\Box$.

As shown in [GK00], when measuring approximation errors in Sobolev spaces with positive smoothness indices, sparse-grid approximation rates can be realized with even smaller spaces (“optimized” sparse grids), with which truly optimal convergence rates are realized. With the introduction of the index sets $\nabla_{L, \nu}^\beta$ for $\nu > 1$, we combine this idea with that of local refinement. With that, we will be able to remove suboptimal log-factors from the corresponding estimates from [Nit05].

The following elementary lemma will be used a couple of times.

LEMMA 4.1. For any $\nu > 0$, $\alpha \geq 0$,

$$\sum_{\{\ell \in \mathbb{N}_0^n : \ell_p + \nu(\|\ell\|_1 - \ell_p) \in [j, j+1]\}} 2^{-\alpha(\|\ell\|_1 - \ell_p)} \approx \begin{cases} j^{n-1} & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0, \end{cases}$$

where the constant factors absorbed by the “ \approx ” symbol may depend on ν , α , and n (but not on j).

On $L^2(\Box)$, we define the projector

$$\mathbf{P}_{L, \nu}^\beta : u = \sum_{\lambda \in \nabla} u_\lambda \psi_\lambda \mapsto \sum_{\lambda \in \nabla_{L, \nu}^\beta} u_\lambda \psi_\lambda.$$

PROPOSITION 4.2. Let $\beta \in [0, 1)$. Then for any $\nu \geq 1$, $\#\nabla_{L, \nu}^\beta \approx \begin{cases} L^{n-1} 2^L & \text{if } \nu=1, \\ 2^L & \text{if } \nu>1. \end{cases}$

Proof. Since by Proposition 2.5, $\#(\nabla_{\ell_i}^{\beta, z_i} \setminus \nabla_{\ell_{i-1}}^{\beta, z_i}) \approx 2^{\ell_i}$, the proof follows from

$$\begin{aligned} \sum_{j=0}^L \sum_{\{\ell: \ell_p + \nu(\|\ell\|_1 - \ell_p) \in [j, j+1]\}} 2^{\|\ell\|_1} &= \sum_{j=0}^L \sum_{\{\ell: \ell_p + \nu(\|\ell\|_1 - \ell_p) \in [j, j+1]\}} 2^{\ell_p + \nu(\|\ell\|_1 - \ell_p)} 2^{(1-\nu)(\|\ell\|_1 - \ell_p)} \\ &\approx \sum_{j=0}^L 2^j \times \begin{cases} j^{n-1} & \text{if } \nu = 1 \\ 1 & \text{if } \nu > 1 \end{cases} = \begin{cases} L^{n-1} 2^L & \text{if } \nu = 1, \\ 2^L & \text{if } \nu > 1 \end{cases} \end{aligned}$$

by an application of Lemma 4.1. \square

We have the following estimates for the approximation error.

THEOREM 4.3. *Let $\theta \in [0, d]$ and $\beta \in (\frac{\theta}{d}, 1)$. Then*

$$(a) \quad \|u - \mathbf{P}_{L,1}^\beta u\|_{L^2(\square)} \lesssim L^{\frac{n-1}{2}} 2^{-dL} \|u\|_{\otimes_{i=1}^n H_\theta^d(\mathbb{I})}$$

for all u in

$$(4.1) \quad \{u \in \otimes_{i=1}^n H_\theta^d(\mathbb{I}) : \partial_n^r u = 0 \text{ on } \Gamma, 0 \leq r \leq \min(m, \lceil d - \theta - \frac{1}{2} \rceil) - 1\},$$

and for $\nu < \frac{d}{d-m}$,

$$(b) \quad \|u - \mathbf{P}_{L,\nu}^\beta u\|_{H^m(\square)} \lesssim 2^{-(d-m)L} \sqrt{\sum_{p=1}^n \|u\|_{\otimes_{i=1}^n H_{\theta - \delta_{pi} \min(m, \theta)}^d(\mathbb{I})}^2}$$

for all $u \in \cap_{p=1}^n \otimes_{i=1}^n H_{\theta - \delta_{pi} \min(m, \theta)}^d(\mathbb{I}) \cap H_{0,\Gamma}^m(\square)$.

Proof. (a) Setting $P_{-1}^{\beta, z_i} := 0$, the mapping $u \mapsto \sum_{\lambda \in \prod_{i=1}^n \nabla_{\ell_i}^{\beta, z_i} \setminus \nabla_{\ell_{i-1}}^{\beta, z_i}} u_\lambda \psi_\lambda$ is equal to $\otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_{i-1}}^{\beta, z_i})$, so that

$$(4.2) \quad \mathbf{I} - \mathbf{P}_{L,\nu}^\beta = \sum_{\{\ell: \nu \|\ell\|_1 + (1-\nu) \|\ell\|_\infty > L\}} \otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_{i-1}}^{\beta, z_i}).$$

From $\{\psi_\lambda : \lambda \in \nabla\}$ being a Riesz basis for $L^2(\square)$, we even have $\|u - \mathbf{P}_{L,1}^\beta u\|_{L^2(\square)}^2 \approx \sum_{\{\ell: \|\ell\|_1 > L\}} \|\otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_{i-1}}^{\beta, z_i})u\|_{L^2(\square)}^2$. The combination of Corollary 2.8(a) and a tensor product argument shows that

$$\|\otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_{i-1}}^{\beta, z_i})u\|_{L^2(\square)} \lesssim 2^{-d\|\ell\|_1} \|u\|_{\otimes_{i=1}^n H_\theta^d(\mathbb{I})} \quad (u \in \otimes_{i=1}^n \tilde{H}_{\theta, z_i}^{d,m}(\mathbb{I})).$$

From (3.3) we know that $\otimes_{i=1}^n \tilde{H}_{\theta, z_i}^{d,m}(\mathbb{I})$ is equal to the space in (4.1). An application of Lemma 4.1 together with $\sum_{j=L+1}^\infty 4^{-dj} j^{n-1} \approx 4^{-dL} L^{n-1}$ completes the proof of part (a).

(b) Using $\theta < d$, in view of (3.4a), (3.2), (4.2), and of the inclusions

$$\{\ell : \nu \|\ell\|_1 + (1-\nu) \|\ell\|_\infty > L\} \subset \{\ell : \ell_p + \nu(\|\ell\|_1 - \ell_p) > L\} \quad (p = 1, \dots, n),$$

w.l.o.g. it suffices to show that

$$\begin{aligned} \sum_{\{\ell: \ell_1 + \nu(\|\ell\|_1 - \ell_1) > L\}} \|\otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_{i-1}}^{\beta, z_i})u\|_{H^m(\mathbb{I}) \otimes L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I})} \\ \lesssim 2^{-(d-m)L} \|u\|_{H_{\max(0, \theta-m)}^d(\mathbb{I}) \otimes H_\theta^d(\mathbb{I}) \otimes \dots \otimes H_\theta^d(\mathbb{I})} \quad (u \in \otimes_{i=1}^n \mathcal{W}_{i1}), \end{aligned}$$

where $\mathcal{W}_{11} = H_{\max(0, \theta - m)}^d(\mathbb{I}) \cap H_{0, z_p}^m(\mathbb{I})$ and, for $i > 1$, $\mathcal{W}_{i1} = \tilde{H}_{\theta, z_i}^{d, m}(\mathbb{I})$. Corollary 2.8 and a tensor product argument show that

$$\begin{aligned} & \| \otimes_{i=1}^n (P_{\ell_i}^{\beta, z_i} - P_{\ell_i - 1}^{\beta, z_i}) u \|_{H^m(\mathbb{I}) \otimes L^2(\mathbb{I}) \otimes \dots \otimes L^2(\mathbb{I})} \\ & \lesssim 2^{-((d-m)\ell_1 + d(\|\ell\|_1 - \ell_1))} \|u\|_{H_{\max(0, \theta - m)}^d(\mathbb{I}) \otimes H_{\theta}^d(\mathbb{I}) \otimes \dots \otimes H_{\theta}^d(\mathbb{I})} \quad (u \in \otimes_{i=1}^n \mathcal{W}_{i1}). \end{aligned}$$

Since $d - \nu(d - m) > 0$, an application of Lemma 4.1 shows that

$$\begin{aligned} & \sum_{\{\ell: \ell_1 + \nu(\|\ell\|_1 - \ell_1) > L\}} 2^{-((d-m)\ell_1 + d(\|\ell\|_1 - \ell_1))} \\ & = \sum_{j=L+1}^{\infty} \sum_{\{\ell: \ell_1 + \nu(\|\ell\|_1 - \ell_1) \in [j, j+1]\}} 2^{-(d-m)(\ell_1 + \nu(\|\ell\|_1 - \ell_1))} 2^{-(d-\nu(d-m))(\|\ell\|_1 - \ell_1)} \\ & \lesssim \sum_{j=L+1}^{\infty} 2^{-(d-m)j} \approx 2^{-(d-m)L}, \end{aligned}$$

which completes the proof. \square

As in Theorem 4.3, let $\theta \in [0, d]$ and $\beta \in (\frac{\theta}{d}, 1)$. Then the combination of this theorem and Proposition 4.2 leads to the conclusion that

$$\|u - \mathbf{P}_{L,1}^{\beta} u\|_{L^2(\square)} \lesssim (\log \#\nabla_{L,1}^{\beta})^{(n-1)(\frac{1}{2}+d)} (\#\nabla_{L,1}^{\beta})^{-d} \|u\|_{\otimes_{i=1}^n H_{\theta}^d(\mathbb{I})}$$

for all u from the space in (4.1), and that for $\nu < \frac{d}{d-m}$,

$$\|u - \mathbf{P}_{L,\nu}^{\beta} u\|_{H^m(\square)} \lesssim (\#\nabla_{L,\nu}^{\beta})^{-(d-m)} \sqrt{\sum_{p=1}^n \|u\|_{\otimes_{i=1}^n H_{\theta - \delta_{p_i} \min(m, \theta)}^d(\mathbb{I})}^2}$$

for all $u \in \cap_{p=1}^n \otimes_{i=1}^n H_{\theta - \delta_{p_i} \min(m, \theta)}^d \cap H_{0,\Gamma}^m(\square)$. To see the first estimate, note that with $N := L^{n-1} 2^L$, we have $N^{\frac{1}{2}} \lesssim 2^L \leq N$, and so $L^{\frac{n-1}{2}} 2^{-dL} = L^{(n-1)(\frac{1}{2}+d)} N^{-d} \approx (\log N)^{(n-1)(\frac{1}{2}+d)} N^{-d}$.

In the next section, we will demonstrate that for u being the solution of the Poisson problem on \square with homogeneous Dirichlet boundary conditions and a sufficiently smooth right-hand side, the norms on the right-hand side of both estimates are bounded.

5. Elliptic regularity of the Laplace–Dirichlet problem in anisotropic weighted Sobolev spaces. In this section, we address the question of the regularity of the solution u of Poisson’s problem on $\square = (0, 1)^n$ with homogeneous Dirichlet boundary conditions and sufficiently smooth right-hand side f :

$$(5.1) \quad u \in H_0^1(\square) \quad \text{and} \quad -\Delta u = f.$$

DEFINITION 5.1. For $i = 1, \dots, n$, let ω_i^0 and ω_i^1 be real numbers. We denote by $\underline{\omega}$ the collection $(\omega_1^0, \omega_1^1, \dots, \omega_n^1)$. For $k \in \mathbb{N}_0$, we introduce the Hilbert space

$$(5.2) \quad M_{\underline{\omega}}^k(\square) = \left\{ u \in L_{\text{loc}}^2(\square) : \prod_{i=1}^n (x_i)^{\omega_i^0 + \alpha_i} (1 - x_i)^{\omega_i^1 + \alpha_i} \partial_x^{\alpha} u \in L^2(\square) \quad \forall \alpha, |\alpha| \leq k \right\}$$

with its natural norm, denoted by $\|u\|_{M_{\underline{\omega}}^k(\square)}$. In particular, $M_{\underline{0}}^k(\square)$ denotes the space of $M_{\underline{\omega}}^k(\square)$ for which $\underline{\omega} = (0, \dots, 0)$.

The main result of this section is the following.

THEOREM 5.2. *Let $\underline{\omega}$ be a collection $(\omega_1^0, \omega_1^1, \dots, \omega_n^1)$ satisfying the conditions*

$$(5.3) \quad \omega_i^0, \omega_i^1 \in (-\frac{3}{2}, 0) \quad \forall i = 1, \dots, n \quad \text{and} \quad \sum_{i=1}^n \omega_i^{\delta_i} > -2 \quad \forall \delta_i \in \{0, 1\}.$$

Let $k \geq 2$ be a natural number. If the right-hand side f in (5.1) belongs to the space $M_{\underline{\omega}}^{k+2(n-2)}(\square)$, then the solution u of problem (5.1) belongs to $M_{\underline{\omega}}^k(\square)$ with the estimate

$$(5.4) \quad \|u\|_{M_{\underline{\omega}}^k(\square)} \lesssim \|f\|_{M_{\underline{\omega}}^{k+2(n-2)}(\square)}.$$

Remark 5.3. We assume that the dimension n is ≥ 2 . Let $d \geq 1$ be an integer. Let $\theta < d$ be such that

$$d - \theta < \frac{1}{n}.$$

If we choose $\underline{\omega}$ such that

$$\omega_1^\delta = \theta - d - 1 \quad \text{and} \quad \omega_2^\delta = \dots = \omega_n^\delta = \theta - d,$$

then $\underline{\omega}$ satisfies assumption (5.3), and we have the embedding

$$(5.5) \quad M_{\underline{\omega}}^{dn}(\square) \subset H_{\theta-1}^d(\mathbb{I}) \otimes H_{\theta}^d(\mathbb{I}) \otimes \dots \otimes H_{\theta}^d(\mathbb{I}),$$

which proves that the solution of Poisson’s problem on \square with sufficiently smooth data satisfies the assumption of Theorem 4.3(b), and thus also that of Theorem 4.3(a).

Note that in the space on the left-hand side of (5.5) nonmixed partial derivatives ∂_i^k up to order dn are measured, whereas in the space on the right-hand side such partial derivatives are measured only up to order d .

The proof of Theorem 5.2 is organized in three steps:

1. Localization.
2. Basic regularity of the solution.
3. Successive estimates obtained by recurrence, each step of the recurrence using
 - (a) a dyadic partition argument;
 - (b) tangential regularity.

5.1. Localization. Let $\sigma \in [\frac{1}{2}, 1)$ be a real number. Let us denote by \square_σ the cube $(0, \sigma)^n$. With obvious notation for functional spaces, by symmetry and covering, the estimate (5.4) will be proved if we show the following local estimate for a $\sigma' \in (\sigma, 1)$:

$$(5.6) \quad \|u\|_{M_{\underline{\omega}}^k(\square_\sigma)} \lesssim \|f\|_{M_{\underline{\omega}}^{k+2(n-2)}(\square_{\sigma'})} + \|u\|_{H^1(\square_{\sigma'})}.$$

Note that on \square_σ the weight $(1 - x_i)$ is equivalent to 1, so that

$$(5.7) \quad M_{\underline{\omega}}^k(\square_\sigma) = \left\{ u \in L_{\text{loc}}^2(\square_\sigma) : \prod_{i=1}^n (x_i)^{\omega_i^0 + \alpha_i} \partial_x^\alpha u \in L^2(\square_\sigma) \quad \forall \alpha, |\alpha| \leq k \right\}.$$

Thus we can denote ω_i^0 by ω_i .

The second localization is obtained by the introduction of an order between the coordinates: The cube is covered by $n!$ domains isomorphic to

$$T = \{x \in \square : x_1 \leq x_2 \leq \dots \leq x_n\}.$$

For $\rho \in [1, 2)$, let us denote by T_ρ the set

$$(5.8) \quad T_\rho = \{x \in \square_{\rho/2} : x_1 \leq \rho x_2 \leq \dots \leq \rho^{n-1} x_n\}.$$

The estimate (5.6) will be proved if we show, for some $1 \leq \rho < \rho' < 2$,

$$(5.9) \quad \|u\|_{M_{\underline{\omega}}^k(T_\rho)} \lesssim \|f\|_{M_{\underline{\omega}}^{k+2(n-2)}(T_{\rho'})} + \|u\|_{H^1(T_{\rho'})}.$$

Finally, we note that we can assume the following conditions on the weight $\underline{\omega}$:

$$(5.10) \quad \omega_1 \in (-\frac{3}{2}, -1] \quad \text{and} \quad \omega_i \in (-1, 0), \quad i = 2, \dots, n, \quad \text{with} \quad \omega_1 + \dots + \omega_n > -2.$$

Indeed, if we prove the estimate (5.9) with condition (5.10), we obviously prove the same estimate for all weights $\underline{\omega}'$ such that

$$(5.11) \quad \prod_{i=1}^n (x_i)^{\omega'_i} \lesssim \prod_{i=1}^n (x_i)^{\omega_i} \quad \text{on} \quad T_\rho.$$

We can see that (5.11) holds for any pair of weights $\underline{\omega}, \underline{\omega}'$ satisfying

$$(5.12) \quad \omega_1 \leq \omega'_1, \quad \omega_1 + \omega_2 \leq \omega'_1 + \omega'_2, \quad \dots \quad \sum_{i=1}^n \omega_i \leq \sum_{i=1}^n \omega'_i$$

(estimate the quotient between the two members of (5.11)). Therefore for any weight $\underline{\omega}'$ such that $\omega'_i \in (-\frac{3}{2}, 0)$ with $\omega'_1 + \dots + \omega'_n > -2$ (cf. conditions of Theorem 5.2), it is easy to find $\underline{\omega}$ satisfying (5.10) and (5.12), which proves the relevance of restricting to conditions (5.10) for the proof of Theorem 5.2.

5.2. Basic regularity of the solution.

LEMMA 5.4. *The solution u of problem (5.1) with a right-hand side $f \in L^2(\square)$ belongs to $H^2(\square)$ and satisfies the estimate*

$$\|u\|_{H^2(\square)} \lesssim \|f\|_{L^2(\square)}.$$

Proof. This regularity result is known to hold for any bounded convex domain [Gri85]. On the hypercube, it can be easily shown by a reflection argument: Odd reflections across the faces of the cube allow us to define an extension \tilde{u} of u to a larger cube $\tilde{\square} \supset \square$, and such that $\Delta \tilde{u} = \tilde{f}$, where \tilde{f} is the corresponding extension of f and belongs to $L^2(\tilde{\square})$. The ellipticity of Δ implies the H^2 regularity in \square . \square

COROLLARY 5.5. *Let $V \subset V' \subset \square$ be some open sets with $\square \cap \bar{V} \subset V'$. Then for any $u \in H_0^1(\square)$ with $\Delta u \in L^2(\square)$, we have*

$$\|u\|_{H^2(V)} \lesssim \|\Delta u\|_{L^2(V')} + \|u\|_{H^1(V')}.$$

Proof. Let χ be a smooth function with support in V' and equal to 1 on V . Then considering the formula

$$\Delta(\chi u) = \chi \Delta u + 2\nabla \chi \cdot \nabla u + u \Delta \chi,$$

we deduce that

$$\|\Delta(\chi u)\|_{L^2(\square)} \lesssim \|\Delta u\|_{L^2(V')} + \|u\|_{H^1(V')}.$$

Since χu obviously belongs to $H_0^1(\square)$, from Lemma 5.4 we deduce that $\chi u \in H^2(\square)$ with

$$\|u\|_{H^2(V)} \leq \|\chi u\|_{H^2(\square)} \lesssim \|\Delta(\chi u)\|_{L^2(\square)} \lesssim \|\Delta u\|_{L^2(V')} + \|u\|_{H^1(V')}. \quad \square$$

We also need some connections of $H^2 \cap H_0^1(\square)$ with weighted spaces.

LEMMA 5.6. *Let $\omega_1, \dots, \omega_n \in (-\frac{3}{2}, 0)$ such that $\omega_1 + \dots + \omega_n > -2$. Let $\rho \in (1, 2)$. With the set T_ρ defined in (5.8), any $u \in H^2 \cap H_0^1(\square)$ satisfies*

$$(5.13) \quad \left\| \left(\prod_{i=1}^n x_i^{\omega_i} \right) u \right\|_{L^2(T_\rho)} + \sum_{|\alpha|=1} \left\| \left(\prod_{i=1}^n x_i^{\omega_i} \right) x_1 \partial_x^\alpha u \right\|_{L^2(T_\rho)} \lesssim \|u\|_{H^2(\square)}.$$

Proof. Let us set $\bar{\omega}_1 = \min\{\omega_1, \dots, \omega_n\}$ and $\bar{\omega}_2 = \omega_1 + \dots + \omega_n - \bar{\omega}_1$. On the set T_ρ there holds

$$\prod_{i=1}^n x_i^{\omega_i} \lesssim x_2^{\bar{\omega}_2} x_1^{\bar{\omega}_1}$$

(depending on ρ). Combining this with the tensor product properties of Sobolev spaces, it suffices to prove

$$\|x_2^{\bar{\omega}_2} x_1^{\bar{\omega}_1} u\|_{L^2(T_\rho)} + \sum_{|\alpha|=1} \|x_2^{\bar{\omega}_2} x_1^{\bar{\omega}_1+1} \partial_x^\alpha u\|_{L^2(T_\rho)} \lesssim \|u\|_{H^2(\square)}$$

for the dimension $n = 2$. In other words we reduce to prove (5.13) in dimension $n = 2$ for all $\omega_1 > -\frac{3}{2}$ and all $\omega_2 \geq \omega_1$ such that $\omega_1 + \omega_2 > -2$.

For such a pair (ω_1, ω_2) , let $\varepsilon > 0$ be such that

$$\varepsilon - \frac{3}{2} \leq \omega_1 \quad \text{and} \quad 2\varepsilon - 2 \leq \omega_1 + \omega_2.$$

Using (5.11)–(5.12) we find $x_2^{\omega_2} x_1^{\omega_1} \lesssim x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{3}{2}}$ on T_ρ . Therefore what we want to prove is equivalent to showing that for all $\varepsilon > 0$ and any $u \in H^2 \cap H_0^1(\mathbb{I}^2)$

$$(5.14) \quad \|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{3}{2}} u\|_{L^2(T_\rho)} + \sum_{|\alpha|=1} \|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{1}{2}} \partial_x^\alpha u\|_{L^2(T_\rho)} \lesssim \|u\|_{H^2(\mathbb{I}^2)}.$$

The first step to proving (5.14) is to show that for any $v \in H^1(\mathbb{I}^2)$,

$$(5.15) \quad \|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{1}{2}} v\|_{L^2(\mathbb{I}^2)} \lesssim \|v\|_{H^1(\mathbb{I}^2)}.$$

This latter estimate relies on the combination of the one-dimensional estimate

$$\|x^{-s} v\|_{L^2(\mathbb{I})} \lesssim \|v\|_{H^s(\mathbb{I})} \quad \text{for} \quad s < \frac{1}{2}$$

(depending on s), which is a consequence of [Dau88, Thm. AA.7], with the embedding $H^1(\mathbb{I}^2) \subset H^s(\mathbb{I}) \otimes H^s(\mathbb{I})$ for all $s \leq \frac{1}{2}$, which is a consequence of [Gri66, Thm. 5.1].

Using (5.15) with $v = \partial_x^\alpha u$ for $|\alpha| = 1$, we obtain

$$\sum_{|\alpha|=1} \|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{1}{2}} \partial_x^\alpha u\|_{L^2(\mathbb{I}^2)} \lesssim \|u\|_{H^2(\mathbb{I}^2)}.$$

Thanks to the fact that $u(0, x_2) \equiv 0$, the Hardy inequality with respect to the variable x_1 yields

$$\|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{3}{2}} u\|_{L^2(\mathbb{I}^2)} \leq \frac{1}{1-\varepsilon} \|x_2^{\varepsilon-\frac{1}{2}} x_1^{\varepsilon-\frac{1}{2}} \partial_{x_1} u\|_{L^2(\mathbb{I}^2)}.$$

Combining the last two estimates, we obtain (5.14) and hence finally (5.13). \square

5.3. Weighted regularity of the solution. Our aim is to prove (5.9). As already mentioned, we can assume condition (5.10) on the weight $\underline{\omega}$. Let us choose $\rho \in [1, 2)$ and $\delta \in (0, \frac{1}{8})$, and set

$$V_\ell = \{x \in T_\rho : \rho^\ell x_{\ell+1} > \delta\}, \quad \ell = 0, \dots, n-1, \quad \text{and} \quad V_n = T_\rho.$$

For $2 > \rho' > \rho$ and $0 < \delta' < \delta$, we define similarly

$$V'_\ell = \{x \in T_{\rho'} : (\rho')^\ell x_{\ell+1} > \delta'\}, \quad \ell = 0, \dots, n-1, \quad \text{and} \quad V_n = T_{\rho'},$$

and also V''_ℓ and V'''_ℓ associated with some $2 > \rho''' > \rho'' > \rho'$ and $0 < \delta''' < \delta'' < \delta'$. We set $\omega_i := \omega_i^0$.

The weight appearing in (5.7) is equivalent to a simpler weight on V_ℓ :

$$\prod_{i=1}^n (x_i)^{\omega_i+\alpha_i} \approx 1 \quad \text{on } V_0 \quad \text{and} \quad \prod_{i=1}^n (x_i)^{\omega_i+\alpha_i} \approx \prod_{i=1}^\ell (x_i)^{\omega_i+\alpha_i} \quad \text{on } V_\ell, \ell \geq 1.$$

In this section, we are going to prove by recurrence over $\ell = 0, \dots, n$ that

$$(5.16) \quad \|u\|_{M_{\underline{\omega}}^k(V_\ell)} \lesssim \|f\|_{M_{\underline{\omega}}^{k+2(\ell-1)}(V'_\ell)} + \|u\|_{H^1(V'_\ell)},$$

and we finally deduce (5.9).

- For $\ell = 0$, the estimate (5.16) is simply

$$(5.17) \quad \|u\|_{H^k(V_0)} \lesssim \|f\|_{H^{k-2}(V'_0)} + \|u\|_{H^1(V'_0)}$$

and is a consequence of the ellipticity of the operator Δ and $V_0 \subset\subset \square$.

- The proof of (5.16) for $\ell = 1$ is done in two steps.

Step 1. With the help of V_0 we construct a locally finite dyadic covering of V_1 : For a chosen $L > 0$, $\mu \in \mathbb{N}_0$, and $\nu \in \mathbb{N}_0^{n-1}$ we denote by $V_1^{\mu,\nu}$ the set

$$V_1^{\mu,\nu} = 2^{-\mu}(V_0 + L(0, \nu)) = \{x = 2^{-\mu}(y + L(0, \nu)), \quad y \in V_0\}.$$

Likewise, we set

$$(V'_1)^{\mu,\nu} = 2^{-\mu}(V'_0 + L(0, \nu)).$$

Then there exists $L > 0$ and for each $\mu \in \mathbb{N}_0$ a subset $\mathbb{N}_{0,\mu}^{n-1}$ of \mathbb{N}_0^{n-1} such that

$$V_1 \subset \bigcup_{\mu \in \mathbb{N}_0} \bigcup_{\nu \in \mathbb{N}_{0,\mu}^{n-1}} V_1^{\mu,\nu} \subset V'_1 \subset \bigcup_{\mu \in \mathbb{N}_0} \bigcup_{\nu \in \mathbb{N}_{0,\mu}^{n-1}} (V'_1)^{\mu,\nu} \subset V''_1.$$

Since $x_1 \approx 1$ on V_0 , from the estimate (5.17) we deduce that for any $v \in H^1(V'_0)$ such that $\Delta v \in H^{k-2}(V'_0)$, there holds

$$\sum_{|\alpha| \leq k} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha v\|_{V_0} \lesssim \sum_{|\alpha| \leq k-2} \|x_1^{|\alpha|} \partial_x^\alpha \Delta v\|_{V'_0} + \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha v\|_{V'_0}.$$

Here the norm $\|\cdot\|_V$ means the L^2 -norm on V .

By the change of variables $y \rightarrow x = 2^{-\mu}(y + L(0, \nu))$ from V_0 onto $V_1^{\mu, \nu}$ we deduce from the previous estimate

$$(5.18) \quad \sum_{|\alpha| \leq k} 2^{\mu\omega_1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V_1^{\mu, \nu}} \lesssim \sum_{|\alpha| \leq k-2} 2^{-2\mu} \|x_1^{|\alpha|} \partial_x^\alpha f\|_{(V'_1)^{\mu, \nu}} + \sum_{|\alpha| \leq 1} 2^{\mu\omega_1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{(V'_1)^{\mu, \nu}}.$$

Since $2 + \omega_1 \geq 0$ and $\mu \geq 0$, this estimate yields

$$\sum_{|\alpha| \leq k} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V_1^{\mu, \nu}} \lesssim \sum_{|\alpha| \leq k-2} \|x_1^{|\alpha|} \partial_x^\alpha f\|_{(V'_1)^{\mu, \nu}} + \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{(V'_1)^{\mu, \nu}}.$$

Summing up the squares of the above estimates for all $\mu \in \mathbb{N}_0$ and $\nu \in \mathbb{N}_{0, \mu}^{n-1}$ we obtain

$$(5.19) \quad \sum_{|\alpha| \leq k} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k-2} \|x_1^{|\alpha|} \partial_x^\alpha f\|_{V'_1} + \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V'_1}.$$

To bound the last term on the right-hand side of this inequality, let χ be a smooth function with support in V'_1 and equal to 1 in V_1 . There holds

$$\sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V'_1} \lesssim \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha \chi u\|_{V'_1}.$$

Using that $V'_1 \subset T_{\rho''}$, from Lemma 5.6 and Corollary 5.5 we deduce that

$$\begin{aligned} \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha \chi u\|_{T_{\rho''}} &\lesssim \|\chi u\|_{H^2(T_{\rho''})} = \|\chi u\|_{H^2(V'_1)} \\ &\lesssim \|\Delta(\chi u)\|_{V_1'''} + \|u\|_{H^1(V_1''')} \lesssim \|f\|_{V_1'''} + \|u\|_{H^1(V_1''')}. \end{aligned}$$

This estimate combined with estimate (5.19) gives

$$\sum_{|\alpha| \leq k} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k-2} \|x_1^{|\alpha|} \partial_x^\alpha f\|_{V_1'''} + \|u\|_{H^1(V_1''')}.$$

Since the choice of the parameters ρ''' and δ''' determining V_1''' is arbitrary (the other parameters $\rho'', \rho', \delta', \delta''$, and L can be tuned accordingly), we may change notation in the above estimate, simply replacing V_1''' by V'_1 :

$$(5.20) \quad \sum_{|\alpha| \leq k} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k-2} \|x_1^{|\alpha|} \partial_x^\alpha f\|_{V'_1} + \|u\|_{H^1(V'_1)}.$$

Step 2. The aim of this step is to replace $|\alpha|$ by α_1 on the left-hand side of (5.20). This corresponds to improving the *tangential* regularity estimate.

Let $\mathbb{N}_0 \ni q < k$. The estimate (5.20) implies in particular

$$\sum_{0 \leq \alpha_1 \leq q} \|x_1^{\omega_1 + \alpha_1} \partial_{x_1}^{\alpha_1} u\|_{V_1} \lesssim \sum_{|\alpha| \leq \max(q-2, 0)} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_1} + \|u\|_{H^1(V'_1)}.$$

Let $\alpha'_2 = (\alpha_2, \dots, \alpha_n)$ be a multi-index of length 1. We denote (x_2, \dots, x_n) by x'_2 . Since

$$\partial V'_1 \cap \partial \square \subset \{x : x_1 = 0\},$$

we can apply the above estimate to $\partial_{x'_2}^{\alpha'_2}$ and deduce

$$\sum_{0 \leq \alpha_1 \leq q} \|x_1^{\omega_1 + \alpha_1} \partial_{x'_2}^{\alpha'_2} \partial_{x_1}^{\alpha_1} u\|_{V_1} \lesssim \sum_{|\alpha| \leq \max(q-1, 1)} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_1} + \|u\|_{H^2(V'_1)}.$$

We combine with the bound $\|u\|_{H^2(V'_1)} \lesssim \|f\|_{V''_1} + \|u\|_{H^1(V''_1)}$ from Corollary 5.5 and obtain

$$\sum_{0 \leq \alpha_1 \leq q} \|x_1^{\omega_1 + \alpha_1} \partial_{x'_2}^{\alpha'_2} \partial_{x_1}^{\alpha_1} u\|_{V_1} \lesssim \sum_{|\alpha| \leq \max(q-1, 1)} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V''_1} + \|u\|_{H^1(V''_1)}.$$

By the same considerations as above, we change V''_1 to V'_1 and can proceed in the same way until $|\alpha'_2| + q = k$, so that we arrive at

$$\sum_{0 \leq \alpha_1 \leq q} \|x_1^{\omega_1 + \alpha_1} \partial_{x'_2}^{\alpha'_2} \partial_{x_1}^{\alpha_1} u\|_{V_1} \lesssim \sum_{|\alpha| \leq \max(k-2, k-q)} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V''_1} + \|u\|_{H^1(V''_1)}.$$

Hence

$$(5.21) \quad \sum_{|\alpha| \leq k} \|x_1^{\omega_1 + \alpha_1} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_1} + \|u\|_{H^1(V'_1)}.$$

Finally, using condition (5.10), which ensures that $\omega_1 + |\alpha| \leq 0$ for all $|\alpha| \leq 1$, we deduce from (5.21) the fully weighted estimate

$$(5.22) \quad \sum_{|\alpha| \leq k} \|x_1^{\omega_1 + \alpha_1} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k} \|x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_1} + \sum_{|\alpha| \leq 1} \|x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V'_1}.$$

• For $\ell = 2$ we have the corresponding two steps: We incorporate the powers of x_2 in estimate (5.22) (which is allowed since $x_2 \approx 1$ in V_1), still denoting $(\alpha_2, \dots, \alpha_n)$ by α'_2 :

$$\sum_{|\alpha| \leq k} \|x_2^{\omega_2 + |\alpha'_2|} x_1^{\omega_1 + \alpha_1} \partial_x^\alpha u\|_{V_1} \lesssim \sum_{|\alpha| \leq k} \|x_2^{|\alpha'_2|} x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_1} + \sum_{|\alpha| \leq 1} \|x_2^{\omega_2} x_1^{\omega_1 + |\alpha|} \partial_x^\alpha u\|_{V'_1}.$$

We construct a dyadic covering of V_2 with domains $V_2^{\mu, \nu}$ similar to V_1 (here ν belongs to \mathbb{N}_0^{n-2}). We use the fact that $2 + \omega_1 + \omega_2 > 0$ and obtain by steps similar to those above that

$$\sum_{|\alpha| \leq k} \|x_2^{\omega_2 + |\alpha'_2|} x_1^{\omega_1 + \alpha_1} \partial_x^\alpha u\|_{V_2} \leq C \left\{ \sum_{|\alpha| \leq k} \|x_2^{|\alpha'_2|} x_1^{\alpha_1} \partial_x^\alpha f\|_{V'_2} + \|u\|_{H^1(V'_2)} \right\}.$$

We end the proof of (5.16) for $\ell = 2$ using tangential derivatives $\partial_{x'_3}^{\alpha'_3}$, with $\alpha'_3 = (\alpha_3, \dots, \alpha_n)$ and $x'_3 = (x_3, \dots, x_n)$.

• For $\ell = n$, only the first step is necessary, which, in the end, proves (5.16), and hence (5.9).

5.4. Comments. There are not many results about regularity in general polyhedra in any space dimension. Let us quote [MP77, MR91] for results in isotropic weighted spaces and [Dau88] for results in standard Sobolev spaces. In addition, for the space dimension $n \geq 3$, regularity in isotropic weighted spaces is by no means sufficient to ensure the assumptions of Theorem 4.3. The claim in [Nit05] that in dimension $n = 3$ a regularity result in isotropic weighted spaces from [MR91] would guarantee the assumptions of Theorem 4.3 seems to be caused by a mistake in the reproduction of the definition of [MR91]’s weighted spaces.

The spaces which we consider in our paper are *anisotropic*, requiring more regularity in directions tangential to edges of any dimensions. Let us quote the note [BCD03], where similar spaces are employed on three-dimensional polyhedra. Thus, to our knowledge, Theorem 5.2 is new.

With suitable definitions for the weighted spaces $M_{\underline{\omega}}^k$ (compare with [BCD03] in dimension $n = 3$) our result on the regularity of solution of Poisson’s problem (5.1) can be extended to any n -dimensional polyhedron Ω along the following lines:

- If the polyhedron Ω is convex, the H^2 regularity (Lemma 5.4) still holds, and Theorem 5.2 is true with the same conditions on $\underline{\omega}$.
- If the polyhedron Ω is not convex, the solution u with a right-hand side $f \in L^2(\Omega)$ does not belong to $H^2(\Omega)$ in general, but there exists $s = s_{\Omega} > 1$ so that $u \in H^s(\Omega)$; see [Dau88]. Then it could be proved that the conclusion of Theorem 5.2 holds under the following stronger condition on $\underline{\omega}$:

$$\omega_i^0, \omega_i^1 \in (-\frac{3}{2}, 0) \quad \forall i = 1, \dots, n \quad \text{and} \quad \sum_{i=1}^n \omega_i^{\delta_i} > -s \quad \forall \delta_i \in \{0, 1\}.$$

Likewise, Theorem 5.2 could be generalized to the Dirichlet problem in H_0^m associated with any coercive sesquilinear form of order m (operator of order $2m$) with smooth coefficients.

In contrast, the solution u of the Neumann problem for the Laplace operator on \square does not belong to $M_{\underline{\omega}}^k(\square)$ under condition (5.3) in general. Nevertheless, if conditions appearing in (5.2) are restricted to the case when $\alpha_1, \dots, \alpha_n \geq 1$, we can still hope that u fulfills such conditions.

6. The approximation of the solution of Poisson’s problem using standard sparse grids. *Without* local refinement, i.e., $\beta = 0$, the estimates from Theorem 2.6 read as

$$\begin{aligned} \|u - P_{\ell}^0 u\|_{L^2(\mathbb{I})} &\lesssim 2^{-d\ell} \|u\|_{H^d(\mathbb{I})} \quad (u \in H^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})), \\ \|u - P_{\ell}^0 u\|_{H^m(\mathbb{I})} &\lesssim 2^{-(d-m)\ell} \|u\|_{H^d(\mathbb{I})} \quad (u \in H^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})). \end{aligned}$$

More generally, with the interpolation space $\mathcal{H}_{0,z}^t(\mathbb{I}) := [L^2(\mathbb{I}), H^d(\mathbb{I}) \cap H_{0,z}^m(\mathbb{I})]_{\frac{t}{d}, 2}$, it holds that

$$(6.1) \quad \|u - P_{\ell}^0 u\|_{L^2(\mathbb{I})} \lesssim 2^{-t\ell} \|u\|_{\mathcal{H}_{0,z}^t(\mathbb{I})} \quad (t \in [0, d], u \in \mathcal{H}_{0,z}^t(\mathbb{I})),$$

$$(6.2) \quad \|u - P_{\ell}^0 u\|_{H^m(\mathbb{I})} \lesssim 2^{-(t-m)\ell} \|u\|_{\mathcal{H}_{0,z}^t(\mathbb{I})} \quad (t \in [m, d], u \in \mathcal{H}_{0,z}^t(\mathbb{I})).$$

Similarly to Theorem 4.3(a), we arrive at

$$\|u - \mathbf{P}_{L,1}^0 u\|_{L^2(\square)} \lesssim L^{\frac{n-1}{2}} 2^{-sL} \|u\|_{\otimes_{i=1}^n \mathcal{H}_{0,z_i}^s(\mathbb{I})} \quad (s \in [0, d], u \in \otimes_{i=1}^n \mathcal{H}_{0,z_i}^s(\mathbb{I})).$$

By applying (6.1) with $t = s$, and (6.2) with $t = s - m + \varepsilon$, similarly to Theorem 4.3(b), we infer that for $s \in [m, d]$, $\varepsilon \in (0, m]$, and $(s - m)\nu < (s - m) + \varepsilon$,

$$(6.3) \quad \|u - \mathbf{P}_{L,\nu}^0 u\|_{H^m(\square)} \lesssim 2^{-(s-m)L} \sqrt{\sum_{p=1}^n \|u\|_{\otimes_{i=1}^n \mathcal{H}_{0,z_i}^{s+(1-\delta_{pi})(\varepsilon-m)}(\mathbf{I})}}^2$$

for all $u \in \cap_{p=1}^n \otimes_{i=1}^n \mathcal{H}_{0,z_i}^{s+(1-\delta_{pi})(\varepsilon-m)}(\mathbf{I})$, where, by taking $\nu > 1$, $2^{-(s-m)L} \asymp (\#\nabla_{L,\nu}^0)^{-(s-m)}$.

The estimates (6.1), (6.2), and, similarly, (6.3) are sharp in the sense that, generally, a decay of the error like $2^{-t\ell}$, $2^{-(t-m)\ell}$, or $2^{-(s-m)L}$ is not valid when only $u \in \mathcal{H}_{0,z_i}^{t'}(\mathbf{I})$ for some $t' < t$, or $u \in \cap_{p=1}^n \otimes_{i=1}^n \mathcal{H}_{0,z_i}^{s'+(1-\delta_{pi})(\varepsilon'-m)}(\mathbf{I})$ for some $s' < s$ or $\varepsilon' \leq 0$. Note that we had to include the word “generally” in the preceding sentence, because higher order convergence for less smooth u does hold when u happens to be a finite linear combination of wavelets ψ_λ , or when it is exceptionally close to such a function.

In the following, we again consider u to be the solution of Poisson’s problem on $\square = (0, 1)^n$ with homogeneous Dirichlet boundary conditions and sufficiently smooth right-hand side f :

$$(6.4) \quad u \in H_0^1(\square) \quad \text{and} \quad -\Delta u = f,$$

i.e., $m = 1$ and $z_i = \{0, 1\}$. We will show that for sufficiently smooth right-hand sides f and $n \geq 2$, for

$$(6.5) \quad s \in [1, \frac{3}{2} + \frac{1}{n}) \implies u \in \cap_{p=1}^n \otimes_{i=1}^n \mathcal{H}_{0,\{0,1\}}^{s-1+\delta_{ip}}(\mathbf{I}),$$

while for $s > \frac{3}{2} + \frac{1}{n}$, generally u is *not* in that space (it will be when additionally f vanishes at a sufficiently high order at the boundary). In view of the estimate (6.3) and the discussion following it, these results mean that the error, measured in $H^1(\square)$, in the best approximation from the optimized sparse-grid space of dimension N of the solution of Poisson’s problem on the n -dimensional hypercube decays like $N^{-(s-1)}$ for any $s < \frac{3}{2} + \frac{1}{n}$, while, generally, this estimate does not hold for $s > \frac{3}{2} + \frac{1}{n}$. Note that here we could write “best approximation,” since the error $u - \mathbf{P}_{L,\nu}^0 u$, measured in $\|\cdot\|_{H^1(\square)}$ is, indeed, up to a constant factor, equal to the error in the best approximation from $\text{span } \mathbf{P}_{L,\nu}^0$, because the projector $\mathbf{P}_{L,\nu}^0$ is uniformly bounded in the $H^1(\square)$ -norm. The reason is that the wavelet basis $\{(\sum_{i=1}^n 4^{|\lambda_i|})^{-1/2} \psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $H_{0,\Gamma}^1(\square)$.

In a way the above result is favorable. Indeed, unlike with standard “full grid” approximation, the rate $\frac{1}{2} + \frac{1}{n}$, although mildly dependent on n , does not decrease to 0 when $n \rightarrow \infty$. On the other hand, however, the rate $\frac{1}{2} + \frac{1}{n}$ does not show any benefit of taking piecewise polynomial wavelets of orders d larger than 2, and depending on the value of d , this rate is (much) worse than the optimal rate $d - 1$ that is realized with the approach of combining optimized sparse grids with the local refinements outlined in the preceding sections.

Before continuing, we note that for $s \geq 1$, $s \neq \frac{3}{2}$,

$$(6.6) \quad \cap_{p=1}^n \otimes_{i=1}^n \mathcal{H}_{0,\{0,1\}}^{s-1+\delta_{ip}}(\mathbf{I}) = \cap_{p=1}^n \otimes_{i=1}^n H^{s-1+\delta_{ip}}(\mathbf{I}) \cap H_0^1(\square).$$

Indeed, since

$$\mathcal{H}_{0,\{0,1\}}^t(\mathbf{I}) = \begin{cases} H^t(\mathbf{I}), & t \in [0, \frac{1}{2}), \\ \{u \in H^t(\mathbf{I}) : \gamma_v u = 0, v \in \{0, 1\}\}, & t > \frac{1}{2}, \end{cases}$$

the proof of (6.6) follows that of (3.4a) for $t \in [0, \frac{1}{2})$ taking the first option in (3.4c), and for $t > \frac{1}{2}$ taking the second option. Since we know that the solution of Poisson's problem is in $H_0^1(\square)$, what is left is to show that for

$$s \in [1, \frac{3}{2} + \frac{1}{n}) \implies \partial_p u \in \otimes_{i=1}^n H^{s-1}(\mathbb{I}) \quad (1 \leq p \leq n),$$

known as a Sobolev space of dominating mixed smoothness, and that this is generally *not true* for $s > \frac{3}{2} + \frac{1}{n}$.

In order to do so, we apply a Fourier analysis. For $k \in \mathbb{N}$, let $\phi_k(x) := \sqrt{2} \sin(k\pi x)$, $\psi_k(x) = \sqrt{2} \cos(k\pi x)$, and let $\psi_0 = 1$. It is clear that $\{\phi_k : k \in \mathbb{N}\}$ is the normed eigenvector basis of the self-adjoint operator $\varphi \mapsto -\varphi''$ with domain $H^2 \cap H_0^1(\mathbb{I})$ (Dirichlet). Likewise, $\{\psi_k : k \in \mathbb{N}_0\}$ is the normed eigenvector basis of the self-adjoint operator $\varphi \mapsto -\varphi''$ with domain $\{\varphi \in H^2(\mathbb{I}), \varphi'(0) = \varphi'(1) = 0\}$ (Neumann). Furthermore $\phi'_k = k\pi\psi_k$ and $\psi'_k = -k\pi\phi_k$. As a consequence we obtain immediately the following.

LEMMA 6.1. $\{\phi_k : k \in \mathbb{N}\}$ and $\{\psi_k : k \in \mathbb{N}_0\}$ are orthonormal bases for $L^2(\mathbb{I})$.

LEMMA 6.2. $\{\frac{\phi_k}{k\pi} : k \in \mathbb{N}\}$ ($\{\frac{\phi_k}{\sqrt{1+k^2\pi^2}} : k \in \mathbb{N}\}$) is an orthonormal basis of $H_0^1(\mathbb{I})$ equipped with $|\cdot|_{H^1(\mathbb{I})}$ ($\|\cdot\|_{H^1(\mathbb{I})}$), and $\{\frac{\psi_k}{\sqrt{1+k^2\pi^2}} : k \in \mathbb{N}_0\}$ is an orthonormal basis of $H^1(\mathbb{I})$.

By taking tensor products, we conclude that

$$\{\phi_{\mathbf{k}} := \otimes_{i=1}^n \phi_{k_i} : \mathbf{k} \in \mathbb{N}^n\}, \quad \left\{ \frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2} : \mathbf{k} \in \mathbb{N}^n \right\}$$

are orthonormal bases for

$$L^2(\square), \quad H_0^1(\square),$$

respectively, the latter space being equipped with $|\cdot|_{H^1(\square)} := \sqrt{\sum_{i=1}^n \|\partial_i \cdot\|_{L^2(\square)}^2}$.

By using this basis, the stiffness matrix resulting from the variational formulation of the Poisson problem is the identity, and we conclude that for a right-hand side $f \in (H_0^1(\square))'$, the solution u is given by

$$(6.7) \quad u = \sum_{\mathbf{k} \in \mathbb{N}^n} f\left(\frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2}\right) \frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2},$$

and so

$$(6.8) \quad \partial_p u(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^n} \frac{k_p}{\|\mathbf{k}\|_2} f\left(\frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2}\right) \psi_{k_p}(x_p) \prod_{i \neq p} \phi_{k_i}(x_i) \quad (1 \leq p \leq n).$$

To estimate the decay of the coefficients in this expansion for $\|\mathbf{k}\|_2 \rightarrow \infty$, we note that for $g \in W^{1,1}(\mathbb{I})$ ($\hookrightarrow L^\infty(\mathbb{I})$),

$$\int_{\mathbb{I}} g(x) \phi_k(x) dx = -g(x) \frac{\psi_k(x)}{k\pi} \Big|_0^1 + \int_{\mathbb{I}} g'(x) \frac{\psi_k(x)}{k\pi} dx,$$

and so $|\int_{\mathbb{I}} g(x) \phi_k(x) dx| \lesssim 1/k$, where a decay of higher order can be expected only when g vanishes at $\partial\mathbb{I}$ (see Remark 6.4). As a consequence, if $f \in \otimes_{i=1}^n W^{1,1}(\mathbb{I})$ ($\hookrightarrow L^\infty(\mathbb{I})$), then

$$(6.9) \quad \left| f\left(\frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2}\right) \right| \lesssim \left(\|\mathbf{k}\|_2 \prod_{i=1}^n k_i \right)^{-1},$$

which for f that do not vanish at $\partial\Omega$ is generally sharp. For example, for $f = 1$,

$$f\left(\frac{\phi_{\mathbf{k}}}{\pi\|\mathbf{k}\|_2}\right) = \begin{cases} \frac{1}{\pi\|\mathbf{k}\|_2} \left(\frac{2\sqrt{2}}{\pi}\right)^n \frac{1}{\prod_{i=1}^n k_i} & \text{when } \mathbf{k} \in (2\mathbb{N} + 1)^n, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 6.3. For $v = \sum_{\mathbf{k} \in \prod_{i=1}^n \mathbb{K}_i} c_{\mathbf{k}} v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_n}^{(n)}$, where $\mathbb{K}_p = \mathbb{N}_0$ and $v_{k_p}^{(p)} = \psi_{k_p}$, and $\mathbb{K}_i = \mathbb{N}$ and $v_{k_i}^{(i)} = \phi_{k_i}$ for $i \neq p$, for $t \in [0, 1] \setminus \{\frac{1}{2}\}$ we have

$$\|v\|_{H^t(\Omega) \otimes \cdots \otimes H^t(\Omega)} \sim \left\| \mathbf{k} \mapsto \left(\prod_{i=1}^n \sqrt{1 + k_i^2 \pi^2} \right)^t c_{\mathbf{k}} \right\|_{\ell^2(\prod_{i=1}^n \mathbb{K}_i)}.$$

Proof. For any $\mathbf{k} \in \prod_{i=1}^n \mathbb{K}_i$, let $w_{\mathbf{k}} := \prod_{i=1}^n \sqrt{1 + k_i^2 \pi^2}$, and

$$\ell_w^2\left(\prod_{i=1}^n \mathbb{K}_i\right) := \left\{ c \in \ell^2\left(\prod_{i=1}^n \mathbb{K}_i\right) : \|c\|_{\ell_w^2(\prod_{i=1}^n \mathbb{K}_i)} := \sqrt{\sum_{\mathbf{k} \in \prod_{i=1}^n \mathbb{K}_i} |w_{\mathbf{k}} c_{\mathbf{k}}|^2} < \infty \right\}.$$

Then $(c_{\mathbf{k}})_{\mathbf{k} \in \prod_{i=1}^n \mathbb{K}_i} \mapsto \sum_{\mathbf{k} \in \prod_{i=1}^n \mathbb{K}_i} c_{\mathbf{k}} v_{k_1}^{(1)} \otimes \cdots \otimes v_{k_n}^{(n)}$ is a boundedly invertible mapping (even an isomorphism) from $\ell^2(\prod_{i=1}^n \mathbb{K}_i) \rightarrow L^2(\Omega)$ as well as from $\ell_w^2(\prod_{i=1}^n \mathbb{K}_i) \rightarrow \mathcal{H}^1(\Omega) := H_0^1(\Omega) \otimes \cdots \otimes H^1(\Omega) \otimes \cdots \otimes H_0^1(\Omega)$, with the space $H^1(\Omega)$ being on the p th position. By definition of an interpolation space, for $t \in [0, 1]$, the mapping is also boundedly invertible from $\ell_{w^t}^2(\prod_{i=1}^n \mathbb{K}_i)$ into

$$[L_2(\Omega), \mathcal{H}^1(\Omega)]_{t,2} = [L_2(\Omega), H_0^1(\Omega)]_{t,2} \otimes \cdots \otimes [L_2(\Omega), H^1(\Omega)]_{t,2} \otimes \cdots \otimes [L_2(\Omega), H_0^1(\Omega)]_{t,2}.$$

Since $[L_2(\Omega), H^1(\Omega)]_{t,2} = H^t(\Omega)$, $[L_2(\Omega), H_0^1(\Omega)]_{t,2} = H_0^t(\Omega)$ when $t > \frac{1}{2}$, and $[L_2(\Omega), H_0^1(\Omega)]_{t,2} = H^t(\Omega)$ when $t < \frac{1}{2}$, in all cases with a norm that is equivalent to the $H^t(\Omega)$ -norm, the proof is completed. \square

The estimate (6.9) is equivalent to

$$\left| \left(\prod_{i=1}^n \sqrt{1 + \pi^2 k_i^2} \right)^{s-1} \frac{k_p}{\|\mathbf{k}\|_2} f\left(\frac{\phi_{\mathbf{k}}}{\pi\|\mathbf{k}\|_2}\right) \right| \lesssim \frac{k_p}{\|\mathbf{k}\|_2^2} \left(\prod_{i=1}^n k_i \right)^{s-2}.$$

We claim that the right-hand side is in $\ell^2(\mathbb{N}^n)$ iff $s - 1 < \frac{1}{2} + \frac{1}{n}$. Indeed,

$$(6.10) \quad \frac{k_p}{\|\mathbf{k}\|_2^2} \left(\prod_{i=1}^n k_i \right)^{s-2} \leq \frac{1}{\|\mathbf{k}\|_2} \left(\prod_{i=1}^n k_i \right)^{s-2} \leq \left(\prod_{i=1}^n k_i \right)^{s-2-\frac{1}{n}},$$

and the right-hand side is in $\ell^2(\mathbb{N}^n)$ iff $2(s - 2 - \frac{1}{n}) < -1$ or $s - 1 < \frac{1}{2} + \frac{1}{n}$. Up to a constant factor, which may depend on n , the estimates from (6.10) are sharp for all $\mathbf{k} \in \mathbb{N}^n$ with, say, $\max_i k_i \leq 2 \min_i k_i$, whereas for those \mathbf{k} , $(\prod_{i=1}^n k_i)^{s-2-\frac{1}{n}} \approx \|\mathbf{k}\|_2^{(s-2)n-1}$. Clearly, $\sum_{\{\mathbf{k} \in \mathbb{N}^n : \max_i k_i \leq 2 \min_i k_i\}} \|\mathbf{k}\|_2^{((s-2)n-1)2} < \infty$ iff $\sum_{\mathbf{k} \in \mathbb{N}^n} \|\mathbf{k}\|_2^{((s-2)n-1)2} < \infty$, the latter being the case iff $2(s - 2)n - 2 + (n - 1) < -1$ or $s - 1 < \frac{1}{2} + \frac{1}{n}$, which proves our claim.

Since $\frac{1}{2} + \frac{1}{n} \leq 1$ when $n \geq 2$, this last result in combination with (6.8), (6.9), and Lemma 6.3 proves our earlier claim about the regularity in Sobolev spaces of

dominating mixed smoothness of the (first order partial derivatives of the) solution of Poisson’s problem on the n -dimensional hypercube, and with that our claim about the rate of convergence of the optimized sparse-grid approximation without local refinement.

Remark 6.4. If $f \in \otimes_{i=1}^n W_0^{d-1,1}(\mathbf{I}) \cap W^{d,1}(\mathbf{I}) = \{g \in \otimes_{i=1}^n W^{d,1}(\mathbf{I}) : \partial_n^r g = 0 \text{ on } \partial\Box, 0 \leq r \leq d - 2\}$, then repeated integration by parts shows that (6.9) can be replaced by

$$\left| f\left(\frac{\phi_{\mathbf{k}}}{\pi \|\mathbf{k}\|_2}\right) \right| \lesssim \left(\|\mathbf{k}\|_2 \prod_{i=1}^n k_i^d \right)^{-1}.$$

Then for $s < d + \frac{1}{2} + \frac{1}{n}$ one shows that $\partial_p u \in \otimes_{i=1}^n H^{s-1}(\mathbf{I})$ ($1 \leq p \leq n$), meaning that for $\nu \in (1, 1 + \frac{1/2+1/n}{d-1})$, $\|u - \mathbf{P}_{L,\nu}^0 u\|_{H^1(\Box)} \lesssim (\#\nabla_{L,\nu}^0)^{-(d-1)}$. So if the right-hand f is sufficiently smooth and *vanishes at the boundary* at a sufficiently high order, then already the optimized sparse-grid approximation *without* local refinement of the solution of Poisson’s problem converges in the $H^1(\Box)$ -norm with the best possible dimension-independent rate.

Remark 6.5. Because of the explicit expression (6.7), one may think of approximating u by the expansion formed by its N largest Fourier coefficients. Assuming a general smooth right-hand side f , based on (6.9) one may verify (cf. [Dij09]) that the error measured in $H^1(\Box)$ of such an approximation with N coefficients is generally not better than that of order $N^{-(\frac{1}{2} + \frac{1}{n})}$, i.e., the approximation rate is equal to that of the optimized sparse-grid approximation without local refinement. The computational complexity of computing the N largest Fourier coefficients is much higher though.

Remark 6.6. The bound $\frac{3}{2} + \frac{1}{n}$ in (6.5) is due to the presence of a singular part in the solutions of the Dirichlet problem (6.4). When $n = 2$, as is well known [Gri85, Dau88], the first contribution to this singular part attached to the corner $(0, 0)$ is the function

$$(x_1, x_2) \mapsto f(0, 0) \left\{ \frac{1}{\pi} r^2 (\log r \sin 2\theta + \theta \cos 2\theta) + \frac{1}{2} x_2^2 \right\},$$

with the polar coordinate (r, θ) . Note the coefficient $f(0, 0)$. Taking similar contributions at four corners into account, we see that if f is not zero at one corner, then $u \notin H^3(\Box)$. In contrast, if f is zero at all corners, $u \in H^{5-\varepsilon}(\Box)$ for all $\varepsilon > 0$, because the next singularity is in $r^4 \log r$. More generally, if f and all its derivatives are zero at all corners, then $u \in H^d(\Box)$ for all $d \in \mathbb{N}$. In fact, the necessary and sufficient condition for the infinite smoothness of u up to corners is even weaker: it requires that at four corners \mathbf{c}_i , $i = 1, 2, 3, 4$, $f(\mathbf{c}_i) = 0$, $(\partial_1^2 - \partial_2^2)f(\mathbf{c}_i) = 0$, and in general $A_i^{2j} f(\mathbf{c}_i) = 0$, where A_i^{2j} are operators of order $2j$, $j \in \mathbb{N}_0$.

In higher dimensions n , the singular part is still related to the traces of f on the edges of \Box . This feature is very particular to the Laplacian and to the domain \Box : It is related to the validity of the reflection principle (cf. the proof of Lemma 5.4): for a smooth right-hand side f with compact support in \Box , this principle yields immediately the infinite smoothness of the solution u .

When $n = 3$, the solutions of the Dirichlet problem (6.4) in more general polyhedral domains than \Box have singular parts which involve distance functions to the singular sets of the domain—edges and corners—and not the distance to the faces of the domain; see [Dau88, Ch. 17]. However, unless all opening angles have the form $\frac{\pi}{\ell}$ with integers ℓ , the fact that $f \equiv 0$ on the boundary does not improve the regularity of the solution in general.

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