Forcing absoluteness and regularity properties
Ikegami, D.

Published in:
Annals of Pure and Applied Logic

DOI:
10.1016/j.apal.2009.10.005

Citation for published version (APA):

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Forcing absoluteness and regularity properties

Daisuke Ikegami*

The Institute for Logic, Language and Computation, Universiteit van Amsterdam, Science Park 904, 1098 XH, Amsterdam, The Netherlands

ARTICLE INFO

Article history:
Received 18 May 2009
Received in revised form 27 October 2009
Accepted 27 October 2009
Available online 13 November 2009
Communicated by A. Kechris

MSC:
03E15
28A05
54H05

Keywords:
Forcing absoluteness
Regularity properties

ABSTRACT

For a large natural class of forcing notions, we prove general equivalence theorems between forcing absoluteness statements, regularity properties, and transcendence properties over L and the core model K. We use our results to answer open questions from set theory of the reals.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction & background

Forcing absoluteness statements have been investigated by Amir, Bagaria, Brendle, Halbeisen, Judah and others [1,3,6,13,19]. These statements of the form “Every $P$-statement is absolute between the ground model and its forcing extensions with $P$” are typically independent of the axioms of ZFC, and can often be proved to be equivalent to statements about regularity properties. Typical equivalence theorems are:

Theorem 1.1 (Bagaria, Woodin, [2,33]). Every $\Sigma_1$-statement is absolute between the ground model and its Cohen forcing extensions if and only if every $\Delta_1^1$-set has the Baire property.

Theorem 1.2 (Ikegami, [15]). Every $\Sigma_1$-statement is absolute between the ground model and its Sacks forcing extensions if and only if every $\Delta_1^1$-set either contains a perfect subset or is disjoint from a perfect set.

The mentioned regularity properties are in turn equivalent to transcendence properties over L. For instance, Judah and Shelah proved that the Baire property of all $\Delta_1^1$-sets is equivalent to the transcendence statement “for all reals $x$, there is a Cohen real over $L[x]$” [20]; similarly, Brendle and Löwe showed that the statement “every $\Delta_1^1$-set either contains a perfect subset or is disjoint from a perfect set” is equivalent to “for all reals $x$, there is a real not in $L[x]$” [8].

In this paper, we shall prove a general abstract result underlying both Theorems 1.1 and 1.2, by connecting (for a large class of forcings $P$) $\Sigma_1^1$-$P$-absoluteness, a regularity property at the $\Delta_1^1$-level, and a transcendence property related to $P$. The case of Cohen forcing might suggest that the right transcendence property is the existence of $P$-generics, but this already fails in the case of Sacks forcing. In order to deal with this situation, Brendle, Halbeisen and Löwe introduced the notion of quasi-generic reals [7]. In many cases of c.c.c. forcings (such as Cohen forcing), the notions of quasi-genericity and genericity

* Corresponding address: Institut Mittag-Leffler, Auravägen 17, 18260 Djursholm, Sweden. Tel.: +46 8 622 0568; fax: +46 8 622 0589.
E-mail address: d.ikegami@uva.nl.

1 In the model after adding $\omega_1$-many Cohen reals to $L$, every projective set either contains or is disjoint from a perfect set, but there is no Sacks real over $L$.
coincide; in general, the existence of quasi-generics gives us the right transcendence property for our general theorem. We prove:

**Theorem 1.3.** For any forcing \( P \) in a large class of forcing notions,\(^2\) the following are equivalent:

1. \( \Sigma^1_1 \)-\( \mathcal{P} \)-absoluteness holds,
2. every \( \Delta^1_1 \)-set of reals is \( \mathcal{P} \)-measurable, and
3. for any real \( a \) and \( T \in \mathcal{P} \), there is a quasi-\( \mathcal{P} \)-generic real \( x \in [T] \) over \( L[a] \).

We shall start by defining and investigating the basic concepts in Section 2 and Section 3. We then state and prove the main result of the paper (the precise version of Theorem 1.3) and its immediate consequences in Section 4. Among the consequences is a general Solovay-style characterization theorem (in the tradition of [30]). In Section 5, we move on to \( \Sigma^1_1 \)-absoluteness and prove the analogues of the results from Section 4 under the assumption of appropriate large cardinal axioms. These proofs use some basic facts of inner model theory. In Section 6, we give applications of our main results, answering an open question from [7]; finally, in Section 7, we list a number of interesting open questions.

2. Basic concepts

From now on, we will work in ZFC. We assume that readers are familiar with the elementary theories of forcing and descriptive set theory. (For basic definitions not given in this paper, see [16,26].) When we are talking about "reals", we mean elements of the Baire space or of the Cantor space.

In this section, we introduce the notions we will need for the rest. We start with introducing the forcing absoluteness we will focus on:

**Definition 2.1 (\( \Sigma^1_1 \)-\( \mathcal{P} \)-Absoluteness).** Let \( P \) be a forcing notion and \( n \) be a natural number with \( n \geq 1 \). Then \( \Sigma^1_1 \)-\( \mathcal{P} \)-absoluteness is the following statement:

"for any \( \Sigma^1_1 \)-formula \( \varphi \), real \( r \) in \( V \), and \( \mathcal{P} \)-generic filter \( G \) over \( V \), \( V \models \varphi(r) \) iff \( V[G] \models \varphi(r) \)."

**Definition 2.2 (Projective Forcings).** Let \( n \) be a natural number with \( n \geq 1 \). A partial order \( P \) is \( \Sigma^1_1 \) (resp. \( \Pi^1_1 \), \( \Delta^1_1 \)) if the sets \( P \), \( \leq_P \), and \( \perp_P \) are \( \Sigma^1_1 \) (resp. \( \Pi^1_1 \), \( \Delta^1_1 \)), where \( P = (P, \leq_P) \) and \( \perp_P \) is the incompatibility relation in \( P \). We say \( P \) is projective if it is \( \Sigma^1_1 \) for some \( n \geq 1 \).

Let \( n \) be a natural number with \( n \geq 1 \). A partial order \( P \) is provably \( \Delta^1_1 \) if there is a \( \Sigma^1_1 \)-formula \( \phi \) and a \( \Pi^1_1 \)-formula \( \psi \) such that the statement "\( \phi \) and \( \psi \) define the same triple \( (P, \leq, \perp) \)" is provable in ZFC.

*Regularity properties* are properties for sets of reals such that sets of reals with such properties can be approximated by some simple sets (e.g. open sets, Borel sets). Typical examples are Lebesgue measurability and the Baire property. All typical forcings related to the known regularity properties are provably \( \Delta^1_1 \). In this paper, we are only interested in projective forcings.

In our main results, we shall need a strengthening of the standard notion of properness for projective forcings that is used in [4, Definition 5]:

**Definition 2.3.** A projective forcing \( P \) is strongly proper if for any countable transitive model \( M \) of a finite fragment of ZFC containing the real parameter in the formula defining \( P \), if \( P^M \), \( \leq_P \), \( \perp_P \) are subsets of \( P, \leq_P, \perp_P \) respectively, then for any condition \( p \) in \( P^M \), there is a \( (M, \mathcal{P}) \)-generic condition \( q \) below \( p \), i.e., if \( M \models "A is a maximal antichain in \( \mathcal{P} \), then \( A \) is predense below \( q \).\(^3\)

Here \( (M, \mathcal{P}) \)-generic conditions are the same as \( (X, \mathcal{P}) \)-generic conditions for countable elementary substructures \( X \) of \( \mathcal{H}_\theta \); if \( P \) is projective, \( X \) is a countable elementary substructure of \( \mathcal{H}_\theta \) for some enough large regular \( \theta \) and \( M \) is the transitive collapse of \( X \), then a condition \( p \) is \( (M, \mathcal{P}) \)-generic iff it is \( (X, \mathcal{P}) \)-generic in the usual sense. In particular, if \( P \) is projective and strongly proper, then \( P \) is proper.

All the typical examples of proper, \( \Delta^1_1 \) forcings are strongly proper. But there is a proper, provably \( \Delta^1_1 \) forcing which is not strongly proper (for the details, see the papers [5,4] by Bagaria and Bosch).

We use strong properness instead of properness, as it allows us to leave out the quantification "\( \in \mathcal{H}_\theta \)" which would increase the complexity of our statements in the relevant results (Proposition 2.17, Theorems 5.3 and 5.6) beyond projective.

Next, we introduce a class of forcings containing all the well-known forcing notions with perfect trees on \( \omega \) or on \( 2 \) as conditions. A partial order \( P \) is arboreal if its conditions are perfect trees on \( \omega \) (resp. 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as \( P = \{ \omega \} \). We need the following stronger notion:

**Definition 2.4.** A partial order \( P \) is strongly arboreal if it is arboreal and the following holds:

\[(\forall T \in \mathcal{P})(\forall t \in T) T_t \in \mathcal{P}, \]

where \( T_t = \{ s \in T | \text{either } s \subseteq t \text{ or } s \supseteq t \} \).

\(^2\) We will give the precise class of forcings in Theorem 4.3. Also we will give precise definitions of the notions used here in Section 2.

\(^3\) Although we will not explicitly mention the finite fragment of ZFC we will use for the definition of strong properness, it will be large enough so that we can proceed all the arguments in this paper as usual. From now on, we say “countable transitive models of ZFC” instead of “countable transitive models of a finite fragment of ZFC” for simplicity.
With strongly arboreal forcings, we can code generic objects by reals in the standard way: let $\mathbb{P}$ be strongly arboreal and $G$ be $\mathbb{P}$-generic over $V$. Let $x_G = \bigcup \{ \text{stem}(T) \mid T \in G \}$, where $\text{stem}(T)$ is the longest $t \in T$ such that $T_t = T$. Then $x_G$ is a real and $G = \{ T \in \mathbb{P} \mid x_G \in [T] \}$, where $[T]$ is the set of all infinite paths through $T$. Hence $V[x_G] = V[G]$. We call such real $x_G$ a $\mathbb{P}$-generic real over $V$.

Almost all typical forcings related to regularity properties are strongly arboreal:

**Example 2.5.** (1) Cohen forcing ($\mathbb{C}$): let $\mathbb{C}$ be $\mathcal{P}^\omega$. Consider the partial order $\langle (\mathbb{C}_n), s \in \mathcal{P}^\omega \rangle$. Then this is strongly arboreal and equivalent to Cohen forcing.

(2) random forcing ($\mathbb{B}$): consider the set of all perfect trees $T$ on $2$ such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.

(3) Hechler forcing ($\mathbb{D}$): for $(n, f) \in \mathbb{D}$, let

$$T_{n,f} = \{ t \in \mathcal{P}^\omega \mid \text{either } t \subseteq f \cup n \text{ or } (t \supseteq f \cap n \text{ and } (\forall m \in \text{dom}(t)) t(m) \geq f(m) ) \}.$$  

Then the partial order $\langle (T_{n,f}) \mid (n, f) \in \mathbb{D} \rangle \subseteq \mathbb{D}$ is strongly arboreal and equivalent to Hechler forcing.

(4) Mathias forcing: for a condition $(s, A)$ of Mathias forcing, let

$$T_{s,A} = \{ t \in \mathcal{P}^\omega \mid t \text{ is strictly increasing and } s \subseteq \text{ran}(t) \subseteq s \cup A \}.$$  

Then $\langle T_{s,A} \rangle$ is a condition of Mathias forcing) is a strongly arboreal forcing equivalent to Mathias forcing.

(5) Sacks forcing, Silver forcing, Miller forcing, Laver forcing ($\mathbb{S}, \mathbb{V}, \mathbb{M}, \mathbb{L}$, respectively): these forcings can be naturally seen as strongly arboreal forcings.

We now introduce a general definition of a regularity property associated with an arbitrary arboreal forcing. Sets of reals with a regularity property should be approximated by some simple sets (e.g., Borel sets) modulo some “smallness” as in the case of the Baire property and Lebesgue measurability. Therefore we first introduce “smallness” for each arboreal forcing by defining a $\sigma$-ideal as follows:

**Definition 2.6.** Let $\mathbb{P}$ be an arboreal forcing. A set of reals $A$ is $\mathbb{P}$-null if for any $T \in \mathbb{P}$ there is a $T' \leq T$ such that $[T] \cap A = \emptyset$. $N_\mathbb{P}$ denotes the set of all $\mathbb{P}$-null sets and $I_\mathbb{P}$ denotes the $\sigma$-ideal generated by $\mathbb{P}$-null sets.

**Example 2.7.** (1) Cohen forcing $\mathbb{C}$: $\mathbb{C}$-null sets are the same as nowhere dense sets and $I_\mathbb{C}$ is the meager ideal.

(2) random forcing $\mathbb{B}$: $\mathbb{B}$-null sets are the same as Lebesgue null sets and $I_\mathbb{B}$ is the Lebesgue null ideal.

(3) Hechler forcing $\mathbb{D}$: $\mathbb{D}$-null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{ [s,f] \mid (s,f) \in \mathbb{D} \}$ where

$$[s,f] = \{ x \in \mathcal{P}^\omega \mid s \subseteq x \text{ and } (\forall n \geq \text{dom}(s)) x(n) \geq f(n) \}.$$  

Hence $I_\mathbb{D}$ is the meager ideal in the dominating topology.

(4) Mathias forcing: a set of reals $A$ is Mathias-null iff $[\text{ran}(x) \mid x \in A \cap A_0]$ is Ramsey null or meager in the Ellentuck topology, where $A_0$ is the set of strictly increasing infinite sequences of natural numbers. Also, Mathias-null sets form a $\sigma$-ideal by a standard fusion argument.

(5) Sacks forcing $\mathbb{S}$: in this case, $I_\mathbb{S} = N_\mathbb{S}$ by a standard fusion argument. The ideal $I_\mathbb{S}$ is called the Marczewski ideal and often denoted by $s_0$.

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_\mathbb{S} = N_\mathbb{S}$. Since $I_\mathbb{S}$ is Borel generated for any ccc arboreal forcing, the condition $(*+)$ in Theorem 4.4 (which we will state in Section 4) holds for all the typical tree-type strongly arboreal forcings.

Now we introduce the regularity property for each arboreal forcing:

**Definition 2.8.** Let $\mathbb{P}$ be arboreal. A set of reals $A$ is $\mathbb{P}$-measurable if for any $T \in \mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_\mathbb{P}$ or $[T'] \setminus A \in I_\mathbb{P}$.

As we expect, $\mathbb{P}$-measurability coincides with the approximation properties via Borel sets for $\mathbb{P}$ in the sense below when $\mathbb{P}$ is ccc:

**Proposition 2.9.** Let $\mathbb{P}$ be a strongly arboreal ccc forcing and let $P$ be a set of reals. Then $P$ is $\mathbb{P}$-measurable iff there is a Borel set $B$ such that $P \triangle B \in I_\mathbb{P}$.

**Proof.** The direction from right to left follows from the fact that every Borel set of reals is $\mathbb{P}$-measurable which will be proved in Lemma 3.5.

For the other direction, suppose $P$ is $\mathbb{P}$-measurable and we will find a Borel set approximating $P$ modulo $I_\mathbb{P}$. Since $P$ is $\mathbb{P}$-measurable, the set $D = \{ T \in \mathbb{P} \mid \text{either } [T] \cap P \in I_\mathbb{P} \text{ or } [T] \setminus P \in I_\mathbb{P} \}$ is dense. We take a maximal antichain $A$ in $D$ and define $B = \bigcup \{ [T] \mid T \in A \}$. Then since $A$ is countable, $B$ is Borel and $P \Delta B \in I_\mathbb{P}$ because $D$ is dense.

This argument does not work for non-ccc forcings such as Sacks forcing. But $\mathbb{P}$-measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals $A$ is Mathias-measurable iff $\{ \text{ran}(x) \mid x \in A \cap A_0 \}$ is completely Ramsey (or has the Baire property in the Ellentuck topology), where $A_0$ is the set of

---

4 For example, assuming every $\Pi_1^1$-set has the perfect set property, every $\Sigma_1^1$-set of reals has the Bernstein property (i.e., either it contains a perfect or there is a perfect set disjoint from the set) but for a $\Sigma_1^1$-set of reals $A$, $A$ is approximated by a Borel set modulo $I_\mathbb{P}$ iff $A$ is Borel. This is because $I_\mathbb{P}$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals.
all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

**Proposition 2.10** (Brendle–Löwe). Let \( \Gamma \) be a topologically reasonable pointclass, i.e., it is closed under continuous preimages and any intersection between a set in \( \Gamma \) and a closed set. Then every set in \( \Gamma \) is \( S \)-measurable iff every set in \( \Gamma \) has the Bernstein property.\(^5\)

**Proof.** See [8, Lemma 2.1]. ■

Next we introduce a technical ideal \( I_p^* \) which we need later:

**Definition 2.11.** Let \( P \) be an arboreal forcing. A set of reals \( A \) is in \( I_p^* \) if for any \( T \) in \( P \) there is a \( T' \leq T \) such that \( [T'] \cap A \in I_p^* \) or \( [T'] \cap A \in I_p^* \) holds. Hence we get the same notion of measurability even if we replace \( I_p^* \) in the definition of \( P \)-measurability.

**Question 2.12.** Let \( P \) be a strongly arboreal, proper forcing. Can we prove \( I_p = I_p^* \)?

We give some easy observations concerning to **Question 2.12:**

**Lemma 2.13.** Let \( P \) be strongly arboreal forcing.

1. The ideal \( I_p \) is a subset of \( I_p^* \).
2. A set of reals \( A \) is \( P \)-measurable iff for any \( T \) in \( P \) there is an \( T' \leq T \) such that either \( [T'] \cap A \in I_p^* \) or \( [T'] \cap A \notin I_p^* \).
3. If \( P \) is ccc, then \( I_p = I_p^* \).
4. If \( I_p = N_p \), then \( I_p = I_p^* \). Hence \( I_p = I_p^* \) for any typical tree-type strongly arboreal forcing admitting a fusion argument.

**Proof.** We will prove only (5). The rest are straightforward. Suppose \( P \) satisfies the above condition and let \( A \) be in \( I_p^* \). We prove \( A \) is in \( I_p \). Since \( A \) is in \( I_p^* \), the set of all \( T \) in \( P \) such that \( [T'] \cap A \in I_p \) is dense in \( P \). Hence we can take a maximal antichain \( \mathcal{A} \) in \( P \) by the definition of \( P \)-measurability. We may assume for any two distinct elements \( T_1, T_2 \) of \( \mathcal{A} \), \( [T_1] \cap A \neq [T_2] \cap A \) are pairwise disjoint. For each \( T \) in \( \mathcal{A} \), \( [T] \cap A \) is in \( I_p^* \). So we can pick \( \{N_{n,T} \mid n \in \omega\} \) such that each \( N_{n,T} \) is \( P \)-null and \( \bigcup_{n \in \omega} N_{n,T} = [T] \cap A \). Let \( N_n = \bigcup_{T \in \mathcal{A}} N_{n,T} \) for each \( n \in \omega \). Since \( A = \bigcup_{n \in \omega} N_n \), the proof is complete if we prove the following

**Claim 2.14.** For each \( n \in \omega \), \( N_n = P \)-null.

**Proof of Claim 2.14.** Take any \( T \) in \( P \). Since \( \mathcal{A} \) is a maximal antichain, we can take a \( T' \in \mathcal{A} \) such that \( T \) and \( T' \) are compatible. Take a common extension \( T'' \). Then \( [T''] \cap A = [T''] \cap N_{n,T} \) because of the property of \( \mathcal{A} \). Hence we know that \( N_n, T \) is \( P \)-null. Hence we can take a further extension of \( T'' \) disjoint from \( N_n \). ■ ■

Next, we introduce quasi-\( P \)-genericity for arboreal forcings \( P \) and compare it with \( P \)-genericity. Quasi-generic reals are obvious generalization of Cohen reals and random reals:

**Definition 2.15.** Let \( P \) be arboreal and \( M \) be a transitive model of ZFC. A real \( x \) is quasi-\( P \)-generic over \( M \) if for any Borel code \( c \in M \) with \( B_c \in I_p^* \), \( x \) is not in \( B_c \), where \( B_c \) is the decoded Borel set from \( c \).

**Example 2.16.** (1) Cohen forcing (C): quasi-C-generic reals are the same as Cohen reals by definition. Hence quasi-C-genericity coincides with \( C \)-genericity.

2. Random forcing (\( R \)): quasi-B-generic reals are the same as random reals by definition. Hence quasi-B-genericity coincides with \( B \)-genericity.

3. Hechler forcing (\( H \)): quasi-D-generic reals are the same as Hechler reals. Hence quasi-D-genericity coincides with \( D \)-genericity.

4. Sacks forcing (\( S \)): if \( M \) is an inner model of ZFC, quasi-S-generic reals over \( M \) are the reals which are not in \( M \) because any Borel set in \( I_p^* = N_S \) is countable and is contained in \( M \) since the statement “A Borel code \( c \) codes a countable set” is absolute between \( M \) and \( V \). Therefore, quasi-S-genericity does not coincide with \( S \)-genericity.

The last example explains the difference between genericity and quasi-genericity and shows that the equivalence for Sacks forcing we mentioned in the introduction is a special case of **Theorem 4.3** which we will prove later.\(^6\)

As is expected, genericity implies quasi-genericity for all the typical strongly arboreal forcings and the converse is true for most ccc forcings:

**Proposition 2.17.** Let \( P \) be a strongly arboreal, strongly proper, provably \( A^\Omega \) forcing. Then

1. The set \( \{c \mid B_c \in I_p^* \} \) is \( \Pi^0_3 \). Hence the statement “\( c \) codes a Borel set in \( I_p^* \)” is absolute between inner models of ZFC.

---

\(^5\) In general, the Bernstein property does not imply \( S \)-measurability while the converse is true. By using the axiom of choice, we can construct a set of reals which is not \( S \)-measurable and has the Bernstein property.

\(^6\) It is easy to check the condition (\( * \)) in **Theorem 4.3** for Sacks forcing by noting that the ideal \( I_p \) restricted to Borel sets is the ideal of countable sets as we mentioned in the last example.
(2) If $M$ is a transitive model of ZFC and a real $x$ is $\mathbb{P}$-generic over $M$, then $x$ is quasi-$\mathbb{P}$-generic over $M$.

(3) Suppose $\mathbb{P}$ is also provably ccc, i.e., there is a formula $\phi$ defining $\mathbb{P}$ and the statement "$\phi$ is ccc" is provable in ZFC. Then if $M$ is an inner model of ZFC and a real $x$ is quasi-$\mathbb{P}$-generic over $M$, then $x$ is $\mathbb{P}$-generic over $M$.

**Proof.** See Section 3. ■

In [35], Zapletal starts from a $\sigma$-ideal $I$ on a Polish space $X$ and considers the quotient of the set of all Borel sets in $X$ modulo $I$ and develops the general theory of this forcing as a Boolean algebra. Let us compare his setting with our setting:

**Proposition 2.18.** Suppose $\mathbb{P}$ is a strongly arboreal, proper forcing. Then the map $i: \mathbb{P} \to (B/I_p)^* \setminus \{0\}$ defined by

$$i(T) = \text{the equivalence class represented by } [T],$$

is a dense embedding, where $B$ denotes the set of all Borel sets of the reals and $B/I_p^*$ is the quotient Boolean algebra via $I_p^*$.

Hence, our situation is a special case of Zapletal’s. ■

**Proof.** See Section 3. ■

3. $\mathbb{P}$-measurability and $\mathbb{P}$-Baireness

In this section, we shall prove the propositions listed in Section 2. In order to do so, we first consider the connection between $\mathbb{P}$-measurability and a property called $\mathbb{P}$-Baireness (which was implicitly introduced by Feng, Magidor and Woodin [12]). This connection will allow us to characterize $I_p^*$ in terms of Banach–Mazur games, which plays an essential role in the proof of Proposition 2.17.

Let $\mathbb{P}$ be a partial order. The Stone space of $\mathbb{P}$ (denoted by $St(\mathbb{P})$) is the set of ultrafilters of $\mathbb{P}$ equipped with the topology generated by $\{O_p \mid p \in \mathbb{P}\}$, where $O_p = \{u \in St(\mathbb{P}) \mid u \ni p\}$.

For example, if $\mathbb{P}$ is Cohen forcing ($\mathbb{C}$), then $St(\mathbb{C})$ is homeomorphic to the Baire space $w^\omega$.

Dense sets in $\mathbb{P}$ are the same as open dense subsets in $St(\mathbb{P})$: if $D$ is a dense subset of $\mathbb{P}$, then the set $\bigcup\{O_p \mid p \in D\}$ is open dense in $St(\mathbb{P})$. Conversely, if $U$ is an open dense subset of $St(\mathbb{P})$, then $\{p \in \mathbb{P} \mid O_p \subseteq U\}$ is a dense open subset of $\mathbb{P}$.

Next, we will talk about meagerness and the Baire property in $St(\mathbb{P})$. The first observation we should make is that this is not nonsense:

**Lemma 3.1.** Let $\mathbb{P}$ be a partial order. Then for any $p \in St(\mathbb{P})$, $O_p$ is not meager.

**Proof.** Take any $p \in \mathbb{P}$ and let $\{U_n \mid n \in \omega\}$ be a countable set of open dense subsets of $St(\mathbb{P})$. We would like to prove that the intersection $\bigcap_{n \in \omega} U_n$ with $O_p$ is nonempty. But this is just the Rasiowa–Sikorsky Theorem. ■

Before defining $\mathbb{P}$-Baireness, let us see the connection between Baire measurable functions from $St(\mathbb{P})$ to the reals and $\mathbb{P}$-names for a real.

Let $X, Y$ be topological spaces. Then a function $f: X \to Y$ is Baire measurable if for any open set $U$ in $Y, f^{-1}(U)$ has the Baire property in $X$. Baire measurable functions are the same as continuous functions modulo meager sets: let $X, Y$ be topological spaces and assume $Y$ is second countable. Then it is fairly easy to see that a function $f: X \to Y$ is Baire measurable iff there is a comeager set $D$ in $X$ such that $f \upharpoonright D$ is continuous.

There is a natural correspondence between Baire measurable functions from $St(\mathbb{P})$ to the reals and $\mathbb{P}$-names for a real:

**Lemma 3.2** (Feng–Magidor–Woodin). Let $\mathbb{P}$ be a partial order.

1. If $f: St(\mathbb{P}) \to w^\omega$ is a $\mathbb{P}$-name for a real, then $\tau_f = \{\langle m, n, p \rangle \mid O_p \setminus \{u \in St(\mathbb{P}) \mid f(u)(m) = n\} \text{ is meager}\}$ is a $\mathbb{P}$-name for a real.

2. Let $\tau$ be a $\mathbb{P}$-name for a real. Define $f_\tau$ as follows. For $u \in St(\mathbb{P})$ and $m, n \in \omega$,

$$f_\tau(u)(m) = n \iff (\exists p \in \mathbb{P}) (u \ni p \iff \tau(m) = n).$$

Then the domain of $f_\tau$ is comeager in $St(\mathbb{P})$ and $f_\tau$ is continuous on the domain. Hence it can be uniquely extended to a Baire measurable function from $St(\mathbb{P})$ to the reals modulo meager sets.

3. If $f: St(\mathbb{P}) \to w^\omega$ is a $\mathbb{P}$-measurable function, then $f_\tau$ and $f$ agree on a comeager set in $St(\mathbb{P})$. Also, if $\tau$ is a $\mathbb{P}$-name for a real, then $\tau_f = \tau$.

**Proof.** See [12, Theorem 3.2]. ■

Recall that we have defined a generic real $x_G$ from a generic object $G$ for any strongly arboreal forcing $\mathbb{P}$. Let $x_G$ be a canonical $\mathbb{P}$-name for $x_G$.

**Example 3.3.** Let $\mathbb{P}$ be strongly arboreal. Then $f_{x_G}(u)(m) = n$ iff there is a $T$ in $u$ such that $\text{stem}(T)(m) = n$. Hence $f_{x_G}(u) = \bigcup\{\text{stem}(T) \mid T \in u\}$ for $u \in \text{dom}(f_{x_G})$ as expected.

Now we define the property $\mathbb{P}$-Baireness. Let $\mathbb{P}$ be a partial order and $A$ be a set of reals. Then $A$ is $\mathbb{P}$-Baire if for any Baire measurable function $f: St(\mathbb{P}) \to w^\omega, f^{-1}(A)$ has the Baire property in $St(\mathbb{P})$. It is easy to see that every Borel set of reals is $\mathbb{P}$-Baire for any $\mathbb{P}$ by the same argument as for the Baire property.

7 In [35, Corollary 2.15], Zapletal proved a more general result. His $I$ is essentially the same as our $I_p^*$ and if we use $b_i = |x_{\text{gen}}(i)| = 1$ ($n \in \omega$) instead of $b_i = |t \leq x_{\text{gen}}| (t \in "2")$ to generate the dense subalgebra $C$ of the completion of $\mathbb{P}$, then Zapletal’s $I$ is exactly the same as our $I_p^*$ on Borel sets.
Example 3.4. Let $C$ be Cohen forcing. A set of reals $A$ is $C$-Baire iff $f^{-1}(A)$ has the Baire property for any continuous function $f: \omega_1 \rightarrow \omega_1$.

**Proof.** As we have remarked in the beginning of this section, $\text{St}(C)$ is homeomorphic to the Baire space $\omega_1$. In the Baire space, any $G_\delta$ comeager set is homeomorphic to the whole space. Hence we can replace Baire measurable functions by continuous functions in the definition of $C$-Baireness.

Before talking about the relation between $\mathbb{P}$-measurability and $\mathbb{P}$-Baireness, let us mention the connection between $\mathbb{P}$-Baireness and universally Baireness. A set of reals $A$ is universally Baire if for any compact Hausdorff space $X$ and any continuous function $f: X \rightarrow \omega_1, f^{-1}(A)$ has the Baire property in $X$. A set of reals $A$ is universally Baire if $A$ is $\mathbb{P}$-Baire for any partial order $\mathbb{P}$. (This is essentially proved in [12].)

Recall that $I_\mathbb{P}^+$ is a technical ideal introduced in Definition 2.11 which is the same as $I_\mathbb{P}$ for most cases.

**Lemma 3.5** ($\mathbb{P}$-measurability vs. $\mathbb{P}$-Baireness). Let $\mathbb{P}$ be a strongly arboreal, proper forcing and $A$ be a set of reals. Then

1. $A$ is in $I_\mathbb{P}^+$ iff $f_{\mathbb{P}}^{-1}(A)$ is meager in $\text{St}(\mathbb{P})$, and
2. $A$ is $\mathbb{P}$-measurable iff $f_{\mathbb{P}}^{-1}(A)$ has the Baire property in $\text{St}(\mathbb{P})$. In particular, if $A$ is $\mathbb{P}$-Baire, then $A$ is $\mathbb{P}$-measurable. Hence every Borel set is $\mathbb{P}$-measurable.

Note that $\mathbb{P}$-measurability does not imply $\mathbb{P}$-Baireness in general.\(^8\)

**Proof of Lemma 3.5.** Let $\pi = f_{\mathbb{P}}$ for abuse of notation.

The following are useful for the proof:

**Claim 3.6.** (a) For $T \in \mathbb{P}$ and $u \in \text{dom}(\pi)$, if $T \in [\pi]$, then $\pi(u) \in [T]$. (b) For $T \in \mathbb{P}$, the converse of (a) holds for comeager many $u \in \text{dom}(\pi)$.

**Proof of Claim 3.6.** (a) Suppose $T \in \mathbb{P}$ and $u \in T$. We prove $\pi(u) \in T$ for each $n \in \omega$. Fix a natural number $n$. Then by Example 3.3, there is a $T'$ in $u$ such that $\text{stem}(T') \supseteq \pi(u) \cap n$. Since both $T$ and $T'$ are in $u$, they are compatible, especially $\text{stem}(T') \cap T = T$. Hence $\pi(u) \cap n \in T$.

(b) Take any $T \in \mathbb{P}$. Then the set $D = \{T' \in \mathbb{P} \mid T' \subseteq T \text{ or } T' \cap T = \emptyset\}$ is dense in $\mathbb{P}$. Take any $T' \subseteq T$, then there is a $T'' \subseteq T' \setminus T$. By strong arboreality of $T'$, $T'' \in \mathbb{P}$ and $T'' \cap T = \emptyset$. Since $D$ is dense, the set $\{u \cup u \cap D = \emptyset\}$ is dense open in $\text{St}(\mathbb{P})$. Hence it suffices to show that if $u$ is in $\text{dom}(\pi)$, $u \cap D \neq \emptyset$ and $\pi(u) \in [T]$, then $T \in u$. Suppose $T \notin u$. Then since $u \cap D = \emptyset$, there is a $T' \in u$ such that $[T'] \cap D = \emptyset$. By (a), $\pi(u) \in [T']$, hence $\pi(u) \notin [T]$, a contradiction.\(\square\)

(1) We prove the direction from left to right.

We first show that $\pi^{-1}(A)$ is meager assuming $A$ is in $I_{\mathbb{P}}^+$. If $A$ is in $I_{\mathbb{P}}$, then the set $D = \{T \mid T \cap \text{dom}(A) = \emptyset\}$ is dense in $\mathbb{P}$. Hence the set of all $u \in \text{dom}(\pi)$ with $u \cap D \neq \emptyset$ is comeager. But if $u$ is in the comeager set, then there is a $T \in u \cap D$ and by Claim 3.6(a), $\pi(u) \in [T]$ and $[T] \cap \text{dom}(A) = \emptyset$, in particular $\pi(u) \notin A$. Therefore $\pi^{-1}(A)$ is meager.

We have seen that $\pi^{-1}(A)$ is meager assuming $A$ is in $I_{\mathbb{P}}^+$. Since $I_{\mathbb{P}}^+$ is the $\sigma$-ideal generated by sets in $N_\mathbb{P}$, $\pi^{-1}(A)$ is meager for all $A$ in $I_{\mathbb{P}}^+$.

We show that $\pi^{-1}(A)$ is meager if $A$ is in $I_\mathbb{P}^+$. Since $A$ is in $I_\mathbb{P}^+$, the set $D' = \{T \mid T \cap \text{dom}(A) \in I_{\mathbb{P}}\}$ is dense in $\mathbb{P}$. We use the following well-known fact:

**Fact 3.7.** Let $X$ be a topological space and $A$ be a subset of $X$. Then $(\bigcup\{U \mid U \text{ is open and } U \cap A \text{ is meager}\}) \cap A$ is meager.

**Proof of Fact 3.7.** See [22, Theorem 8.29].\(\square\)

Since $D'$ is dense, $\bigcup\{O_T \mid T \in D'\}$ is open dense. By the above fact, it suffices to prove that $O_T \cap \pi^{-1}(A)$ is meager for any $T \in D'$.

Take any $T \in D'$. By the definition of $O_T$, we know that $[T] \cap A \in I_{\mathbb{P}}$. Hence $\pi^{-1}([T] \cap A)$ is meager in $\text{St}(\mathbb{P})$. But by Claim 3.6(a), $O_T \cap \pi^{-1}(A) \subseteq \pi^{-1}([T] \cap A)$. Therefore, $O_T \cap \pi^{-1}(A)$ is meager as desired.

Next, we see the direction from right to left. Suppose $\pi^{-1}(A)$ is meager. Take any $T \in \mathbb{P}$ and we will find an extension $T'$ of $T$ such that $[T'] \cap A \in I_{\mathbb{P}}$. Since $\pi^{-1}(A)$ is meager, then there is a sequence $(U_n \mid n \in \omega)$ of open dense sets in $\text{St}(\mathbb{P})$ such that $\bigcap_{n \in \omega} U_n \cap \pi^{-1}(A) = \emptyset$. For each $n \in \omega$, let $D_n = \{S \in \mathbb{P} \mid O_S \subseteq U_n\}$. Since $U_n$ is open dense in $\text{St}(\mathbb{P})$, $D_n$ is dense open in $\mathbb{P}$. We choose a sequence $(A_n \mid n \in \omega)$ of maximal antichains such that $A_n \subseteq D_n$, for each element of $A_n$, the length of $\text{stem}(S)$ is greater than $n$, and $A_{n+1}$ refines $A_n$, i.e., every element of $A_{n+1}$ is below some element in $A_n$.

Now we use the properness of $\mathbb{P}$ to treat each $A_n$ as “countable”. Let $\theta$ be a sufficiently large regular cardinal and $X$ be a countable elementary substructure of $\mathcal{H}_\theta$ such that $\mathbb{P}$, $\langle A_n \mid n \in \omega \rangle$ are in $X$. By properness, there is an $(X, \mathbb{P})$-generic condition $T$ below $T$. We show that $[T'] \cap A \in I_{\mathbb{P}}$, which will complete the proof of (1).

Consider the set $B = \bigcap_{n \in \omega} \{S \in \mathcal{S} \mid S \in A_n \cap X \} \cap \bigcup_{n \in \omega} \{S \in \mathcal{S} \mid S \neq S' \in A_n \cap X \}$. So $B$ is the set of all $x$ uniquely deciding which condition from $A_n$ contains it for each $n$. By the property of $(A_n \mid n \in \omega)$, it will generate a filter coming from elements in the $A_n$. The point is that any ultrafilter $u$ extending that filter satisfies $\pi(u) = x$, the given element, and that $u$ is in $U_n$ for each $n$. This will play a role for the argument.

---

\(^8\) For example, if $A$ is a $\Sigma_1^1$ (lightface) set of reals universal for $\Sigma_1^1$ (boldface) sets of reals and if every $\Sigma_1^1$ (lightface) set of reals has the Baire property but there is a $\Sigma_1^1$ (boldface) set of reals without the Baire property, then $A$ is $\mathbb{P}$-measurable by Proposition 2.9, but $A$ is not $\mathbb{P}$-Baire by Example 3.4.
Now we claim $[T'] \setminus B \in I_{P}$ and $B \cap A = \emptyset$. We will be done if we prove this. The fact that $[T'] \setminus B \in I_{P}$ follows from the fact that $[S \setminus S \in A_{n} \cap X]$ is predense below $[T']$ for each $n$ because $T'$ is $(X, P)$-generic and from that $[S \setminus [S'] \in I_{P}$ for each $S, S' \in A_{n} \cap X$ with $S \neq S'$ because $A_{n}$ is an antichain, and from that $A_{n} \cap X$ is countable for each $n$.

To prove $B \cap A = \emptyset$, take any element $x$ from $B$. As we mentioned above, for each $n \in \omega$, there is a unique element $S_{n}$ in $A_{n} \cap X$ with $x \in [S_{n}]$. Since $A_{n} \cap X$ refines $A_{n}, S_{n+1} \leq S_{n}$ for each $n$. Hence the set $[S_{n} \setminus n \in \omega]$ generate a filter $F_{0}$. Take any ultrafilter $u$ extending $F_{0}$. We claim that $\pi(u) = x$ and $u \subseteq U_{n}$ for each $n$. By the property of $\pi_{*}(A) \setminus \omega$, the length of stem($S_{n}$) is greater than $n$. Hence, by Example 3.3, $\pi(u)$ is already decided to be $x$ by $S_{n}$. The fact that $u \subseteq U_{n}$ for each $n$ follows from the fact that $S_{n} \subseteq A_{n} \subseteq D_{n}$ and the definition of $D_{n}$.

Since we have assumed that $\bigcap_{n \in \omega}U_{n}\pi^{-1}(A) = \emptyset$, $x$ does not belong to $A$ because $\pi(u) \subseteq U_{n}$ for each $n$ by Claim 3.6. Hence we have seen $B \cap A = \emptyset$ as desired.

(2) For left-right, we assume $A$ is $P$-measurable. Then the set $D = \{T \in P \mid \exists T \cup A \in I_{P} \}$ is dense. Then the set $U = \bigcup \{O_{T} \mid T \in D \}$ is dense in $St(P)$. Hence, $\bigcap_{n \in \omega}U_{n}\pi^{-1}(A)$ is meager. The case for $U_{\omega} \pi^{-1}(A)$ being meager is similar. By Fact 3.7, it suffices to see that $T \cap \pi^{-1}(A)$ is meager when $[T] \cap A \in I_{P}$. But if $[T] \cap A \in I_{P}$, then $T \cap \pi^{-1}(A) \subseteq \pi^{-1}(T \cap A)$ and $\pi^{-1}(T \cap A)$ is meager by Claim 3.6(a), Lemma 2.13(1), and (1) in this lemma. Hence we are done.

Now we see the direction from right to left. Assume $\pi^{-1}(A)$ has the Baire property in $St(P)$. Then there are open sets $U_{1}, U_{2}$ such that $\pi_{\omega}^{-1}(A), \pi_{\omega}^{-1}([\omega \setminus A]$ are meager. By Lemma 3.1, $U_{1}, U_{2} = \emptyset$ and $U_{1} \cup U_{2}$ is open dense in $St(P)$. Let $D_{1} = \{T \in P \mid T \cap U_{1} \neq \emptyset \}$ for $i = 1, 2$. Then $D_{1} \cup D_{2}$ is dense in $P$. Hence by Lemma 2.13(2), it suffices to prove that $[T] \cap A \in I_{P}$ for each $T \in D_{1}$ and that $[T] \cap A \in I_{P}$ for each $T \in D_{2}$.

We only prove $[T] \cap A \in I_{P}$ for each $T \in D_{1}$. By (1) in this lemma, it is enough to see that $\pi^{-1}(T \cap A)$ is meager in $St(P)$. But by Claim 3.6(b), $\pi^{-1}(T \cap A)$ is almost the same as $\pi_{\omega}^{-1}(T \cap A)$. Since $T$ is in $D_{1}$, by the definition of $U_{1}, T \cap \pi^{-1}(A)$ is meager. This completes the proof of (2).

Note that if $P$ satisfies the condition in Proposition 2.13(5), we do not need the properness of $P$ for the proof of Lemma 3.5. Now we are ready to prove Propositions 2.17 and 2.18. We first see the proof of Proposition 2.18:

**Proof of Proposition 2.18.** First we see that the map $i$ is well defined, i.e., $T$ is not in $I_{P}$ for each $T \in P$. If it were in $I_{P}$, then by Lemma 3.5(1), $\pi^{-1}(T)$ would be meager and $\pi^{-1}(T) \subseteq \pi^{-1}(T)$ by Claim 3.6(a). Hence $\pi^{-1}(A)$ is meager, which contradicts Lemma 3.1. Therefore $T$ is not in $I_{P}$.

It is clear that if $T_{1} \leq T_{2}$ then $i(T_{1}) \leq i(T_{2})$. To show the converse, assume $T_{1} \leq T_{2}$ and we prove that $i(T_{1}) \leq i(T_{2})$. Since $T_{1} \neq T_{2}$, there is a $t \in T_{1}$ which is not in $T_{2}$. By strong arborescalness of $P$, $(T_{1}, t) \in P$ and $(i(T_{1}), t \in T_{1}) \cap T_{2} = \emptyset$. Hence $i(T_{1}, t) \leq i(T_{2})$. Since $T_{1} \leq T_{2}$, we have $i(T_{1}, t) \leq i(T_{2})$. Therefore, $i(T_{1}) \leq i(T_{2})$.

So it suffices to see that $i^{*}$ is dense in $(B / I_{P}^{*}) \setminus \{0\}$. Let $B$ be a Borel set which is not in $I_{P}$. We will find a $T \in P$ with $[T] \setminus B \in I_{P}$.

Since every Borel set is $P$-Baire, by Lemma 3.5(2), $B$ is $P$-measurable. Since $B$ is not in $I_{P}$, there is a $T$ such that $[T] \setminus B \in I_{P}$, hence $[T] \setminus B \in I_{P}$ by Lemma 2.13 (1), as desired.

**Proof of Proposition 2.17.** (1) Let $\pi = f_{c}$ be as in the proof of Lemma 3.5. By Lemma 3.5, a set of reals $A$ is in $I_{P}$, iff $\pi^{-1}(A)$ is meager in $St(P)$. Hence, it suffices to show that $[c \setminus \pi^{-1}(B_{c})]$ is meager in $\Pi_{2}$.

We will prove the following:

$$\pi^{-1}(B_{c}) \text{ is meager } \iff (\forall M \ni c) (M \text{ is a countable transitive model of ZFC } \implies M \vDash \pi^{-1}(B_{c}) \text{ is meager}).$$

To be rigorous, $M$ is a model of a large enough fragment of ZFC as noted in the footnote in Definition 2.3.

First note that the right hand side makes sense because the statement "$P$ is a strongly arborescally forcing" is $\Pi_{1}$ by the assumption that $P$ is provably $\Delta_{1}$, so by downward absoluteness, this is also true in $M$. Since the right hand side is $\Pi_{2}$, it suffices to show the above equivalence.

The following claim is the key point:

**Claim 3.8.** Let $M$ be a countable transitive model of a large fragment of ZFC with $c \in M$. If $M \vDash \pi^{-1}(B_{c})$ is meager", then for any $T \in M^{P}$ (or $P \cap M$), there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_{c})$ is meager in $V$.

**Proof of Claim 3.8.** Take any $T$ in $M^{P}$. Since $P$ is provably $\Delta_{1}, P^{M}, S^{M}$ and $\perp^{M}$ are subsets of $P$, $\leq$ and $\perp$ respectively. Hence, by strong properness, there is a $T' \leq T$ such that $T'$ is $(M, P)$-generic.

We will show that $T'$ satisfies the desired property. For that, we will use the unfolded Banach–Mazur game. Let $U$ be a tree on $\omega \times \omega$, recursive in $c$ such that $B_{c} = \pi[U]$ holds in any transitive model of ZFC N with $c \in N$. Consider the following game $G$: player I and II produce a decreasing sequence $\langle S_{n} \mid n \in \omega \rangle$ one by one and in addition, player II produces a real $\langle y_{n} \mid n \in \omega \rangle$. Player II wins if $(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n \in \omega}S_{n}$. Note that we may assume that $\pi$ is defined for any $u \in \bigcap_{n \in \omega}S_{n}$ and the value of $\pi$ only depends on the sequence $\langle S_{n} \mid n \in \omega \rangle$ because we can arrange $\pi(u) = \bigcup_{n \in \omega}\text{stem}(S_{n})$ by strong arborescalness of $P$ and Example 3.3.

By the well-known characterization of meagerness for the topological space $O_{T}$ via Banach–Mazur games, it suffices to show that player II has a winning strategy in the game $G$ in $V$. Since $M \vDash \pi^{-1}(B_{c})$ is meager", in $M$, player II has a winning
strategy $\sigma$ in the game $G$ which is the same as $G'$ except that player I can start from any condition in $\mathbb{P}$. The idea is to transfer $\sigma$ to a winning strategy for player II in $G'$ in $V$. Instead of writing down a winning strategy for player II in $G'$, we will describe how to win the game $G'$ for player II as follows:

\[
\begin{array}{c|c|c|c}
V & S_0 \leq T' & S_2' & \cdots \\
\hline
I & (S_1', y_0) & (S_3', y_1) & \cdots \\
M & (S_1, y_0) & (S_3, y_1) & \cdots \\
\end{array}
\]

We will construct sequences $\langle S_n \mid n \in \omega \rangle$, $\langle S_n' \mid n \in \omega \rangle$, $\langle y_n \mid n \in \omega \rangle$ with the following properties:

- $\langle (S_n' \mid n \in \omega), (y_n \mid n \in \omega) \rangle$ is a run in the game $G'$ in $V$,
- $\langle (S_n \mid n \in \omega), (y_n \mid n \in \omega) \rangle$ is a run in the game $G^M$ in $V$,
- $S_{2n}'$ is arbitrarily chosen by player I for each $n$,
- player II follows $\sigma$ in $G^M$, and
- $S_{2n+1}' \leq S_{2n+1}$ for each $n$.

Assuming we have constructed the above sequences, we prove that player II wins in the game $G'$. First note that $G^M$ is a closed game for player II, hence the strategy $\sigma$ remains winning in $V$. Therefore, $(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n \in \omega} O_{S_n}$ in $V$. But since $S_{2n+1} \leq S_{2n+1}$ for each $n$, $(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n \in \omega} O_{S_n}'$, hence player II wins the game $G'$.

We describe how to construct the above sequences. Suppose we have got $\langle (S_i, y_i) \mid i < 2n \rangle$ for some $n$. We will decide $S_{2n}'$, $S_{2n+1}'$, $S_{2n+1}$, and $y_n$. By the above properties, $S_{2n}$ is arbitrarily chosen by player I and $S_{2n+1}, y_n$ will be decided by the rest and $\sigma$. So let us decide $S_{2n}$ and $S_{2n+1}'.

Let $D$ be the set of all possible candidates for $S_{2n+1}'$ by $\sigma$ and the previous play $(S_i \mid i < 2n)$, $(y_i \mid i < n)$. Then in $D$, $M$ is dense below $S_{2n-1}'$ (if it exists). Since $S_{2n}' \leq S_{2n-1}' \leq S_{2n-1}$ and $T'$ is $(M, \mathbb{P})$-generic, $D \cap M = D$ is predense below $S_{2n}'$. Take an element from $D$ which is compatible with $S_{2n}'$ and choose $S_{2n}$ so that the element we have taken becomes $S_{2n+1}'$ by $\sigma$ and let $S_{2n+1}'$ be a common extension (in $V$) of $S_{2n}'$ and $S_{2n+1}$; This finishes the construction of the sequences.

Now let us prove the equivalence (*):

Suppose $\pi^{-1}(B_i)$ is meager and assume there is a countable transitive model $M$ of ZFC with $c \in M$ such that $M \vDash \pi^{-1}(B_i)$ is not meager''. We will derive a contradiction. Since every Borel set is $\mathbb{P}$-Baire, $\pi^{-1}(B_i)$ has the Baire property. Hence there is a $T \in \mathbb{P}^M$ such that in $M$, $\pi^{-1}(B_i)$ is comeager in $O_T$. By Claim 3.6(a), $\pi^{-1}((T \cup B_i) \setminus B_i)$ is almost included in $O_T \setminus \pi^{-1}(B_i)$, hence, in $M$, $\pi^{-1}((T \cup B_i) \setminus B_i)$ is meager in $St(\mathbb{P})$. Now apply the claim for $[T] \setminus B_i$. Then we get a $T' \leq T$ such that $O_T \cap \pi^{-1}((T \cup B_i) \setminus B_i)$ is meager. But this means that $O_T$ is almost included in $\pi^{-1}(B_i)$. Since $O_T$ is not meager by Lemma 3.1, $\pi^{-1}(B_i)$ is not meager, which contradicts the assumption that $\pi^{-1}(B_i)$ is meager.

For the other direction, by Fact 3.7, it suffices to show that for any $T$ in $\mathbb{P}$, there is a $T' \leq T$ such that $O_T \cap \pi^{-1}(B_i)$ is meager. So fix any $T$. Then pick a countable transitive model $M$ with $c \in M$. Then by Claim 3.8, there is a $T' \leq T$ such that $O_T \cap \pi^{-1}(B_i)$ is meager, as desired.

(2) Let $x$ be $\mathbb{P}$-generic over $M$. Then the set $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$ is a $\Pi^1_2$-generic filter over $M$. We show that $x \notin B_c$ when $c$ is a Borel code in $M$ with $B_c \in I^\mathbb{P}^x$.

Let $c$ be such a Borel code. By (1) and the downward absoluteness for $\Pi^1_2$-formulas, $M \vDash \langle B_c \in I^\mathbb{P}^x \rangle$. Let $\mathcal{M}$ be the dense embedding from $\mathbb{P}^M$ to $\bigl\langle (B/\langle b \rangle) \setminus \{0\} \bigr\rangle^M$ defined in Proposition 2.18 and $\mathcal{M}^x(G_x)$ be the $\bigl\langle B/\langle b \rangle \bigr\rangle$-generic filter over $M$ induced by $\mathcal{M}$ and $G_x$. Using the fact that $I^\mathbb{P}^x$ is a $\sigma$-ideal, it is routine to check that $B \in \mathcal{M}^x(G_x)$ iff $x \in B$ for any Borel set $B$ with a code in $M$. But the left hand side of the above equivalence implies $M \vDash \langle B \notin I^\mathbb{P}^x \rangle$, hence by upward absoluteness for $\Sigma^1_2$-formulas, $B \notin I^\mathbb{P}^x$. Since $B_c \in I^\mathbb{P}^x$, $x \notin B_c$, as desired.

(3) Let $x$ be a quasi-$\mathbb{P}$-generic real over $M$ and put $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$. We show that $G_x$ is a $\Pi^M_2$-generic filter over $M$.

We first see that $G_x$ meets every maximal antichain of $\mathbb{P}^M$ in $M$. Take any maximal antichain $A$ of $\mathbb{P}^M$ in $M$. Since $\mathbb{P}$ is provably ccc, $A$ is countable in $M$. Now consider $B = \bigcup_{T \in A} \{T \mid T \in A\}$. Then $B$ is a Borel set with a code in $M$ and $M \vDash \forall \omega \langle B \notin I^\mathbb{P}^x \rangle$. By (1), this is also true in $V$. Since $x$ is quasi-$\mathbb{P}$-generic over $M$, $x \notin B_c$, i.e., $x$ is in $B$. So $G_x$ meets $A$.

Now we see that $G_x$ is a filter. Take any two elements $T_1, T_2$ in $G_x$. We will find a common extension of $T_1, T_2$ in $G_x$. Consider $D = \{S \in \mathbb{P} \mid \langle S \cap [T_1] \cap B \rangle = \emptyset \}$. Then $D$ is a countable in $M$. Since $D$ is Borel, we can find a common extension of $T_1, T_2$ in $G_x$. Then by strong absoluteness of $\mathbb{P}$, $D$ is dense in $M$. Hence $G_x$ meets $D$. Therefore, $G_x$ is a $\Pi^M_2$-generic filter over $M$.

There is a close connection between forcing absoluteness for $\mathbb{P}$ and $\mathbb{P}$-Baireness. The author would like to thank Neus Castells for telling him about the following theorem:

**Theorem 3.9** (Castells). Let $\mathbb{P}$ be a partial order. Then the following are equivalent:

1. $\Sigma^1_2$-absoluteness holds, and
2. every $\Delta^1_2$-set of reals is $\mathbb{P}$-Baire.

**Proof.** The argument is essentially the same as in [12, Theorem 3.1].


4. $\Sigma^1_3$-absoluteness

Now we give a precise statement of Theorem 1.3 and prove it. Also we will prove related results.

**Theorem 4.1.** Let $P$ be a strongly arboreal, proper forcing. Then the following are equivalent:

1. $\Sigma^1_3$-absoluteness holds, and
2. every $\Delta^1_2$-set of reals is $P$-measurable.

**Proof.** By Theorem 3.9, it suffices to show that every $\Delta^1_2$-set of reals is $P$-measurable if every $\Delta^1_2$-set of reals is $P$-Baire. By Lemma 3.5, it is enough to see that every $\Delta^1_2$-set of reals is $P$-Baire assuming every $\Delta^1_2$-set of reals is $P$-measurable.

The following claim is the key point:

**Claim 4.2.** Let $P$ be a strongly arboreal proper forcing and $\tau$ be a $P$-name for a real. Then for any $T$ in $P$, there is a $T' \leq T$ and a Borel function $g : [T'] \rightarrow \mathbb{R}$ such that $T' \Vdash \tau = g(\check{c})$.

**Proof of Claim 4.2.** This is a combination of Proposition 2.18 in this paper and [34, Proposition 2.3.1].

Now take any $\Delta^1_2$-set $A$ and a Baire measurable function $f$ from $\text{St}(P)$ to the reals. We show that $f^{-1}(A)$ has the Baire property. It suffices to show that $[T \mid O_T \cap f^{-1}(A)\text{ is meager or } O_T \setminus f^{-1}(A)\text{ is meager}]$ is dense in $P$.

So take any $T$ in $P$ and we will find an extension $S$ of $T$ with the above property. By the above claim, there is a $T' \leq T$ and a Borel function $g : [T'] \rightarrow \mathbb{R}$ such that $T' \Vdash \tau_T = g(\check{c}_T)$, where $\tau_T$ is the $P$-name for a real defined in Lemma 3.2(1).

Hence, by Lemma 5.2(3), $f = g \circ f_{\check{c}_T}$ almost everywhere in $O_{T'}$. Since $g^{-1}(A)$ is $\Delta^1_2$, it is $P$-measurable by the assumption. By Lemma 3.5(2), $f^{-1}_{\check{c}_T}^{-1}(g^{-1}(A)) = (g \circ f_{\check{c}_T})^{-1}(A)$ has the Baire property. Hence $f^{-1}(A)$ has the Baire property in $O_{T'}$. In particular, there is an $S \subseteq T'$ such that either $O_S \cap f^{-1}(A)$ is meager or $O_S \setminus f^{-1}(A)$ is meager as desired. □

**Theorem 4.3.** Let $P$ be a strongly arboreal, strongly proper, $\Sigma^1_3$ forcing. Assume the following:

\[
\{c \mid c \text{ is a Borel code and } B_c \in I_{P^*} \} \in \Sigma^1_2.
\]

Then the following are equivalent:

1. $\Sigma^1_3$-absoluteness holds,
2. every $\Delta^1_2$-set of reals is $P$-measurable, and
3. for any real $a$ and $T \in P$, there is a quasi-$P$-generic $x \in [T]$ over $L[a]$.

**Proof.** We have seen the equivalence between (1) and (2). We will show the direction from (1) to (3) and the direction from (3) to (2).

For (1) implies (3), take a real $a$ and $T \in P$. We will find a quasi-$P$-generic real $x$ over $L[a]$ with $x \in [T]$. But by the assumption (\ast), the statement “There is a quasi-$P$-generic real $x$ over $L[a]$ with $x \in [T]$” is $\Sigma^1_3$ and this is true in a generic extension $V[C]$ with $T \subseteq G$ by the same argument as in Proposition 2.17. (Although $P$ might not be provably $\Delta^1_2$ as we assumed in Proposition 2.17, we used it only to see $M \models \exists x \in I_{P^*} \models \exists x \in I_{P^*}$ in $V$ and this is ensured by the assumption (\ast) and Shoenfield absoluteness without using $P$ being provably $\Delta^1_2$.) Hence by $\Sigma^1_3$-forcing absoluteness, the statement is also true in $V$ as desired.

For (3) implies (2), take any $\Delta^1_2$-set $A$ and we will show that $A$ is $P$-measurable. Take any $T \in P$.

**Case 1:** $\omega_1^1[a] < \omega_1^Y$ for any real $a$.

Pick a real $a$ with $T \in L[a]$. By the assumption, $\omega_1^Y$ is inaccessible in $L[a]$. Take an $\alpha < \omega_1^Y$ so that $L_{\omega_1}[a]$ contains all the dense subsets of $P$ in $L[a]$ and $L_\alpha[a]$ is an elementary substructure of $L_{\omega_1}[a]$ and let $M = L_\alpha[a]$. Since $\alpha$ is $\Sigma^1_2$, $M$ is a countable transitive model of ZFC computing $P$ correctly. Hence by strong properness of $P$, there is an $(M, P)$-generic condition $T'$ below $T$. Then any dense subset of $P$ in $M$ is predense below $T'$ and the set of all $P$-generic reals over $M$ is of measure one w.r.t. $I_{P^*}$ below $T'$, (i.e., the complement of that set intersected with $[T']$ is in $I_{P^*}$). By the property of $M$, this means that the set of all $P$-generic reals over $M$ is of measure one w.r.t. $I_{P^*}$ below $T'$. The rest is a standard Solovay argument to prove regularity properties in Solovay models. (Actually, every $\Sigma^1_2$-set of reals is $P$-measurable in this case.)

**Case 2:** $\omega_1^1[a] = \omega_1^Y$ for some real $a$.

The argument is basically the same as in [7, Proposition 2.1]. Pick a real $a$ with $T \in L[a]$ and such that $\omega_1^1[a] = \omega_1^Y$ and $A$ is $\Delta^1_2(a)$. The idea is to decompose $[T] \cap A$ and $[T] \setminus A$ into Borel sets in an absolute way between $L[a]$ and $V$, and a Borel set containing a quasi-$P$-generic real over $L[a]$ must be $P^*$-positive and below that Borel set we will find an extension of $T$ as a witness for $P^*$-measurability of $A$.

Since $[T] \cap A$ and $[T] \setminus A$ are $\Sigma^1_2(a)$ sets, there are Shoenfield trees $U_1$ and $U_2$ in $L[a]$ for $[T] \cap A$ and $[T] \setminus A$ respectively. From these trees, we can naturally decompose $[T] \cap A$ and $[T] \setminus A$ into $\omega_1$-many Borel sets as in [26, 2F.1-2F.3], i.e., there are sequences $(c_\alpha \mid \alpha < \omega_1), (d_\alpha \mid \alpha < \omega_1)$ of Borel codes in $L[a]$ such that $[T] \cap A = \bigcup_{\alpha < \omega_1} B_{c_\alpha}$ and $[T] \setminus A = \bigcup_{\alpha < \omega_1} B_{d_\alpha}$. The point is that the above equations are absolute between $L[a]$ and $V$ because those two sequences only depend on $U_1, U_2$ and $\omega_1$ and $\omega_1^1[a] = \omega_1^Y$ as we assumed.
By assumption, there is a quasi-$\mathbb{P}$-generic real $x$ over $[a]$ with $x \in [T]$. Hence there is an $\alpha < \omega_1$ such that either $x \in B_{c_\alpha}$ or $x \in B_{c_\alpha}$. Without loss of generality, we may assume $x \in B_{c_\alpha}$. Since $c_\alpha$ is in $[a]$, by the definition of quasi-$\mathbb{P}$-genericity, $B_{c_\alpha}$ is not in $I_\mathbb{P}^*$. Since every Borel set is $\mathbb{P}$-Baire, it is $\mathbb{P}$-measurable by Lemma 3.5 (2). Hence there is a condition $T'$ such that $[T'] \setminus B_{c_\alpha} \in I_\mathbb{P}$. Since $B_{c_\alpha} \subseteq [T] \cap A$, $T' \subseteq T$ and $[T'] \setminus A \in I_\mathbb{P}$, as desired. 

**Theorem 4.4.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, $\Sigma^1_2$ forcing. Assume

\[ \{ c \mid c \text{ is a Borel code and } B_c \in I_\mathbb{P}^* \} \in \Sigma^1_2, \]

and

$I_\mathbb{P}$ is Borel generated or $I_\mathbb{P} = N_\mathbb{P}$.

Then the following are equivalent:

1. every $\Sigma^1_2$-set of reals is $\mathbb{P}$-measurable, and
2. for any real $a$, $\mathbb{P} \setminus \{ x \mid x$ is quasi-$\mathbb{P}$-generic over $[a] \} \in I_\mathbb{P}^*$.

**Proof.** (1) implies (2), take any real $a$ and we show that $A = \{ x \mid x \text{ is quasi-$\mathbb{P}$-generic over } [a] \}$ is of measure one w.r.t. $I_\mathbb{P}^*$. Suppose not. Then $\omega \setminus A \notin I_\mathbb{P}^*$. By the assumption (1), $\omega \setminus A$ is $\Sigma^1_2$. So by (1), it is $\mathbb{P}$-measurable. Hence there is a $T$ in $\mathbb{P}$ such that $[T] \setminus (\omega \setminus A) = [T] \cap A \in I_\mathbb{P}$. We show that this cannot happen.

Case 1: $I_\mathbb{P}$ is Borel generated, i.e., for any $N \in I_\mathbb{P}$ there is a Borel set $B \in I_\mathbb{P}$ such that $N \subseteq B$.

Since $[T] \cap A \in I_\mathbb{P}$, there is a Borel set $B \subseteq [T]$ in $I_\mathbb{P}$ such that $[T] \cap A \in B$. Let $c$ be a Borel code for $B$. By Theorem 4.3, there is a quasi-$\mathbb{P}$-generic real $x$ over $[a]$, $c$ with $x \in [T]$. Since $B \in I_\mathbb{P}$, $x \notin B$. But this is impossible because $x$ is also quasi-$\mathbb{P}$-generic over $[a]$ and hence $x \in [T] \cap A \subseteq B$.

Case 2: $I_\mathbb{P} = N_\mathbb{P}$.

In this case, $[T] \cap A$ is $\mathbb{P}$-null, hence there is a $T' \subseteq T$ such that $[T'] \cap A = \emptyset$. By Theorem 4.3, there is a quasi-$\mathbb{P}$-generic real $x$ over $[a]$ with $x \in [T']$. Hence $x \in [T'] \cap A$, a contradiction.

For (2) implies (1), take any $\Sigma^1_2$-set $A$. We show that $A$ is $\mathbb{P}$-measurable. Let $T$ be in $\mathbb{P}$. We will find an extension $T'$ of $T$ approximating $A$ as in the definition of $\mathbb{P}$-measurability. If $[T] \cap A \in I_\mathbb{P}^*$, we are done. So we assume $[T] \cap A \notin I_\mathbb{P}^*$.

Case 1: $\omega_{\mathbb{P}} \mid [a] < \omega_1^\mathbb{P}$ for any real $a$.

As in (3) to (2) in Theorem 4.3, in this case, every $\Sigma^1_2$-set of reals is $\mathbb{P}$-measurable by a standard Solovay argument.

Case 2: $\omega_{\mathbb{P}} \mid [a] = \omega_1^\mathbb{P}$ for some real $a$.

Let $a$ be a real such that $[T] \cap A$ is $\Sigma^1_2(a)$ and $\omega_{\mathbb{P}} \mid [a] = \omega_1^\mathbb{P}$. Then we have a Shoenfield tree in $[a]$ for $[T] \cap A$ and we get an $\omega_1$-many Borel decomposition of $[T] \cap A$ into Borel sets $\{ B_{c_\alpha} \mid \alpha < \omega_1 \}$ with $c_\alpha \in [a]$ for each $\alpha$ as in the proof of Theorem 4.3. Since $[T] \cap A \notin I_\mathbb{P}^*$ and the set of quasi-$\mathbb{P}$-generic reals over $[a]$ is of measure one w.r.t. $I_\mathbb{P}^*$ by (2), there is a quasi-$\mathbb{P}$-generic real $x$ over $[a]$ with $x \in [T] \cap A$, so there is an $\alpha$ such that $x \in B_{c_\alpha}$.

The rest is the same as in the proof for (3) to (2) in Theorem 4.3. Since $c_\alpha \in [a]$ and $x$ is quasi-$\mathbb{P}$-generic over $[a]$, $B_{c_\alpha} \notin I_\mathbb{P}^*$. Since Borel set is $\mathbb{P}$-measurable, there is a $T'$ in $\mathbb{P}$ such that $[T'] \setminus B_{c_\alpha} \in I_\mathbb{P}$. Since $B_{c_\alpha} \subseteq [T] \cap A$, $T' \subseteq T$ and $[T'] \setminus A \in I_\mathbb{P}$, as desired. 

**5. \Sigma^1_3$-absoluteness**

It is natural to try to generalize the equivalence results proved in Section 4 up to the ones between $\Sigma^1_3$-forcing absoluteness and the regularity properties for $\Delta^1_3$-sets of reals and $\Sigma^1_3$-sets of reals. But these analogues cannot be proved in ZFC.\footnote{Start from $L$ and add $\omega_1$-many Cohen reals, then in this model, $\Sigma^1_3$-forcing absoluteness for Cohen forcing holds but there is a $\Sigma^1_2$-set of reals without the Baire property.}\footnote{For the basic definitions and theories for sharps, the reader can refer to [21] and [10].}\footnote{Martin and Solovay [24] constructed so-called Martin–Solovay trees assuming a measurable cardinal when they proved all the $\Sigma^1_3$-formulas are absolute between the ground model and small generic extensions. For the details, the reader can refer to [21].}

In this section, with an additional assumption (sharps for sets), we will prove the analogues of the results in Section 4. We use inner model theory for some arguments. For the basics of inner model theory, the reader can consult [36,25,32].

**Theorem 5.1.** Let $\mathbb{P}$ be a strongly arboreal proper forcing. Suppose every set has a sharp.\footnote{Martin and Solovay [24] constructed so-called Martin–Solovay trees assuming a measurable cardinal when they proved all the $\Sigma^1_3$-formulas are absolute between the ground model and small generic extensions. For the details, the reader can refer to [21].} Then either $\Delta^1_3$-determinacy holds or the following are equivalent:

1. $\Sigma^1_3$-$\mathbb{P}$-absoluteness holds, and
2. every $\Delta^1_3$-set of reals is $\mathbb{P}$-measurable.

**Proof.** For (1) implies (2), the argument is the same as for (1) implies (2) in [12, Theorem 3.1]. What we should check is that we get the absolute tree representation for $\Sigma^1_3$-sets between $V$ and $V^{\mathbb{P}}$. The rest is exactly the same.

For such tree representation, Feng, Magidor and Woodin used Shoenfield trees for $\Sigma^1_2$-sets. With the help of sharps for sets, now we use Martin–Solovay trees for $\Sigma^1_3$-sets.\footnote{Martin and Solovay [24] constructed so-called Martin–Solovay trees assuming a measurable cardinal when they proved all the $\Sigma^1_3$-formulas are absolute between the ground model and small generic extensions. For the details, the reader can refer to [21].} By [14, Theorem 2.1], it suffices to see that $u_2^V = u_2^{V^{\mathbb{P}}}$ for the absoluteness
of Martin–Solovay trees between $V$ and $V^\mathcal{P}$.

But this is true assuming every set has a sharp and $\mathbb{P}$ being proper by [29, Theorem 2.1.9, Example 3.2.7].

For (2) implies (1), first note that we may assume that every $\Delta^1_3$-set is $\mathbb{P}$-Baire by the same argument for (2) implies (1) in Theorem 4.1. The argument is the same as for in [12, Theorem 3.1]. What we need is to uniformize a $\Pi^1_2$-relation by a $\Sigma^1_3$-function (in [12, Theorem 3.1], Feng, Magidor and Woodin uniformized a $\Pi^1_2$-relation by a $\Sigma^1_3$-function). The rest is exactly the same. But such uniformization is possible assuming the failure of $\Delta^1_3$-determinacy.

The author would like to thank Hugh Woodin for pointing out the following fact to him:

**Theorem 5.2.** Suppose every real has a sharp. Then either $\Delta^1_3$-determinacy holds or $\Sigma^1_3$ has the uniformization property, i.e., any $\Sigma^1_3$-relation can be uniformized by a $\Sigma^1_3$-function.

**Proof.** The reader who is not familiar with inner model theory may skip this proof and take the above statement as a black box.

It suffices to show that every $\Pi^1_2$-relation can be uniformized by a $\Sigma^1_3$-function. Suppose $\Delta^1_3$-determinacy fails.

**Case 1:** For any real $a$, $a^\dagger$ exists.

In this case, by the result of Steel [32], there is a real $a_0$ such that the Mitchell–Steel core model $K_0$ exists and that $K_0$ is $\Sigma^1_3$-correct for any $a \geq_T a_0$, where $\leq_T$ is the Turing order on the reals.\(^{15,16}\)

For each $a \geq_T a_0$, let $<_a$ be the canonical good $\Delta^1_3(a)$-well-ordering on the reals in $K_0$.\(^{15}\) Given a real $b$ and a $\Pi^1_2(b)$-relation $R$, define the uniformization $f$ as follows:

$$f(x) = y \iff y \text{ is the first } <_{(x,a_0,b)} \text{-element with } (x,y) \in R,$$

where $(x, a_0, b)$ is the real coding $x$, $a_0$ and $b$. For each $x \in \text{dom}(R)$, such a $y$ always exists because $K_{(x,a_0,b)}$ is $\Sigma^1_3$-correct. So $f$ uniformizes $R$ and regarding the fact that $<_a$ is a good $\Delta^1_3(a)$-well-ordering in $K_0$ for each $a \geq_T a_0$, it is easy to see that $f$ is $\Sigma^1_3$.

**Case 2:** There is a real $a$ such that $a^\dagger$ does not exist.

Then there is a real $a_1$ such that for any real $a \geq_T a_1$, $a^\dagger$ does not exist. By the result of Dodd–Jensen in [11], the Dodd–Jensen core model $K_0$ exists and it enjoys the same properties as the Mitchell–Steel core model mentioned above for any $a \geq_T a_1$.\(^{17}\) The rest is the same as Case 1. □ □

**Theorem 5.3.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_3$ forcing. Suppose every set has a sharp. Then either $\Delta^1_3$-determinacy holds or the following are equivalent:

1. $\Sigma^1_2$-$\mathbb{P}$-absoluteness holds,
2. every $\Delta^1_3$-set of reals is $\mathbb{P}$-measurable, and
3. for any real $a$ and any $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over $K_0$, where

$$K_0 = \begin{cases} 
\text{the Mitchell–Steel core model}^{15} & \text{if } a^\dagger \text{ exists;} \\
\text{the Dodd–Jensen core model}^{17} & \text{otherwise.}
\end{cases}$$

**Proof.** In Theorem 5.1, we have seen the equivalence between (1) and (2). We show the direction from (1) to (3) and the one from (3) to (2).

For (1) implies (3), all we need is that the statement "there is a quasi-$\mathbb{P}$-generic real $x$ over $K_0$ with $x \in [T]$" is $\Sigma^1_3$ for each real $a$ and each $T \in \mathbb{P}$. But this is true by Proposition 2.17(1) and the fact that the set of reals in $K_0$ is $\Sigma^1_3(a)$ in $V$.

The argument for (3) to (2) is basically the same as the one in Theorem 4.3. Assume $\Delta^1_3$-determinacy fails. Then there is a real $a$ such that there is no inner model with a Woodin cardinal containing $a$.\(^{18}\) For simplicity, we assume that there is no inner model with a Woodin cardinal. The general case can be dealt with in the same way.

---

12 The second uniform indiscernible $u_2$ is the second element of $\bigcap_{x \in \mathcal{L}_1} l_x$, where $l_x$ is the indiscernible sequence over $\mathcal{L}[x]$ derived from $x^\dagger$. For the details, the reader can consult [21].

13 Since $\Delta^1_3$-determinacy implies that $\Pi^1_1$ has the uniformization property, this fact states the dichotomy of the uniformization property for $\Sigma^1_1$ and $\Pi^1_1$.

14 The theory of daggers ($a^\dagger$) was developed by Solovay to analyze the theory of $L_x$ as is done for the theory of $L$ by $0^\sharp$. For the basics, the reader can consult [21].

15 Assuming daggers for reals and the failure of $\Delta^1_3$-determinacy, there is a real $a_0$ such that the Mitchell–Steel core model $K_0$ exists, $K_0$ is $\Sigma^1_3$-correct, the set of reals in $K_0$ is $\Sigma^1_3(a)$ in $V$ and the canonical order of $K_0$ gives a good $\Delta^1_3(a)$-well-ordering on the reals in $K_0$ for any $a \geq_T a_0$. For the basics of the Mitchell–Steel core model, the reader can consult [32].

16 In [32, Theorem 7.9], Steel assumed the existence of two measurable cardinals to prove $\Sigma^1_3$-correctness of $K_0$. We can replace the lower measurable by $a^\dagger$ and the greater measurable by $a^\sharp$. (Recent development of inner model theory even allows one to omit this sharp. Jensen and Steel [18,17] constructed the core model $K$ without using measurable cardinals.) For the details, see [27].

17 For the basics of the Dodd–Jensen core model, the reader can consult [11].

18 For the proof, see [32, Corollary 7.14].
Case 1. $\omega_1^K < \omega_1^V$ for any real $a$.

As in Theorem 4.3, we can conclude that every $\Delta^1_1$-set of reals (even $\Delta^1_1$-set of reals) is $\mathbb{P}$-measurable by using $\Sigma^1_1$-correctness of $K$. To see $\Sigma^1_1$-correctness of $K$, we need the case distinction whether $a^i$ exists or not. If $a^i$ does not exist, this is due to Dodd–Jensen in [11]. When $a^i$ exists, this is due to Steel.16

Case 2. $\omega_1^K = \omega_1^V$ for some real $a$.

We need the absolute decomposition of $\Sigma^1_1$-sets into Borel sets between $K_0$ and $V$ for some real $a$ as we had for $\Sigma^1_1$-sets between $L[a]$ and $V$ in Theorem 4.3. The following result is essential; its proof was communicated to us by Ralf Schindler:

**Theorem 5.4 (Schindler).** Suppose there is no inner model with a Woodin cardinal. Then if $u_2^K < u_2^V$ for any real $a$, then $\omega_1^K < \omega_1^V$ for any real $a$.

**Proof.** Here we heavily use the machinery of inner model theory. All the definitions and theorems we will use without any mention can be found in [36]. The reader who is not familiar with inner model theory may skip this proof and take the above statement as a black box.

For simplicity, we only prove $\omega_1^K < \omega_1^V$ assuming $u_2^K < u_2^V$ for each real $a$. To derive a contradiction, we assume $\omega_1^K = \omega_1^V$.

The following is the first point:

**Claim 5.5.** The mouse $K_{|\omega_1}$ is universal for countable mice, i.e., $M \leq^* K_{|\omega_1}$ for any countable mouse $M$, where $\leq^*$ is the mouse order.

**Proof of Claim 5.5.** Suppose there is a countable mouse $M$ with $M >^* K_{|\omega_1}$. Coiterate them and let $T$. $U$ be the resulting trees for $M$ and $K_{|\omega_1}$ respectively.

**Case 1:** $\text{lh}(T)$ is countable.

Since $M >^* K_{|\omega_1}$, $U$ does not have a drop. But then the last model of $U$ cannot be an initial segment of the last model of $T$ since the length of $T$ is countable, a contradiction.

**Case 2:** $\text{lh}(T)$ is uncountable.

Since $M >^* K_{|\omega_1}$, $U$ does not have a drop. If $U$ was non-trivial, then the final model of $U$ would be non-sound and could not be a proper initial segment of the final model of $T$. Hence $U$ is trivial and $K_{|\omega_1}$ is an initial segment of the final model of $T$. But this means $\omega_1$ is a limit of critical points of embeddings via $T$, hence $\omega_1$ is inaccessible in $K$, contradicting the assumption $\omega_1^K = \omega_1^V$. \(\blacksquare\) Theorem 5.5.

By the same argument, we can prove that $K_{|\omega_1}$ is universal for countable $a$-mice for each real $a$. We now have two cases:

**Case 1:** There is a real $a$ such that $a^i$ does not exist.

This case was taken care of by Steel and Welch. In [31, Lemma 3.6], they assumed $u_2 = \omega_2$, which is stronger than $u_2^K < u_2^V$ for each real $a$, and proved there is a countable mouse stronger than $K_{|\omega_1}$ w.r.t. mouse order. But assuming $\omega_1^K = \omega_1^V$ and the non-existence of $\Theta^a$, we can run their argument only assuming $u_2^K < u_2^V$ and get the same conclusion.

Furthermore, we can easily relativize this argument to $K$. Hence assuming $\omega_1^K = \omega_1^V$ (even $\omega_1^K = \omega_1^V$) and the non-existence of $a^i$, if $u_2^K < u_2^V$, then there is an $a$-mouse stronger than $K_{|\omega_1}$ w.r.t. mouse order, which contradicts the $a$-relativized version of Claim 5.5.

**Case 2:** For any real $a$, $a^i$ exists.

This case is new. Since $u_2^K < u_2^V$, there is a real $a$ such that $u_2^K < (\omega_1^a)^{L[a]}$. The idea is to use $a^i$ (that exists since $a^i$ exists) and linearly iterate it with the lower measure in $a^i$ with length $\omega_1$. Then the height of the last model is bigger than $u_2^K$ since $u_2^K < (\omega_1^a)^{L[a]}$. Now we restrict this linear iteration map to $K$ in $a^i$ constructed up to the point with the top measure. The point is this is an iteration map on it and the final model of this iteration has height bigger than $u_2^K$. Since it is a countable mouse, by Claim 5.5, we get a countable mouse in $K$ with the same property, which yields a contradiction by a standard boundedness argument.

We will discuss this idea in detail. Let $i$ be the linear iteration map of $a^i$ derived from the iterated ultrapower starting from the lower measure in it with length $\omega_1$. Then the target of $i$ of $N$ has height bigger than $u_2^K$ since $u_2^K < (\omega_1^a)^{L[a]}$ and the critical point of $i$ goes to $\omega_1$ and $N$ has a cardinal bigger than $\omega_1$ and $a \in N$. Let $K_{|\Omega}$ be the $K$ in $a^i$ constructed up to $\Omega$, the critical point of the top measure in $a^i$. Then $K_{|\Omega}$ is a mouse and we call it $M$.

We claim that if we restrict $i$ to $M$, then it is an iteration map on $M$. Since $i$ is from a linear iteration of ultrapowers via measures, by applying the result of Schindler [28] in each ultrapower in the iteration, we can prove that the restriction of $i$ to $M$ is an iteration with length $\omega_1$ (which itself might be quite complicated). Moreover, the final model of this iteration has height greater than $u_2^K$ because $i$ maps $\Omega$ greater or equal to $\omega_1^a$. Let $M_{\alpha}$ be the $\alpha$-th iterate via $\mathcal{T}$ and $\overline{M}_{\alpha, \beta}$ be the induced maps for $\alpha \leq \beta < \omega_1$, $\Omega$.

Since $M$ is a countable mouse, by Claim 5.5, there is an $a_0 < \omega_1$ such that $M \leq^* K_{|a_0}$. We will show that $K_{|a_0}$ has the same property, i.e., there is an iteration from $K_{|a_0}$ with length $\omega_1$ such that the height of the final model is greater than $u_2^K$. (Note that there might be a drop.) Coiterate $K_{|a_0}$ and $M$ and let $\pi : M \rightarrow N$ be the resulting map. Note that there is no drop from the $M$-side because $M \leq^* K_{|a_0}$.
We will construct \( \langle N_\alpha \mid \alpha \leq \omega_1 \rangle \), \( \langle \pi_\alpha : M_\alpha \to N_\alpha \mid \alpha \leq \omega_1 \rangle \), and \( \langle i^{U}_{\alpha, \beta} : N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1 \rangle \) with the following properties:

1. The diagrams below all commute,
2. \( M_\alpha \sim^* N_\alpha \sim^* M_{\alpha + 1} \) for each \( \alpha \),
3. \( N_\alpha \) is the direct limit of \( N_\beta \) (\( \beta < \alpha \)) for limit \( \alpha \), and
4. \( i^{U}_{\alpha, \alpha + 1} \) and \( \pi_{\alpha + 1} \) are the reported maps by the comparison between \( N_\alpha \) and \( M_{\alpha + 1} \) for each \( \alpha \).

\[
\begin{array}{c}
K|\alpha_0 \sim \longrightarrow N = N_0 \xrightarrow{i^{U}_{0, 1}} N_1 \xrightarrow{i^{U}_{1, 2}} \cdots \xrightarrow{i^{U}_{\alpha, \alpha + 1}} N_\alpha \xrightarrow{i^{U}_{\alpha + 1, \alpha + 2}} N_{\alpha + 1} \\
M = M_0 \xrightarrow{i^{U}_{0, 1}} M_1 \xrightarrow{i^{U}_{1, 2}} \cdots \xrightarrow{i^{U}_{\alpha, \alpha + 1}} M_\alpha \xrightarrow{i^{U}_{\alpha + 1, \alpha + 2}} M_{\alpha + 1} \\
\pi = \pi_0 \quad \pi_1 \quad \pi_\alpha \quad \pi_{\alpha + 1}
\end{array}
\]

The above properties uniquely specify \( \langle N_\alpha \mid \alpha \leq \omega_1 \rangle \), \( \langle \pi_\alpha : M_\alpha \to N_\alpha \mid \alpha \leq \omega_1 \rangle \), and \( \langle i^{U}_{\alpha, \beta} : N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1 \rangle \). Hence it suffices to check (1) and (2) above for this construction.

For (1), it suffices to show that \( i^{U}_{\alpha + 1, \alpha} \circ \pi_\alpha = \pi_{\alpha + 1} \circ i^{U}_{\alpha, \alpha + 1} \) for each \( \alpha \). By the Dodd–Jensen Lemma (e.g., in [36, Theorem 9.2.10]), any two simple iteration maps from a mouse to a mouse are the same. By (2) for \( \alpha, \pi_\alpha, \pi_{\alpha + 1}, i^{U}_{\alpha, \alpha + 1} \), and \( i^{U}_{\alpha + 1, \alpha} \) are all simple iteration maps. Hence we get the desired commutativity. (2) follows from the fact that all the maps constructed before are simple iteration maps.

Since the height of \( N_{\omega_1} \) is greater than or equal to that of \( M_{\omega_1} \), there is an iteration from \( K|\alpha_0 \) with length \( \omega_1 \) whose final model has height greater than \( u_2^K \), as desired.

Since \( K|\alpha_0 \) is in \( K \) and \( \alpha_0 \) is countable in \( K \), there is a real \( x \) in \( K \) coding \( K|\alpha_0 \). We show that the height of \( N_{\omega_1} \) is less than \( (\alpha_0^+)^{[\omega]} \). In \( L[x] \), we collapse \( \omega_1^x \) with the forcing \( \text{Coll}(\omega, \omega_1^x) \). Let \( g : \omega \to \omega_1^x \) be a generic surjection over \( L[x] \). Since \( K|\alpha_0 \) is coded by \( x \) and the length of iteration is \( \omega_1^x \) which is countable witnessed by \( g \), by the boundedness lemma in \( L[x]\{g\} \), the height of \( N_{\omega_1} \) is less than \( \alpha_0^{[\omega]\{g\}} = (\alpha_0^+)^{[\omega]} \) as desired. Since \( x \) is in \( K \), \( (\alpha_0^+)^{[\omega]} < u_2^K \) and hence the height of \( N_{\omega_1} \) is less than \( u_2^K \). But the height was greater than \( u_2^K \). Contradiction! — Theorem 5.4.

Now by the assumption and the above fact, there is a real \( a \) such that \( \alpha_0^K = \omega_1^a \) and \( u_2^K = u_2^a \). By the result of Hjorth [14, Theorem 2.1], the Martin–Solovay trees for \( \Sigma_1^1 \)-sets are absolute between \( K_a \) and \( V \). Since the trees are on \( \omega \times u_0 \) and \( u_0 \) is absolute between \( K_a \) and \( V \), we get the absolute decomposition of \( \Sigma_1^1 \)-sets into Borel sets between \( K_a \) and \( V \) as desired. The rest is exactly the same as in Theorem 4.3. — Theorem 5.6.

**Theorem 5.6.** Let \( \mathbb{P} \) be a strongly arboreal, strongly proper, provably \( \Delta^1_2 \)-forcing. Suppose every set has a sharp. Assume

\[
I_\mathbb{P} \text{ is Borel generated or } I_\mathbb{P} = N_\mathbb{P}.
\]

Then either \( \Delta^1_2 \)-determinacy holds or the following are equivalent:

1. Every \( \Sigma_1^1 \)-set of reals is \( \mathbb{P} \)-measurable, and
2. for any real \( a \), \( \mathbb{R} \setminus \{ x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } K_a \} \in I_\mathbb{P}^* \), where

\[
K_a = \begin{cases} 
\text{the Mitchell–Steel core model} & \text{if } a^1 \text{ exists,} \\
\text{the Dodd–Jensen core model} & \text{otherwise.}
\end{cases}
\]

**Proof.** The argument is exactly the same as Theorem 4.4 by replacing \( L[a] \) by \( K_a \) and using the analogous facts about \( K_a \) we have already stated. — Theorem 5.6.

**6. Applications**

In this section, we mention two applications of our theorems to particular cases. One will be proved here and the other is in [9].

The first one is about Silver forcing. The conditions of Silver forcings are uniform perfect trees on 2 where a tree on 2 is uniform if for any \( s, t \in T \) of the same length, we have \( s^- T (0) \in T \iff t^- T (0) \in T \) and \( s^- T (1) \in T \iff t^- T (1) \in T \). The order of Silver forcing is inclusion. It is easy to check that Silver forcing is strongly arboreal and strongly proper. Hence we can talk about the regularity property for Silver forcing as defined in Section 2.

Brendle, Halbeisen and Löwe [7] proved the following:

**Proposition 6.1** (Brendle–Halbeisen–Löwe). Let \( \mathbb{V} \) be Silver forcing. Suppose for any real \( a \) there is a quasi-\( \mathbb{V} \)-generic real over \( L[a] \). Then every \( \Delta^1_2 \)-set of reals is \( \mathbb{V} \)-measurable.\(^{19}\)

\(^{19}\) See [7, Proposition 2.1]. Regarding \( I_\mathbb{V} = N_\mathbb{V} \), it is easy to check that Silver measurability in their sense coincides with our \( \mathbb{V} \)-measurability.
**Question 6.2 (Brendle–Halbeisen–Löwe).** Does the converse of Proposition 6.1 hold? 20

We answer the above question positively:

**Proposition 6.3.** Assume every $\Delta^1_2$-set of reals is $\forall$-measurable. Then for any real $a$, there is a quasi-$\forall$-generic real over $L[a]$.

**Proof.** Since Silver forcing is strongly arboreal and strongly proper, by Theorem 4.3, it suffices to show that the set of Borel codes with $B_c \in I$ is $\Sigma^1_2$. We use the following fact:

**Fact 6.4 (Zapletal).** Let $G$ be the graph on $2^\omega$ connecting two binary sequences if they differ in exactly one place. Let $I$ be the $\sigma$-ideal generated by Borel $G$-independent sets (i.e., Borel sets in $2^\omega$ such that any two distinct elements of them are not connected by $G$). Then every analytic set is either in $I$ or contains $[T]$ for some $T \in \forall$.

**Proof.** See [34, Lemma 2.3.37]. ■

We show how to use Fact 6.4 to prove Proposition 6.3. We first show that $I \subseteq I$. It suffices to see that every Borel $G$-invariant set is in $N$. Take such a Borel set $B$. Since every Borel set is $\forall$-measurable and $I = N$, for each $T \in \forall$, there is a $T' \leq T$ such that either $[T'] \subseteq B$ or $[T'] \cap B = \emptyset$. But the former case cannot happen because $[T']$ contains many $G$-connected elements. Hence $[T'] \cap B = \emptyset$. Therefore $B$ is $\forall$-null.

With the above fact, this means every Borel set is either in $I$ or contains $[T]$ for some $T \in \forall$. Hence $B_c \in I$ iff $B_c$ is in $I$, i.e., it is the union of a countable set of $G$-invariant Borel sets. This is easily $\Sigma^1_2$ as desired. ■

Regarding $I = N$, the following is a direct consequence of Theorem 4.4 and Proposition 6.3 (or an easy consequence of [7, Lemma 3.1]) 21:

**Corollary 6.5.** The following are equivalent:

1. Every $\Sigma^1_2$-set of reals is $\forall$-measurable, and
2. For any real $a$, the set of quasi-$\forall$-generic reals over $L[a]$ is of measure one w.r.t $N$.

Another application is for eventually different forcing by Brendle–Löwe [9]. They used Theorem 4.4 to prove that the Baire property in eventually different topology for every $\Delta^1_2$-set of reals is equivalent to the statement “$\omega_1$ is inaccessible by reals”. For the basic definitions and properties for eventually different forcing and its topology, the reader can consult [23].

7. Questions and discussions

We close this paper by raising questions and discussing them.

7.1. On $I$ and $I^*$

Although $I^*$ is the same as $I$ for most cases as we have seen in Lemma 2.13, as in Question 2.12, we still do not know whether this is true in general. What we could wish is that this is true at least for Borel sets:

**Question 7.1.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then can we prove $B \in I$ iff $B \in I^*$ for any Borel set $B$?

If this is true, we do not have to mention $I^*$ in our theorems.

7.2. On the condition $(\ast)$ in Theorem 4.3

It is interesting to give sufficient conditions for $\mathbb{P}$ satisfying $(\ast)$ in Theorem 4.3, i.e., the set of all Borel codes with $B_c \in I^*$ is $\Sigma^1_2$. These conditions could be definability conditions on $I^*$ or directly on $\mathbb{P}$.

For the first case, we have a useful sufficient condition: we say that a $\sigma$-ideal $I$ on the reals is $\Sigma^1_2$ on $\Sigma^1_1$ if for any analytic set $B \subseteq 2^{\omega} \times \omega_1$, the set $\{c \mid B_c \in I\}$ is $\Sigma^1_2$. It is easy to check that if $I^*$ is $\Sigma^1_2$ on $\Sigma^1_1$, then $(\ast)$ holds. Since $I$ is $\Sigma^1_2$ on $\Sigma^1_1$ and $I^* = I^*$ for most cases, $(\ast)$ is true for most $\mathbb{P}$.

For the second case, we ask the following:

**Question 7.2.** Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably $\Delta^1_2$ forcing. Then can we prove $(\ast)$?

7.3. $\Delta^1_2$-determinacy and $\Sigma^1_2$-forcing absoluteness

In Theorem 5.1, we use the failure of $\Delta^1_2$-determinacy to prove the equivalence between (1) and (2). But it could be that both (1) and (2) are consequences of $\Delta^1_3$-determinacy. Since we have only used sharps for sets for the direction from (1) to (2),

---

20 See [7, Question 4].
21 This answers [7, Question 3] positively.
it is enough to see whether $\Delta_1^1$-determinacy implies $\Sigma_4^1$-forcing absoluteness:

**Question 7.3.** Suppose $\Delta_1^1$-determinacy holds. Then can we prove $\Sigma_4^1$-P-absoluteness for each strongly arboreal, proper, provably $\Delta_1^1$ forcing $\mathbb{P}$?

### 7.4. Sharps for sets vs sharps for reals

In Theorems 5.1, 5.3 and 5.6, we have assumed the existence of sharps for sets. It is natural to ask whether we can reduce this assumption to sharps for reals. The obstacle is whether proper forcings preserve the statement “every real has a sharp” and $u_2$:

**Question 7.4.** Suppose every real has a sharp. Let $\mathbb{P}$ be a strongly arboreal, proper, provably $\Delta_1^1$ forcing. Then can we prove that every real has a sharp in $V^\mathbb{P}$ and $u_2^V = u_2^\mathbb{P}$?

Finally, we show that in the case of provably ccc, $\Sigma_1^1$ forcings, things work perfectly:

**Proposition 7.5.** Let $\mathbb{P}$ be a strongly arboreal, provably ccc, $\Sigma_1^1$ forcing. Then

1. $I_\mathbb{P} = I_\mathbb{P}^*$.
2. $I_\mathbb{P}$ is Borel generated.
3. The condition $(\ast)$ holds. Moreover, $\{c \mid B_c \in I_\mathbb{P}^*\} \in \Delta_1^1$.
4. Let $M$ be a transitive model of $\text{ZFC}$. Then a real $x$ is $\mathbb{P}$-generic over $M$ if and only if $x$ is quasi-$\mathbb{P}$-generic over $M$. Moreover, $\mathbb{P}$-$\text{genericity}$ implies $\mathbb{P}$-$\text{forking}$.
5. If $\Delta_1^1$-determinacy holds, then so does $\Sigma_4^1$-P-absoluteness.
6. If every real has a sharp, then every real has a sharp also in $V^\mathbb{P}$ and $u_2^V = u_2^\mathbb{P}$.

**Proof.** (1) is already mentioned in Lemma 2.13(3) and (2) is immediate since $\mathbb{P}$ is ccc. For (3), it suffices to see the following by Lemma 3.5(1):

$$\pi^{-1}(B_c) \text{ is meager } \iff (\exists M \ni c) (M \text{ is a countable transitive model of } \text{ZFC} \text{ and } M \models \text{“} \pi^{-1}(B_c) \text{ is meager”})$$

$$\iff (\forall M \ni c) (M \text{ is a countable transitive model of } \text{ZFC} \implies M \models \text{“} \pi^{-1}(B_c) \text{ is meager”}),$$

where $\pi = f_{\mathcal{K}}$ as before.

We only show the first equivalence. For left to right, if we take a countable elementary substructure $X$ of $\mathcal{K}_\theta$ for enough large $\theta$, and $X$ has all the essential elements, then the transitive collapse of $X$ will do the job for $M$ in the right hand side.

For right to left, take an $M$ with the property in the right hand side. The idea is the same as the proof of Claim 3.8 in Lemma 2.17(1). This time, we use $G$, the Banach–Mazur game with a witness for $\pi^{-1}(B_c)$ starting from any element of $\mathbb{P}$, both in $M$ and $V$ and translate a winning strategy in $G^M$ to the one in $G$.

By the assumption, in $M$, player II has a winning strategy $\sigma'$ in $G$. The construction of a winning strategy for $\Pi$ in $G$ in $V$ from $\sigma'$ is exactly the same as Claim 3.8. But instead of using the $(M, \mathbb{P})$-genericty for a condition $T'$, we use the following:

**Claim 7.6.** Let $D$ be a dense subset of $\mathbb{P}$ in $M$. Then $D$ is predense in $\mathbb{P}$ in $V$.

**Proof of Claim 7.6.** Let $D$ be a dense subset of $\mathbb{P}$ in $M$. Then since $\mathbb{P}$ is provably ccc, in $M$, there is a countable maximal antichain $A \subseteq D$. But since $\mathbb{P}$ is $\Sigma_1^1$, the statement “a real codes a maximal antichain” is $\Sigma_1^1 \land \Pi_1^1$ and therefore $A$ remains a maximal antichain in $V$. Hence $D$ is predense in $\mathbb{P}$ in $V$.

The rest is exactly the same as Claim 3.8. The argument for (4) is exactly the same as for Lemma 2.17(2) and (3). For (5), see [29, Lemma 2.2.4]. For (6), see [29, Lemma 2.2.2, Theorem 2.2.7, Example 3.2.7].

**Acknowledgements**

The author would like to thank Joan Bagaria, Jörg Brendle, Neus Castells, Qi Feng, Greg Hjorth, Yuri Khomski, Benedikt Löwe, Ralf Schindler, Philipp Schlicht, John Steel, Hugh Woodin and Jindřich Zapletal for their helpful comments and discussions. He is also grateful to Sara Uckelmann for proofreading this paper and giving helpful comments.

This research was supported by a GLoRiClass fellowship funded by the European Commission (Early Stage Research Training Mono-Host Fellowship MEST-CT-2005-020841).

**References**


[27] Ralf Schindler, A simple proof of \(\Sigma_1^1\)-correctness of \(K\). Available at: http://wwwmath.uni-muenster.de/logik/Personen/rds/.