Chapter 1

Introduction

This thesis deals with the interface of low-dimensional topology, geometry and representation theory. These fields come together in the interpretation of the quantum invariants of knots and 3-manifolds such as the Jones polynomial. Although these invariants derived from quantum groups have revolutionized the field of low-dimensional topology, their geometric meaning is still not understood well [RT], [Wit1], [Wit2]. Great progress towards a geometrical interpretation of the quantum invariants was made in the formulation of the volume conjecture [Ka2], [MM].

In the volume conjecture the quantum invariants are connected to Thurston’s geometrization program [Th2]. Recently settled by Perelman, this program implies that every 3-manifold can be decomposed into pieces, each of which admits one of eight homogeneous geometries. Of these geometries hyperbolic geometry is the most relevant to knot theory. According to the volume conjecture, the Jones polynomial are related to the hyperbolic geometry of the knot complement. More precisely, in its simplest form the volume conjecture asserts the following:

**Conjecture 1.1. (Volume Conjecture)** [Ka2], [MM] Let \( K \) be a knot and denote its \( N \)-colored Jones polynomial by \( J_N(K)(q) \).

\[
\lim_{N \to \infty} \frac{2\pi}{N} \log |J_N(K)(e^{\frac{2\pi i}{N}})| = \text{Vol}(S^3 - K)
\]

More precisely the claim is the limit on the left hand side exists and is equal to the right hand side.

1.1 Background

In this section we briefly explain some of the basic notions necessary for understanding the definition of the terms used in the description of the volume conjecture above. This section can safely be skipped by people familiar with the area.

1.1.1 Geometrization of knot complements

Recall that a knot \( K \) in \( S^3 \) is called hyperbolic if its complement \( S^3 - K \) admits a complete metric of constant sectional curvature \(-1\). Alternatively we can...
say that in that case the complement is homeomorphic to $\mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is standard hyperbolic space and $\Gamma$ is a discrete torsion free subgroup of the group of its orientation preserving isometries.

It is clear that if two hyperbolic knot complements are isometric then the complements are homeomorphic as well. What is far from obvious is that the converse should be true. One might expect non trivial parameter spaces or moduli spaces of complete hyperbolic structures on a knot complement. However by the Mostow-Prasad Rigidity theorem this is not the case. If it exists at all, the complete hyperbolic metric is unique [BP]. More precisely, two knot complements $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ are homeomorphic if and only if $\Gamma_1$ and $\Gamma_2$ are conjugate in the isometry group.

One can also phrase the property of hyperbolicity of a knot in terms of representations of its fundamental group $\rho : \pi_1(S^3 - K) \to SL(2, \mathbb{C})$. The complete hyperbolic structure gives rise to a unique (up to conjugation) faithful representation. More general representations of the knot group also have a geometric role to play and indeed the geometry of the corresponding character variety seems to play a dominant role in the volume conjecture.

The main point of all this is that the hyperbolic structure on the knot complement is a topological property of the knot. One can thus use geometric information to distinguish knots and indeed this is employed by the fastest known algorithms to tell knots apart [Wee].

It may seem that being hyperbolic is a rather restrictive condition on a knot but this is not quite the case. It follows from Thurston’s geometrization theorem that the only prime knots that are non-hyperbolic are torus knots or satellite knots [BP]. By a torus knot we mean a knot that can be realized as a simple closed curve on the surface of a standard torus. A satellite knot $K$ is constructed from a non-trivial knot $P$ inside a solid torus $T$ and embedding it in $S^3$ such that $T$ becomes the solid torus neighborhood of a second nontrivial knot $C$. The knot $P$ is called the pattern knot and can also be viewed as a two component link in $S^3$ by replacing the solid torus by its meridian. $C$ is called the companion knot. In the figure below we have drawn an example of a satellite knot (left).

![Figure 1.1: Left: a satellite knot of the figure eight knot. The companion knot is the figure eight knot and the pattern link is the Whitehead link shown on the right.](image-url)
It should be clear now that for hyperbolic knots the hyperbolic volume is a topological invariant and this is what the right hand side of the volume conjecture refers to. What is not yet clear is how to define the volume in case the knot is not hyperbolic. It is standard to extend the hyperbolic volume by stating that it is additive under connected sum so we can restrict to prime knots.

So suppose we have a non-hyperbolic prime knot. By Thurston’s theorem such a knot is either a torus knot or a satellite knot. We now define the volume of a torus knot to be zero and the volume of a satellite knot to be the volume of its companion plus the volume of its pattern. This provides a natural extension of the volume to all knot complements that is known to agree with the Gromov norm of knot complement.

If one would be allowed only one real number to describe a knot, the extended hyperbolic volume would be a natural choice that retains a lot of information. By the above definition the family of zero volume knots is exactly those that can be obtained from the unknot by repeating the operations connected sum and cabling. By cabling we mean the satellite operation where the pattern is a torus knot. On the other hand, it is known that given any \( B > 0 \) there are only finitely many hyperbolic knots whose volume is less than \( B \), see [BP]. Although the volume is not a complete invariant it does quite well at distinguishing the hyperbolic ones.

1.1.2 The colored Jones polynomial

The Jones polynomial is the simplest of the Reshetikhin Turaev invariants or quantum knot invariants that are derived from quantum groups. In this elementary account we will base our definition of the Jones polynomial on the Kauffman bracket. As is usual we will work with framed knots throughout, which can be thought of as knotted ribbons instead of strings. The Kauffman bracket of a knot diagram, notation \( \langle K \rangle \), is defined by the following relations.

\[
\begin{align*}
\langle \emptyset \rangle &= 1 \\
\langle \text{ } \rangle &= A \langle \text{ } \rangle + A^{-1} \langle \text{ } \rangle \\
\langle \bigcirc \cup D \rangle &= (-A^2 - A^{-2}) \langle D \rangle
\end{align*}
\]

It is readily shown that the Kauffman bracket is a Laurent polynomial in \( A \) which is invariant under the second and third Reidemeister moves. Therefore it is an actual invariant of (framed) knots. The Jones polynomial is a modification of the Kauffman bracket which makes it an invariant of oriented unframed links.

But this distinction is irrelevant for the applications we have in mind so we will use (as is customary in the literature) the terminology Kauffman bracket and Jones polynomial interchangeably to refer to the framed version of the invariant.

The (framing dependent) colored Jones polynomial is a more elaborate version of the same construction where one now considers not only the knot but also its parallels. Given a knot \( K \) we define its (0-framed) \( n \)-th parallel \( K^n \) to be the satellite link whose pattern is \( n \) parallel unknots and whose companion knot is \( K \). Alternatively \( K^n \) is what one gets when one ties the knot \( K \) in \( n \) parallel strands. \( K^0 \) is by definition the empty link.
The (framing dependent) unnormalized $N$-colored Jones polynomial is defined to be the following linear combination of Kauffman brackets of parallels of $K$:

$$J_{N+1}(K) = \sum_{j=0}^{N/2} (-1)^j \binom{N-2j}{j} (K^{N-2j})$$

One can give a more natural definition of the $N$-colored Jones polynomial as the quantum invariant corresponding to the $N$-dimensional irreducible representation of $sl_2$ [KM]. Note that the Kauffman bracket itself corresponds to the standard representation and the above formula becomes nothing more than an expression of the $N$-dimensional representation in terms of tensor powers of the standard representation.

Finally we normalize the invariant by dividing by the value of the unknot $O$:

$$J_N(K) = \frac{J_{N+1}(K)}{J_N(O)}$$

In the formulation of the volume conjecture it is usual to change the variable from $A$ to $q$ by setting $q = A^4$. This then is the version of the colored Jones polynomial referred to above.

Although we work with framed knots, there is no mention of framing in the statement of the volume conjecture. The zero framing on the knot is intended, but the framing does not matter for this version of the conjecture. This is because framing change will cause the colored Jones polynomial to change only by a power of $q$. Since $q$ is on the unit circle the absolute value will not change.

As indicated before one can also define an unframed version of the Jones polynomial for oriented links by correcting with the writhe. Indeed this is in many cases the more standard terminology. We warn the reader however that in this thesis by colored Jones polynomial we always refer to the colored Kauffman bracket polynomial also known as the framing dependent colored Jones polynomial as this choice seems more natural in our context.

### 1.2 Main results

Despite many efforts in the last decade and a half, the volume conjecture is still open. We start with a list of knots and links for which the conjecture has been settled independently of this thesis. So far the conjecture has been proven for the figure eight knot, its $(2,q)$-cables and Borromean doubles [MI] [LT] [YY], torus knots [KT], Whitehead doubles of $(2,q)$ torus knots and twisted Whitehead links [Zh], $(2,q)$-torus links [Hi2], and finally the Borromean rings [GL2].

The central question addressed in this thesis is to gain elementary understanding of the volume conjecture by examining the simplest non-trivial cases and to make simpler toy models for the conjecture. The toy models we have studied are the classical spin networks and the quantum spin networks at a fixed root of unity. The spin networks we study are closely related to the Jones polynomial, but instead of knots they deal with graphs. Quantum spin networks are a generalization of the Jones polynomial to embedded graphs. Everything done in this thesis can be described as studying the asymptotics of quantum spin network evaluations. At a glance here are some of the main results in the thesis.
Theorem 1.1. (Chapter 1)
The volume conjecture is true for the infinite family of all Whitehead chains.

Theorem 1.2. (Chapter 2)
For any knot \( K \) one can add additional unlinked components to get a link \( L \) for which the volume conjecture is true.

The theorem holds equally well for when \( K \) is a link a knotted trivalent graph.

Theorem 1.3. (Chapters 3, 4)
For any quantum spin network \( (\Gamma, \gamma) \) and any root of unity \( \zeta \), the sequence \( \langle \langle \Gamma, N\gamma \rangle \rangle(\zeta) \) allows an asymptotic expansion of Nilsson type.

Theorem 1.4. (Chapter 5)
For all links \( L \) of volume zero, \( J_N(L)(e^{2\pi i N}) = O(N^c) \) as \( N \to \infty \) for some \( c \) depending on \( L \).

The last theorem means that for all such links the volume conjecture holds provided that we can also give a lower bound to exclude it from decaying exponentially or being zero and we can prove the limit in the left hand side of (1.1) exists.

1.3 Brief history of the conjecture

Let us now give a short overview of the history of the volume conjecture and explain how our approach fits into this picture.

The volume conjecture originated around 1995 as R. Kashaev’s conjecture on the asymptotics of his link invariant [Ka1][Ka2]. Kashaev constructed a new knot invariant in a way reminiscent of the definition of the Turaev-Viro 3-manifold invariant [TV]. Starting from a special triangulation of the knot complement he assigned a modified 6j-symbol to every tetrahedron in such a way that the result was independent of the chosen triangulation. However Kashaev’s modified 6j-symbols contain extra structure to deal with the knot in the three manifold and are closely related to the quantum dilogarithm function. The quantum dilogarithm derives its name from the fact that after taking the appropriate classical limit it reduces to the actual dilogarithm function. Kashaev now observed that the dilogarithm function can be used to express the hyperbolic volume of an ideal tetrahedron in such a way that the result was independent of the chosen triangulation. However Kashaev’s modified 6j-symbols contain extra structure to deal with the knot in the three manifold and are closely related to the quantum dilogarithm function. The quantum dilogarithm derives its name from the fact that after taking the appropriate classical limit it reduces to the actual dilogarithm function. Kashaev now observed that the dilogarithm function can be used to express the hyperbolic volume of an ideal tetrahedron in terms of its cross ratio. This was one of his motivations for expecting the classical limit of his invariant would produce the hyperbolic volume of the knot complement. This approach has inspired quantum hyperbolic field theory that has given rise to many interesting new conjectures [BB].

Five years later H. Murakami and J. Murakami identified Kashaev’s invariant as the \( N \)-colored Jones polynomial at the \( N \)-th root of unity and reformulated Kashaev’s conjecture as we have done above. Together with Y. Yokota they amplified the above motivation for the volume conjecture as follows. Given a knot diagram one can compute the colored Jones polynomial as a trace of a composition of \( R \)-matrices and twist parameters, where each \( R \)-matrix corresponds to a crossing [MM]. On the topological side we can also use the crossings to
subdivide the complement of the knot. For each crossing we get an ideal octahedron. Supposing the rate of growth of each $R$-matrix contributes the volume of the corresponding octahedron one would obtain the volume conjecture. Indeed treating the composition of $R$-matrices as an integral and applying the method of stationary phase, they derive equations for the parameters of the $R$-matrices. These equations turn out to be exactly the equations on the shape parameters of the octahedra that ensure that one gets the complete hyperbolic structure [Ch]. Although progress has been made in this direction to make this approach rigorous, there seem to be serious problems in terms of the contour of integration and the choice of diagram.

The volume conjecture can also be fruitfully approached and motivated from the point of view of physics, in particular Chern-Simons theory. Witten [Wit1] gave an interpretation of the Jones polynomial in terms of Chern-Simons theory with gauge group $SU(2)$. By complexifying the gauge group the hyperbolic volume was also interpreted in terms of Chern-Simons theory [Gu]. However, how exactly one passes to the complexification as one takes the classical limit seems as yet unclear [Wit2], [DG], [DGLZ]. Recently the volume conjecture was also interpreted in terms of topological string theory [DF].

1.4 Approach

The common thread that runs through all the chapters in the thesis is to study the colored Jones polynomial by expressing it in terms of $6j$-symbols. The associated objects in the graphical calculus are called quantum spin networks and this is where the title comes from. Although this way of calculating the Jones polynomial is far from new [KR], [MV], [KL], [Tu], the above theorems show that this approach does have some merit for studying the volume conjecture. Below we briefly sketch the main points of this approach, our motivation for taking it and how it fits with what had already been done.

Figure 1.2: The $Y$ shaped figure depicts a projector mapping the bottom representation $a$ into the tensor product of the top two $b \otimes c$. The crossing depicts the $R$-matrix composed with the flip.

The essence of the approach is depicted graphically in the above picture. As is usual in graphical calculus, the strands are colored by representations of the quantum group. When read from bottom to top, a $Y$ shaped figure (second from the left) depicts the canonical projection that maps the representation $a$ on the bottom edge into the tensor product $b \otimes c$ of the representations present at the top edges. A very important point to make is that since we will always work with quantum $sl_2$ the tensor product of two irreducible representations decomposes in a multiplicity free way. This fact simplifies matters enormously.
The map \( Y \) can also be read as a change of basis that diagonalizes the \( R \)-matrix. Indeed by Schur’s lemma it is clear that the two left most maps in the figure can only differ by a scalar. We can apply this to any knot diagram, by fusing the strands close to any crossing. After having removed the crossings in this way we are left with a labeled planar graph that represents a composition of \( Y \) maps. To evaluate such graphs (also known as spin networks) one can employ the orthogonality relations of the maps but only after making a series of base changes. These base changes connect the two natural bases of the triple tensor product, graphically depicted in the rightmost two pictures of the figure. The coefficients of these base changes are by definition the \( 6j \)-symbols. The conclusion is that every colored Jones polynomial can be expressed in terms of a sum of products of \( 6j \)-symbols and powers of \( q \).

In the thesis we choose to shift our attention from the calculation of the colored Jones polynomial for knots to calculating it for trivalent graphs embedded in the three sphere. When such graphs are colored with representations and to be evaluated by the same Reshetikhin-Turaev functor that is used to compute the ordinary Jones polynomials the graphs are called quantum spin networks. Taking spin networks as the fundamental objects has as an advantage that all stages of the computation of the colored Jones polynomial can be treated on equal footing, knot or not. This also puts the volume conjecture within a larger context of questions about the asymptotics of quantum spin networks.

We were led to the viewpoint of quantum spin networks by an attempt to apply a variation of the strategy of Murakami, Murakami and Yokota. Instead of using the crossings as a fundamental building block for both the quantum invariant and the geometry of the knot complement we turn to the standard quantum \( 6j \)-symbols. Although the quantum \( 6j \)-symbols we are referring to differ from Kashaev’s modified \( 6j \)-symbols, they are still known to be related to volumes of geometric tetrahedra, [Co3], [CM]. In trying to match the combinatorial triangulation to the \( 6j \)-symbols in the expression of the colored Jones polynomial it becomes very natural to consider graphs as more fundamental than knots.

Another advantage of viewing the volume conjecture in terms of quantum spin networks is that non-trivial toy models become available by leaving the root of unity fixed. In the case of knots and links this leads to trivial results but the study of the asymptotics of quantum spin networks at a fixed root of unity is far from settled. The simplest toy model that appears in such a way are the classical spin networks. These are quantum spin networks evaluated at \( q = 1 \).

By going back to classical spin networks we reconnect the volume conjecture mathematics that already existed before the Jones polynomial. Classical spin networks have been studied since the beginning of quantum mechanics under the name of addition of quantum angular momentum. And indeed their asymptotics have been studied intensively resulting in a conjecture by Ponzano and Regge [PR] relating the asymptotics of the tetrahedral spin network to the Euclidean geometric tetrahedron. This conjecture then can be interpreted as a toy version of the volume conjecture and studying it will tell us something about the original conjecture. Later on classical spin networks were used by Penrose [Pe1] in attempts to quantize gravity combinatorially and they still play an important role in three-dimensional quantum gravity [Ro].
1.5 Overview of the thesis

Let us now give a brief overview of the chapters to come in the light of the common theme asymptotics of spin networks described above.

In the second chapter we deal with a family of links obtained by composing three simple 2-2 tangles. By composing these tangles with the \( Y \) map as sketched above we find explicit formulas for the colored Jones polynomial and relate them to the geometry of the complement. One of these 2-2 tangles, the belt tangle, plays a significant role in simplifying matters even further, as does the arithmeticity of the links involved and these two aspects motivated the results in the next chapter.

The third chapter is largely a continuation and expansion of the ideas presented in chapter one. Here the belt tangle introduced in chapter one is exploited in full to augment arbitrary knots. This augmentation dramatically reduces the complexity of both the Jones polynomial of the knot and the geometry of its complement.

Furthermore we present an extension of the volume conjecture to links and quantum spin networks where all components have equal color. From the point of view of the calculation this is natural and the same proof that works in the knot case works extends to quantum spin networks. The main point of the proof is that changing the diagram step by step one can simultaneously observe how the colored Jones polynomial and the geometry of the complement change. Both are described by the same graphical formalism.

In chapter four we turn away from the full volume conjecture to focus on the toy model that deals with classical spin networks. Here we give a new semi explicit evaluation for classical spin networks in terms of a generating function. We then remark that the classical spin network evaluations can be expressed as sums of hypergeometric terms. Using Wilf-Zeilberger theory \([PWZ]\) we thus conclude that every spin network satisfies a recursion relation when one scales the labels. Coupled with the theory of (Siegel) \( \zeta \)-functions this allows us to prove the existence of a full asymptotic expansion of any classical spin network evaluation. We study the special case of the tetrahedral spin network in detail and settle rigorously the previously unknown cases of the Ponzano-Regge conjecture using Borel transforms.

Chapter five is largely a generalization to quantum spin networks of the main results on classical spin networks obtained in the previous chapter. However the quantum spin networks are still required to be evaluated at a fixed root of unity. In order to be able to evaluate at a root of unity at all we first prove an integrality result extending work of Costantino \([Co2]\). It should be noted that there also exists a different notion of evaluating at a root of unity where one works with the quantum group itself at a root of unity, \([GR]\). These evaluations gives rise more refined behavior and are not covered in our work. Finally we formulate a conjecture about how the leading asymptotics at any fixed root of unity is controlled by the leading asymptotics of the corresponding classical spin network.

In the final chapter we return to the original volume conjecture and examine some limitations of our approach using \( 6j \)-symbols. Here we study the class of knots and links whose (generalized) hyperbolic volume equals zero. In many ways such knots possess the simplest topology and there exists a full classification of them. Yet the method of calculating the colored Jones polynomial we
consider gives very unsatisfying results in this case. Most of the terms in the resulting sum of $6j$-symbols show exponential growth, but we know the volume is zero. So if the volume conjecture is to hold all such terms should cancel to very high order. And indeed they do.

Using different approach based on Schur-Weyl duality we prove a general cabling formula for the evaluation of all zero volume links and knots. From this formula it is immediate that the invariants are growing at most polynomially when $q$ is on the unit circle. An interesting point about the cabling formula is that it motivates considering more general asymptotics than the one we concentrate on in the volume conjecture.

It should be noted that most of the results of chapters three and four generalize naturally from spin network evaluations to holonomic and $q$-holonomic sequences, where $q$ is set to a fixed root of unity. (By a ($q$)-holonomic sequence we mean a sequence satisfying a ($q$)-difference equation). In some sense spin networks merely provide natural geometric examples for studying such sequences. This theme persists in the case of the original volume conjecture in the sense that it was shown that the sequence of colored Jones polynomials is always $q$-holonomic [GaLe]. In the AJ-conjecture this $q$-recursion was related to the $SL(2, \mathbb{C})$ character variety of the knot in [Ga0]. This is a theme that plays a central role in the volume conjecture.

1.6 Generalizations

We end this introduction with a brief account of the various ways the volume conjecture has been generalized and extended so far. We have already seen the volume conjecture can be generalized to links and graphs (spin networks) to some extent but in the case of general colors problems may arise [LT, GR].

The conjecture can also be generalized by varying the ambient 3-manifold. Specializing to the case of the empty link in a general 3-manifold this becomes of the asymptotics of quantum invariants of 3-manifolds for which there are related conjectures [Gu], [And]. One can reformulate the question in terms of links in the three–sphere by presenting the 3-manifold as surgery on an auxiliary link. Of course the surgery link has to be colored in the appropriate way, so we have to consider more general mixed asymptotics of Jones polynomials of links.

A different generalization involves changing the evaluation of $q$ from $q = \frac{2\pi i}{N}$ to $q = \frac{c}{N}$. It is conjectured that the asymptotics one now sees is related to incomplete geometric structures on the complement or more generally to other points on the character variety [GuM, MuE]. Indeed we have already mentioned that the complete hyperbolic structure corresponds to a very special point on the character variety. Also for $c$ very close to 0 this has been verified to correspond to the abelian representations. This is an extension of the proof of the Melvin–Morton–Rozanski conjecture [BG, GL2].

The last generalization we mention is to change the Lie algebra $sl_2$ and consider the asymptotics of quantum invariants related to other (simple) Lie algebras. Due to technical difficulties there are not many concrete indications of what geometric quantity the asymptotics should measure [HL, DG]. To gain intuition in this case it would be fruitful to extend the toy model of classical spin networks to this case but even this seems to be a daunting problem.

Chapters one and two were published as two papers in [V1], [V2]. Chapters
three and four are joint papers with Stavros Garoufalidis that are submitted. Chapter five appeared as a preprint [V3].