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Chapter 19

Dual Addition Formulas Associated with Dual Product Formulas

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*Dedicated to Mourad E. H. Ismail on the occasion
of his 70th birthday*

We observe that the linearization coefficients for Gegenbauer polynomials are the orthogonality weights for Racah polynomials with special parameters. Then it turns out that the linearization sum with such a Racah polynomial as extra factor inserted, can also be evaluated. The corresponding Fourier–Racah expansion is an addition-type formula which is dual to the well-known addition formula for Gegenbauer polynomials. The limit to the case of Hermite polynomials of this dual addition formula is also considered. Similar results as for Gegenbauer polynomials, although only formal, are given by taking the Ruijsenaars–Hallnäs dual product formula for Gegenbauer functions as a starting point and by working with Wilson polynomials.

Keywords: Addition formula for Gegenbauer polynomials; dual addition formula; Racah polynomials.

Mathematics Subject Classification 2010: 33C45, 33-03, 01A55

1. Introduction

A prototype for an addition formula for a family of special orthogonal polynomials is the *addition formula for Legendre polynomials* [17, (18.18.9)]

$$\begin{aligned} & P_n(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) \\ &= P_n(\cos \theta_1)P_n(\cos \theta_2) + 2 \sum_{k=1}^n \frac{(n-k)!(n+k)!}{2^{2k}(n!)^2} \\ & \quad \times (\sin \theta_1)^k P_{n-k}^{(k,k)}(\cos \theta_1) (\sin \theta_2)^k P_{n-k}^{(k,k)}(\cos \theta_2) \cos(k\phi). \end{aligned} \quad (1.1)$$

The right-hand side is the Fourier-cosine expansion of the left-hand side as a function of ϕ . Integration with respect to ϕ over $[0, \pi]$ gives the constant term in this expansion, [17, (18.17.6)]

$$P_n(\cos \theta_1)P_n(\cos \theta_2) = \frac{1}{\pi} \int_0^\pi P_n(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) d\phi, \quad (1.2)$$

which is known as the *product formula for Legendre polynomials*.

A formula dual to (1.2) is the *linearization formula for Legendre polynomials*, see [17, (18.18.22)] for $\lambda = \frac{1}{2}$ together with [17, (18.7.9)] and see [17, (5.2.4)] for the shifted factorial $(a)_n$. It reads

$$\begin{aligned} P_l(x)P_m(x) &= \sum_{j=0}^{\min(l,m)} \frac{(\frac{1}{2})_j (\frac{1}{2})_{l-j} (\frac{1}{2})_{m-j} (l+m-j)!}{j! (l-j)! (m-j)! (\frac{3}{2})_{l+m-j}} \\ & \quad \times (2(l+m-2j)+1) P_{l+m-2j}(x). \end{aligned} \quad (1.3)$$

On several occasions, during his lectures at conferences, Richard Askey raised the problem to find an addition-type formula associated with (1.3) in a similar way as the addition formula (1.1) is associated with the product formula (1.2). It is the purpose of the present paper to give such a formula, more generally associated with the linearization formula for Gegenbauer polynomials, and also a (formal) addition-type formula associated with the dual product formula for Gegenbauer functions which was recently given by Hallnäs and Ruijsenaars [9, (4.17)].

In order to get a better feeling for what a dual addition formula should look like, we first rewrite (1.1) and (1.2) by substituting $z = \sin \theta_1 \sin \theta_2 \cos \phi$, $x = \cos \theta_1$, $y = \cos \theta_2$, and putting $\cos(k\phi) = T_k(\cos \phi)$ (T_k is a Chebyshev polynomial [17, (18.5.1)]). Assume that $x, y \in [-1, 1]$

and $z \in [-\sqrt{1-x^2}\sqrt{1-y^2}, \sqrt{1-x^2}\sqrt{1-y^2}]$. We obtain

$$\begin{aligned}
 P_n(z+xy) &= P_n(x)P_n(y) + 2 \sum_{k=1}^n \frac{(n-k)!(n+k)!}{2^{2k}(n!)^2} \\
 &\quad \times (1-x^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(x) (1-y^2)^{\frac{1}{2}k} P_{n-k}^{(k,k)}(y) \\
 &\quad \times T_k \left(\frac{z}{\sqrt{1-x^2}\sqrt{1-y^2}} \right)
 \end{aligned} \tag{1.4}$$

and

$$P_n(x)P_n(y) = \frac{1}{\pi} \int_{-\sqrt{1-x^2}\sqrt{1-y^2}}^{\sqrt{1-x^2}\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}} dz. \tag{1.5}$$

The rewritten addition formula (1.4) expands the left-hand side as a function of z in terms of Chebyshev polynomials of dilated argument, where the dilation factor depends on x, y , and the product formula (1.5) recovers the constant term in this expansion by integration with respect to the weight function over the orthogonality interval for these dilated Chebyshev polynomials. The linearization formula (1.3) very much looks as a dual formula with respect to (1.5). If we can recognize the coefficients in the sum in (1.3) as weights for some finite system of orthogonal polynomials, then the dual addition formula associated with (1.3) should be the corresponding orthogonal expansion of $P_{l+m-2j}(x)$ as a function of j .

It is not so easy to recognize the coefficients in (1.3) as weights for known orthogonal polynomials, but a strong hint was provided by the Hallnäs–Ruijsenaars dual product formula [9, (4.17)] for Gegenbauer functions, which can be rewritten as (6.2). There the weight function in the integral is clearly the weight function for Wilson polynomials [11, Section 9.1] with suitable parameters. This suggests that in (1.3) we should have weights of Racah polynomials [11, Section 9.2], which are the discrete analogues of the Wilson polynomials. Indeed, this turns out to work, see (4.2) (more generally for Gegenbauer polynomials), and a nice expansion (4.6) in terms of these Racah polynomials can be derived, which is the dual addition formula for Gegenbauer polynomials predicted by Askey. In a remark at the end of Section 4, we point to a paper by Koelink *et al.* [12] from 2013 which already has the dual addition formula in disguised form.

For better comparison of the dual results in Section 4, we state the addition formula for Gegenbauer polynomials and related formulas in Section 3. There we also mention the quite unknown paper [1] by Allé from

1865 which already gives the addition formula for Gegenbauer polynomials, much earlier than Gegenbauer’s paper [7] from 1874.

In Section 5 we obtain a limit of the dual addition formula for Gegenbauer polynomials corresponding to the limit from Gegenbauer to Hermite polynomials. The resulting formulas for Hermite polynomials are well known. Remarkable is that the orthogonality for special Racah polynomials tends in the limit to a (well-known) biorthogonality for shifted factorials.

In Section 2 we introduce the needed special functions. The paper concludes in Section 7 with a list of possible follow-up work on dual addition formulas.

2. Preliminaries

2.1. Jacobi, Gegenbauer and Hermite polynomials

We will work with renormalized *Jacobi polynomials* [11, Section 9.8]

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1\left(\begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix}; \frac{1}{2}(1-x)\right), \tag{2.1}$$

where $P_n^{(\alpha,\beta)}(1) = (\alpha + 1)_n/n!$ and $R_n^{(\alpha,\beta)}(1) = 1$ (see [17, (16.2.1)] for the definition of a ${}_pF_q$ hypergeometric series). These are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ ($\alpha, \beta > -1$). In particular, for $\alpha = \beta$, the polynomials are called *Gegenbauer polynomials*, then the weight function is even and we have

$$R_n^{(\alpha,\alpha)}(-x) = (-1)^n R_n^{(\alpha,\alpha)}(x).$$

The precise orthogonality relation is

$$\int_{-1}^1 R_m^{(\alpha,\alpha)}(x) R_n^{(\alpha,\alpha)}(x) (1-x^2)^\alpha dx = h_n^{(\alpha,\alpha)} \delta_{m,n}, \tag{2.2}$$

$$h_n^{(\alpha,\alpha)} = \frac{2^{2\alpha+1}\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)} \frac{n+2\alpha+1}{2n+2\alpha+1} \frac{n!}{(2\alpha+2)_n}.$$

From (2.1) we see that

$$R_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n(\alpha+1)_n} x^n + \text{terms of lower degree}. \tag{2.3}$$

The connection of $R_n^{(\alpha,\alpha)}$ with the usual $C_n^{(\lambda)}$ notation for Gegenbauer polynomials is

$$R_n^{(\alpha,\alpha)}(x) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x) = \frac{n!}{(2\alpha+1)_n} C_n^{(\alpha+\frac{1}{2})}(x).$$

From [17, (18.5.10)] we have the power series

$$R_n^{(\alpha, \alpha)}(x) = \frac{n!}{(2\alpha + 1)_n} \sum_{k=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^k (\alpha + \frac{1}{2})_{n-k}}{k! (n - 2k)!} (2x)^{n-2k}. \tag{2.4}$$

We will need the difference formula

$$R_n^{(\alpha, \alpha)}(x) - R_{n-2}^{(\alpha, \alpha)}(x) = \frac{n + \alpha - \frac{1}{2}}{\alpha + 1} (x^2 - 1) R_{n-2}^{(\alpha+1, \alpha+1)}(x) \quad (n \geq 2). \tag{2.5}$$

Proof of (2.5). More generally, let $w(x) = w(-x)$ be an even weight function on $[-1, 1]$, let $p_n(x) = k_n x^n + \dots$ be orthogonal polynomials on $[-1, 1]$ with respect to the weight function $w(x)$, and let $q_n(x) = k'_n x^n + \dots$ be orthogonal polynomials on $[-1, 1]$ with respect to the weight function $w(x)(1 - x^2)$. Assume that p_n and q_n are normalized by $p_n(1) = 1 = q_n(1)$. Let $n \geq 2$. Then $p_n(x) - p_{n-2}(x)$ vanishes for $x = \pm 1$. Hence $(p_n(x) - p_{n-2}(x))/(1 - x^2)$ is a polynomial of degree $n - 2$. It is seen immediately that x^k ($k < n - 2$) is orthogonal to this polynomial with respect to the weight function $w(x)(1 - x^2)$ on $[-1, 1]$. We conclude that

$$p_n(x) - p_{n-2}(x) = \frac{k_n}{k'_{n-2}} (x^2 - 1) q_{n-2}(x) \quad (n \geq 2).$$

Now specialize to $w(x) = (1 - x^2)^\alpha$ and use (2.3). □

Hermite polynomials [11, Section 9.15] are orthogonal polynomials H_n on $(-\infty, \infty)$ with respect to the weight function e^{-x^2} and normalized such that $H_n(x) = 2^n x^n +$ terms of lower degree. From [17, (18.5.13)] we have the power series

$$H_n(x) = n! \sum_{k=0}^n \frac{(-1)^k}{k! (n - 2k)!} (2x)^{n-2k}. \tag{2.6}$$

It follows from (2.4) and (2.6) that

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \alpha^{\frac{1}{2}n} R_n^{(\alpha, \alpha)}(\alpha^{-\frac{1}{2}}x) &= 2^{-n} H_n(x), \\ \lim_{\alpha \rightarrow \infty} \alpha^{\mu n} R_n^{(\alpha, \alpha)}(\alpha^{-\mu}x) &= x^n \quad \left(\mu < \frac{1}{2} \right), \end{aligned} \tag{2.7}$$

in particular,

$$\lim_{\alpha \rightarrow \infty} R_n^{(\alpha, \alpha)}(x) = x^n. \tag{2.8}$$

2.2. Racah polynomials

We will consider *Racah polynomials* [11, Section 9.2]

$$R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) := {}_4F_3\left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix}; 1\right) \quad (2.9)$$

for $\gamma = -N - 1$, where $N \in \{1, 2, \dots\}$, and for $n \in \{0, 1, \dots, N\}$. These are orthogonal polynomials on the finite quadratic set $\{x(x + \gamma + \delta + 1) \mid x \in \{0, 1, \dots, N\}\}$:

$$\begin{aligned} & \sum_{x=0}^N (R_m R_n)(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) w_{\alpha, \beta, \gamma, \delta}(x) \\ &= h_{n; \alpha, \beta, \gamma, \delta} \delta_{m, n} \quad (m, n \in \{0, 1, \dots, N\}) \end{aligned} \quad (2.10)$$

with

$$w_{\alpha, \beta, \gamma, \delta}(x) = \frac{(\alpha + 1)_x (\beta + \delta + 1)_x (\gamma + 1)_x (\gamma + \delta + 1)_x}{(-\alpha + \gamma + \delta + 1)_x (-\beta + \gamma + 1)_x (\delta + 1)_x x!} \frac{\gamma + \delta + 1 + 2x}{\gamma + \delta + 1}, \quad (2.11)$$

$$\frac{h_{n; \alpha, \beta, \gamma, \delta}}{h_{0; \alpha, \beta, \gamma, \delta}} = \frac{\alpha + \beta + 1}{\alpha + \beta + 2n + 1} \frac{(\beta + 1)_n (\alpha + \beta - \gamma + 1)_n (\alpha - \delta + 1)_n n!}{(\alpha + 1)_n (\alpha + \beta + 1)_n (\beta + \delta + 1)_n (\gamma + 1)_n}, \quad (2.12)$$

$$h_{0; \alpha, \beta, \gamma, \delta} = \sum_{x=0}^N w_{\alpha, \beta, \gamma, \delta}(x) = \frac{(\alpha + \beta + 2)_N (-\delta)_N}{(\alpha - \delta + 1)_N (\beta + 1)_N}. \quad (2.13)$$

Clearly $R_n(0; \alpha, \beta, \gamma, \delta) = 1$ while, by (2.9) and the Saalschütz formula [17, (16.4.3)], we can evaluate the Racah polynomial for $x = N$:

$$R_n(N\delta; \alpha, \beta, \gamma, \delta) = \frac{(\beta + 1)_n (\alpha - \delta + 1)_n}{(\alpha + 1)_n (\beta + \delta + 1)_n}. \quad (2.14)$$

The backward shift operator equation [11, (9.2.8)] can be rewritten as

$$\begin{aligned} & w_{\alpha, \beta, \gamma, \delta}(x) R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) \\ &= \frac{\gamma + \delta + 2}{\gamma + \delta + 2 + 2x} w_{\alpha+1, \beta+1, \gamma+1, \delta}(x) \\ & \quad \times R_{n-1}(x(x + \gamma + \delta + 2); \alpha + 1, \beta + 1, \gamma + 1, \delta) \end{aligned}$$

$$\begin{aligned}
 & - \frac{\gamma + \delta + 2}{\gamma + \delta + 2x} w_{\alpha+1, \beta+1, \gamma+1, \delta}(x-1) \\
 & \times R_{n-1}((x-1)(x + \gamma + \delta + 1); \alpha + 1, \beta + 1, \gamma + 1, \delta). \quad (2.15)
 \end{aligned}$$

This holds for $x = 0, \dots, N$. For $x = 0$, (2.15) remains true if we put the second term on the right equal to 0, while for $x = N$ the first term on the right can be assumed to vanish. In this last case, the identity (2.15) can be checked by using (2.14) and (2.11).

Hence, for a function f on $\{0, 1, \dots, N\}$ we have

$$\begin{aligned}
 & \sum_{x=0}^N w_{\alpha, \beta, \gamma, \delta}(x) R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) f(x) \\
 & = \sum_{x=0}^{N-1} \frac{\gamma + \delta + 2}{\gamma + \delta + 2 + 2x} w_{\alpha+1, \beta+1, \gamma+1, \delta}(x) \\
 & \times R_{n-1}(x(x + \gamma + \delta + 2); \alpha + 1, \beta + 1, \gamma + 1, \delta)(f(x) - f(x + 1)). \quad (2.16)
 \end{aligned}$$

2.3. Jacobi and Gegenbauer functions

In [9, (4.3)] Hallnäs and Ruijsenaars define a *conical function*

$$F(g; r, 2k) := \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{\Gamma(g + ik)\Gamma(g - ik)}{\Gamma(g)(2 \sinh r)^{g-\frac{1}{2}}} P_{ik-\frac{1}{2}}^{\frac{1}{2}-g}(\cosh r) \quad (r > 0, \operatorname{Re} g > 0). \quad (2.17)$$

Here the P -function is the *associated Legendre function of the first kind* which is expressed by [17, (14.3.6) and (15.1.2)] as Gauss hypergeometric function:

$$P_{\nu}^{\mu}(x) = \frac{1}{\Gamma(1 - \mu)} \left(\frac{x + 1}{x - 1}\right)^{\frac{1}{2}\mu} {}_2F_1\left(\begin{matrix} \nu + 1, -\nu \\ 1 - \mu \end{matrix}; \frac{1}{2} - \frac{1}{2}x\right) \quad (x > 1). \quad (2.18)$$

Substitute (2.18) in (2.17) and also use Euler’s transformation formula [17, (15.8.1)] and Legendre’s duplication formula [17, (5.5.5)]. Then we obtain

$$F(g; r, 2k) = \frac{\Gamma(g + ik)\Gamma(g - ik)}{2\Gamma(2g)} {}_2F_1\left(\begin{matrix} g + ik, g - ik \\ g + \frac{1}{2} \end{matrix}; -\sinh^2 \frac{1}{2}r\right). \quad (2.19)$$

By [17, (15.9.11)] this can be written in terms of *Jacobi functions* [14, 15]

$$\begin{aligned} \phi_\lambda^{(\alpha, \beta)}(t) &:= {}_2F_1\left(\begin{matrix} \frac{1}{2}(\alpha + \beta + 1 + i\lambda), \frac{1}{2}(\alpha + \beta + 1 - i\lambda) \\ \alpha + 1 \end{matrix}; -\sinh^2 t\right) \quad (t \in \mathbb{R}) \end{aligned} \tag{2.20}$$

(called *Gegenbauer functions* if $\alpha = \beta$) as

$$F(g; r, 2k) = \frac{\Gamma(g + ik)\Gamma(g - ik)}{2\Gamma(2g)} \phi_{2k}^{(g-\frac{1}{2}, g-\frac{1}{2})}(\frac{1}{2}r).$$

By [14, (2.8)]

$$\phi_{2\lambda}^{(\alpha, \alpha)}(t) = \phi_\lambda^{(\alpha, -\frac{1}{2})}(2t), \tag{2.21}$$

this becomes

$$F(g; r, 2k) = \frac{\Gamma(g + ik)\Gamma(g - ik)}{2\Gamma(2g)} \phi_k^{(g-\frac{1}{2}, -\frac{1}{2})}(r) \tag{2.22}$$

or equivalently,

$$F(\alpha + \frac{1}{2}; t, 2\lambda) = \frac{\Gamma(\alpha + \frac{1}{2} + i\lambda)\Gamma(\alpha + \frac{1}{2} - i\lambda)}{2\Gamma(2\alpha + 1)} \phi_\lambda^{(\alpha, -\frac{1}{2})}(t). \tag{2.23}$$

Note that, by (2.20), $\phi_\lambda^{(\alpha, \beta)}(0) = 1$. From [15, (6.1)] we have

$$|\phi_\lambda^{(\alpha, \beta)}(t)| \leq 1 \quad (\alpha \geq \beta \geq -\frac{1}{2}, t \in \mathbb{R}, |\text{Im } \lambda| \leq \alpha + \beta + 1). \tag{2.24}$$

The contiguous relation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) - {}_2F_1\left(\begin{matrix} a - 1, b + 1 \\ c \end{matrix}; z\right) = \frac{(b - a + 1)z}{c} {}_2F_1\left(\begin{matrix} a, b + 1 \\ c + 1 \end{matrix}; z\right),$$

which follows immediately by substitution of the power series for the three ${}_2F_1$ functions, can be rewritten by (2.20) in terms of Jacobi functions:

$$\frac{\phi_{\lambda-i}^{(\alpha, \beta)}(t) - \phi_{\lambda+i}^{(\alpha, \beta)}(t)}{i\lambda} = \frac{\sinh^2 t}{\alpha + 1} \phi_\lambda^{(\alpha+1, \beta)}(t). \tag{2.25}$$

2.4. Wilson polynomials

Wilson polynomials [11, Section 9.1] are defined by

$$\frac{W_n(x^2; a, b, c, d)}{(a + b)_n (a + c)_n (a + d)_n} := {}_4F_3 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix}; 1 \right). \tag{2.26}$$

We need these polynomials with parameters $\pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}$ ($\alpha > -\frac{1}{2}$, $\lambda, \mu \in \mathbb{R}$). Then the orthogonality relation becomes [11, (9.1.2)]

$$\begin{aligned} & \frac{1}{4\pi} \int_{-\infty}^{\infty} (W_m W_n)(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}) \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 d\nu \\ &= \frac{\Gamma(\alpha + \frac{1}{2})^2 |\Gamma(n + \alpha + \frac{1}{2} + 2i\lambda)|^2 |\Gamma(n + \alpha + \frac{1}{2} + 2i\mu)|^2}{\Gamma(2n + 2\alpha + 1)} \\ & \quad \times (n + 2\alpha)_n n! \delta_{m,n}. \end{aligned} \tag{2.27}$$

Here and later $\pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}$ in the parameter list means four elements in the list with the four possibilities given by the two \pm signs. Similarly $\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})$ stands for a product of four Gamma functions. We also wrote $(4\pi)^{-1} \int_{-\infty}^{\infty}$ instead of the usual $(2\pi)^{-1} \int_0^{\infty}$, which is allowed because the integrand is an even function of ν .

The backward shift operator equation [11, (9.1.9)] can be rewritten as

$$\begin{aligned} & \frac{\Gamma(a \pm ix) \cdots \Gamma(d \pm ix)}{\Gamma(\pm 2ix)} W_n(x^2; a, b, c, d) \\ &= \frac{\Gamma(a + \frac{1}{2} \pm i(x + \frac{1}{2}i)) \cdots \Gamma(d + \frac{1}{2} \pm i(x + \frac{1}{2}i))}{2i(x + \frac{1}{2}i) \Gamma(\pm 2i(x + \frac{1}{2}i))} \\ & \quad \times W_n((x + \frac{1}{2}i)^2; a + \frac{1}{2}, b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2}) \\ & \quad - \frac{\Gamma(a + \frac{1}{2} \pm i(x - \frac{1}{2}i)) \cdots \Gamma(d + \frac{1}{2} \pm i(x - \frac{1}{2}i))}{2i(x - \frac{1}{2}i) \Gamma(\pm 2i(x - \frac{1}{2}i))} \\ & \quad \times W_n((x - \frac{1}{2}i)^2; a + \frac{1}{2}, b + \frac{1}{2}, c + \frac{1}{2}, d + \frac{1}{2}). \end{aligned} \tag{2.28}$$

3. The Addition Formula for Gegenbauer Polynomials

In this section we briefly review the addition formula for Gegenbauer polynomials and formulas associated with it. This extends the discussion

of the Legendre case in Section 1. As a reference, see for instance [2, Section 9.8]. We use the notation (2.1).

Product formula ($\alpha > -\frac{1}{2}$)

$$R_n^{(\alpha,\alpha)}(x)R_n^{(\alpha,\alpha)}(y) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})\Gamma(\frac{1}{2})} \times \int_{-1}^1 R_n^{(\alpha,\alpha)}(xy + (1 - x^2)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}}t) (1 - t^2)^{\alpha - \frac{1}{2}} dt. \tag{3.1}$$

Addition formula

$$R_n^{(\alpha,\alpha)}(xy + (1 - x^2)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}}t) = \sum_{k=0}^n \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2}k} \frac{(n + 2\alpha + 1)_k(2\alpha + 1)_k}{2^{2k}(\alpha + 1)_k^2} \times (1 - x^2)^{\frac{1}{2}k} R_{n-k}^{(\alpha+k,\alpha+k)}(x) (1 - y^2)^{\frac{1}{2}k} R_{n-k}^{(\alpha+k,\alpha+k)}(y) R_k^{(\alpha - \frac{1}{2}, \alpha - \frac{1}{2})}(t). \tag{3.2}$$

Addition formula for $t = 1$

$$R_n^{(\alpha,\alpha)}(xy + (1 - x^2)^{\frac{1}{2}}(1 - y^2)^{\frac{1}{2}}) = \sum_{k=0}^n \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2}k} \frac{(n + 2\alpha + 1)_k(2\alpha + 1)_k}{2^{2k}(\alpha + 1)_k^2} \times (1 - x^2)^{\frac{1}{2}k} R_{n-k}^{(\alpha+k,\alpha+k)}(x) (1 - y^2)^{\frac{1}{2}k} R_{n-k}^{(\alpha+k,\alpha+k)}(y). \tag{3.3}$$

For $x = \cos \theta_1$, $y = \cos \theta_2$ the left-hand side takes the form $R_n^{(\alpha,\alpha)}(\cos(\theta_1 - \theta_2))$.

Addition formula for $t = 1$, $x = y$

$$1 = \sum_{k=0}^n \binom{n}{k} \frac{\alpha + k}{\alpha + \frac{1}{2}k} \frac{(n + 2\alpha + 1)_k(2\alpha + 1)_k}{2^{2k}(\alpha + 1)_k^2} (1 - x^2)^k (R_{n-k}^{(\alpha+k,\alpha+k)}(x))^2. \tag{3.4}$$

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This shows in particular that $|R_n^{(\alpha,\alpha)}(x)| \leq 1$ if $x \in [-1, 1]$ and $\alpha > -\frac{1}{2}$, [17, (18.14.1)]. This is also well known by several other methods, including as a corollary of (3.1).

Limit to Hermite polynomials

In the addition formula (3.2) replace x by $\alpha^{-\frac{1}{2}}x$, t by $\alpha^{-\frac{1}{2}}t$, multiply both sides of (3.2) by $\alpha^{\frac{1}{2}n}$ and let $\alpha \rightarrow \infty$. By (2.7) and (2.8) we obtain

$$H_n(xy + (1 - y^2)^{\frac{1}{2}}t) = \sum_{k=0}^n \binom{n}{k} H_{n-k}(x) H_k(t) (1 - y^2)^{\frac{1}{2}k} y^{n-k}. \quad (3.5)$$

This is the case $n = 2$ of [5, 10.13(40)]. The corresponding limit of the product formula (3.1) is

$$H_n(x) y^n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(xy + (1 - y^2)^{\frac{1}{2}}t) e^{-t^2} dt. \quad (3.6)$$

History of the addition formula for Gegenbauer polynomials

The addition formula (3.2) for Gegenbauer polynomials is usually ascribed to Gegenbauer [7] in (1874). However, it is already stated and proved by Allé [1] in 1865. The subsequent proofs by Gegenbauer [7, 8] in 1874 and 1893, and by Heine [10, p. 455] in (1878) do not mention Allé’s result.

4. The Dual Addition Formula for Gegenbauer Polynomials

The linearization formula for Gegenbauer polynomials, see [3, (5.7)], can be written as

$$\begin{aligned} R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x) &= \frac{l! m!}{(2\alpha + 1)_l(2\alpha + 1)_m} \sum_{j=0}^{\min(l,m)} \frac{l + m + \alpha + \frac{1}{2} - 2j}{\alpha + \frac{1}{2}} \\ &\times \frac{(\alpha + \frac{1}{2})_j(\alpha + \frac{1}{2})_{l-j}(\alpha + \frac{1}{2})_{m-j}(2\alpha + 1)_{l+m-j}}{j!(l - j)!(m - j)!(\alpha + \frac{3}{2})_{l+m-j}} \\ &\times R_{l+m-2j}^{(\alpha,\alpha)}(x). \end{aligned} \quad (4.1)$$

As mentioned in [4, (4.18)], Rogers already gave the analogous linearization formula for q -ultraspherical polynomials in 1895 and observed (4.1) as a special case. Then (4.1) was independently given by Dougall in 1919 without proof. See [3, p. 40] for a discussion of further treatments of (4.1). See also [2, Theorem 6.8.2] and the proof and discussion following the theorem.

From now on assume $\alpha > -\frac{1}{2}$. Then the linearization coefficients in (4.1) are nonnegative (as they are in the degenerate case $\alpha = -\frac{1}{2}$). We also assume, without loss of generality, that $l \geq m$.

It is rather hidden in (4.1) that the linearization coefficients are special cases of orthogonality weights (2.11) for Racah polynomials. But indeed, a further rewriting of (4.1) and substitution of (2.11) and (2.13) gives

$$R_l^{(\alpha, \alpha)}(x)R_m^{(\alpha, \alpha)}(x) = \sum_{j=0}^m \frac{w_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}}(j)}{h_{0; \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}}}$$

$$\times R_{l+m-2j}^{(\alpha, \alpha)}(x) \quad (l \geq m). \tag{4.2}$$

This identity can be considered as giving the constant term of an expansion of $R_{l+m-2j}^{(\alpha, \alpha)}(x)$ as a function of j in terms of Racah polynomials

$$R_n(j(j-l-m-\alpha-\frac{1}{2}); \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2})$$

$$= {}_4F_3\left(\begin{matrix} -n, n+2\alpha, -j, j-l-m-\alpha-\frac{1}{2} \\ \alpha+\frac{1}{2}, -l, -m \end{matrix}; 1\right). \tag{4.3}$$

The general terms of this expansion will be obtained by evaluating the sum

$$S_{n,l,m}^{(\alpha)}(x) := \sum_{j=0}^m w_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}}(j) R_{l+m-2j}^{(\alpha, \alpha)}(x)$$

$$\times R_n(j(j-l-m-\alpha-\frac{1}{2});$$

$$\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}), \tag{4.4}$$

where we still assume $l \geq m$ and where $n \in \{0, \dots, m\}$.

Theorem 4.1. *The sum (4.4) can be evaluated as*

$$S_{n,l,m}^{(\alpha)}(x) = \frac{(2\alpha+1)_{l+n}(2\alpha+1)_{m+n}(\alpha+\frac{1}{2})_{l+m}}{2^{2n}(\alpha+\frac{1}{2})_l(\alpha+\frac{1}{2})_m(2\alpha+1)_{l+m}(\alpha+1)_n^2}$$

$$\times (x^2-1)^n R_{l-n}^{(\alpha+n, \alpha+n)}(x) R_{m-n}^{(\alpha+n, \alpha+n)}(x). \tag{4.5}$$

Proof. By (4.4), (2.16) and (2.5) we obtain the recurrence

$$S_{n,l,m}^{(\alpha)} = \frac{-l-m-\alpha+\frac{1}{2}}{\alpha+1} (1-x^2) S_{n-1, l-1, m-1}^{(\alpha+1)}.$$

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Iteration gives

$$S_{n,l,m}^{(\alpha)} = \frac{(-l - m - \alpha + \frac{1}{2})_n}{(\alpha + 1)_n} (1 - x^2)^n S_{0,l-n,m-n}^{(\alpha+n)}.$$

Now use (4.4), (4.1) and (2.13). □

As an immediate corollary, by the orthogonality relation (2.10) for Racah polynomials and by substitution of (2.12) and (2.13), we obtain the following theorem.

Theorem 4.2 (Dual addition formula). *For $l \geq m$ and for $j \in \{0, \dots, m\}$, there is the expansion*

$$\begin{aligned} R_{l+m-2j}^{(\alpha,\alpha)}(x) &= \sum_{n=0}^m \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(-l)_n (-m)_n (2\alpha + 1)_n}{2^{2n} (\alpha + 1)_n^2 n!} \\ &\quad \times (x^2 - 1)^n R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x) \\ &\quad \times R_n(j(j - l - m - \alpha - \frac{1}{2}); \\ &\quad \alpha - \frac{1}{2}, \alpha - \frac{1}{2}, -m - 1, -l - \alpha - \frac{1}{2}). \end{aligned} \tag{4.6}$$

In particular, for $j = 0$,

$$\begin{aligned} R_{l+m}^{(\alpha,\alpha)}(x) &= \sum_{n=0}^m \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(-l)_n (-m)_n (2\alpha + 1)_n}{2^{2n} (\alpha + 1)_n^2 n!} \\ &\quad \times (x^2 - 1)^n R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x), \end{aligned} \tag{4.7}$$

and for $j = m$ we obtain by (2.14) that

$$\begin{aligned} R_{l-m}^{(\alpha,\alpha)}(x) &= \sum_{n=0}^m \binom{m}{n} \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(l + 2\alpha + 1)_n (2\alpha + 1)_n}{2^{2n} (\alpha + 1)_n^2} \\ &\quad \times (1 - x^2)^n R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x), \end{aligned} \tag{4.8}$$

which is dual to (3.3) and which has a further specialization to

$$\begin{aligned} 1 &= \sum_{n=0}^m \binom{m}{n} \frac{\alpha + n}{\alpha + \frac{1}{2}n} \frac{(m + 2\alpha + 1)_n (2\alpha + 1)_n}{2^{2n} (\alpha + 1)_n^2} \\ &\quad \times (1 - x^2)^n (R_{m-n}^{(\alpha+n,\alpha+n)}(x))^2. \end{aligned} \tag{4.9}$$

Note that (4.9) coincides with formula (3.4), which is a specialization of the addition formula (3.2) for Gegenbauer polynomials.

Remark 4.3. It follows from (4.4) and (2.3) that

$$\begin{aligned} & \int_{-1}^1 S_{n,l,m}^{(\alpha)}(x) R_{l+m-2j}^{(\alpha,\alpha)}(x) (1-x^2)^\alpha dx \\ &= w_{\alpha-\frac{1}{2},\alpha-\frac{1}{2},-m-1,-l-\alpha-\frac{1}{2}}(j) h_{l+m-2j}^{(\alpha,\alpha)} \\ & \quad \times R_n(j(j-l-m-\alpha-\frac{1}{2}); \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}). \end{aligned} \tag{4.10}$$

By (4.5) and (4.3) we can rewrite this as

$$\begin{aligned} & \int_{-1}^1 R_{l-n}^{(\alpha+n,\alpha+n)}(x) R_{m-n}^{(\alpha+n,\alpha+n)}(x) R_{l+m-2j}^{(\alpha,\alpha)}(x) (1-x^2)^{\alpha+n} dx \\ &= \text{const. } {}_4F_3 \left(\begin{matrix} -n, n+2\alpha, -j, j-l-m-\alpha-\frac{1}{2} \\ \alpha+\frac{1}{2}, -l, -m \end{matrix}; 1 \right) \\ &= \text{const. } {}_4F_3 \left(\begin{matrix} -m+n, -m-n-2\alpha, j-m, l-j+\alpha+\frac{1}{2} \\ -m, -m-\alpha+\frac{1}{2}, l-m+1 \end{matrix}; 1 \right), \end{aligned}$$

where the second equality follows by twofold application of Whipple’s identity [2, Theorem 3.3.3] and where the constants can be given as explicit, elementary, but somewhat tedious expressions. It turns out that the second ${}_4F_3$ evaluation of the integral above precisely matches the formula given by Koelink *et al.* [12, (2.6)]. Just put there (without loss of generality) $k = 0$ and replace α, β, n, m, t by $\alpha+n-\frac{1}{2}, \alpha+\frac{1}{2}, l-n, m-n, j-n$, respectively. So, in a sense, the dual addition formula for Gegenbauer polynomials was already derived there in disguised form.

5. A Limit to Hermite Polynomials

We will do a rescaling in the dual addition formula (4.6) such that we can take the limit for $\alpha \rightarrow \infty$. For this purpose observe that the Racah

polynomial (4.3) (where $l \geq m \geq \max(j, n)$) has limits

$$\begin{aligned} &\lim_{\alpha \rightarrow \infty} \alpha^{-j} R_n(j(j-l-m-\alpha-\frac{1}{2}); \\ &\quad \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}) = \frac{2^j(-n)_j}{(-l)_j(-m)_j}, \\ &\lim_{\alpha \rightarrow \infty} \alpha^{-n} R_n(j(j-l-m-\alpha-\frac{1}{2}); \\ &\quad \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}) = \frac{2^n(-j)_n}{(-l)_n(-m)_n}. \end{aligned} \tag{5.1}$$

Otherwise said, $R_n(j(j-l-m-\alpha-\frac{1}{2}); \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}) = O(\alpha^{\min(n,j)})$ as $\alpha \rightarrow \infty$ with the order constant given in (5.1).

Now, in (4.6), replace x by $\alpha^{-\frac{1}{2}}x$, multiply both sides by $\alpha^{\frac{1}{2}(l+m-2j)}$ and let $\alpha \rightarrow \infty$. By (2.7) and (5.1) we obtain

$$\begin{aligned} &2^j(-l)_j(-m)_j H_{l+m-2j}(x) \\ &= \sum_{n=j}^m \frac{(-n)_j}{n!} (-2)^n (-l)_n (-m)_n H_{l-n}(x) H_{m-n}(x) \quad (l \geq m), \end{aligned} \tag{5.2}$$

which may be called the dual addition formula for Hermite polynomials. Formula (5.2) for arbitrary j is equivalent to its case $j = 0$,

$$H_{l+m}(x) = \sum_{n=0}^m \frac{(-2)^n (-l)_n (-m)_n}{n!} H_{l-n}(x) H_{m-n}(x) \quad (l \geq m), \tag{5.3}$$

and this is precisely [5, 10.13(36)].

Next we want to consider the limit as $\alpha \rightarrow \infty$ of (4.5) with $S_{n,l,m}^{(\alpha)}$ given by (4.4). Recall that (4.5) together with (4.4) is the dual of (4.6) in the sense of Fourier–Racah inversion. Observe from (2.11)–(2.13) that

$$\lim_{\alpha \rightarrow \infty} \alpha^j w_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}}(j) = \frac{(-l)_j(-m)_j}{2^j j!}, \tag{5.4}$$

$$\lim_{\alpha \rightarrow \infty} \alpha^{-n} h_{n; \alpha-\frac{1}{2}, \alpha-\frac{1}{2}, -m-1, -l-\alpha-\frac{1}{2}} = \frac{2^n n!}{(-l)_n(-m)_n}. \tag{5.5}$$

In (4.5), replace x by $\alpha^{-\frac{1}{2}}x$, multiply both sides by $\alpha^{\frac{1}{2}(l+m-2n)}$ and let $\alpha \rightarrow \infty$. By (2.7), (5.1) and (5.4) we obtain

$$\sum_{j=n}^m \frac{(-j)_n}{j!} 2^j (-l)_j (-m)_j H_{l+m-2j}(x) = (-2)^n (-l)_n (-m)_n H_{l-n}(x) H_{m-n}(x) \quad (l \geq m). \tag{5.6}$$

Again, as with (5.2), formula (5.6) for arbitrary n is equivalent to its case $n = 0$,

$$\sum_{j=0}^m \frac{2^j (-l)_j (-m)_j}{j!} H_{l+m-2j}(x) = H_l(x) H_m(x) \quad (l \geq m), \tag{5.7}$$

and this is precisely the linearization formula [3, p. 42] for Hermite polynomials.

Just as with (4.6) and (4.5), the identities (5.2) and (5.6) can be obtained from each other by a Fourier type inversion. This no longer involves an orthogonal system as the Racah polynomials but a biorthogonal system implied by the *biorthogonality relation* (see [20, Section 2.1])

$$\sum_{j=0}^{\infty} \frac{(-n)_j}{n!} \frac{(-j)_k}{k!} = \delta_{n,k}. \tag{5.8}$$

Note that the above sum in fact runs from $j = k$ to n . The biorthogonality (5.8) is also a limit case of the Racah orthogonality relation (2.10). Indeed, replace $\alpha, \beta, \gamma, \delta$ by $\alpha - \frac{1}{2}, \alpha - \frac{1}{2}, -m - 1, -l - \alpha - \frac{1}{2}$, multiply both sides of (2.10) by α^{-n} , let $\alpha \rightarrow \infty$, and use (5.1), (5.4) and (5.5). It is quite remarkable that a biorthogonal (and essentially nonorthogonal) system can be obtained as a limit case of an orthogonal system. Of course, before the limit it taken, the orthogonal system already has to be prepared as a biorthogonal system by rescaling.

6. The Dual Addition Formula for Gegenbauer Functions

The dual product formula for the functions (2.17) is given in [9, (4.17)] as

$$F(g; r, 2p)F(g; r, 2q) = \frac{1}{8\pi} \int_0^\infty F(g; r, 2k) \times \frac{\prod_{\delta_1, \delta_2, \delta_3 = +, -} \Gamma(\frac{1}{2}(g + i\delta_1 p + i\delta_2 q + i\delta_3 k))}{\Gamma(g)^2 \prod_{\delta = +, -} \Gamma(i\delta k) \Gamma(g + i\delta k)} dk, \tag{6.1}$$

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where $g \in (0, \infty)$ and $r, p, q \in \mathbb{R}$. The formula is obtained there as a limit case of a similar formula for a q -analogue (or relativistic analogue) of the Gegenbauer function. By (2.22) we can rewrite (6.1) as

$$\frac{\Gamma(\alpha + \frac{1}{2})^2 |\Gamma(\alpha + \frac{1}{2} + 2i\lambda)|^2 |\Gamma(\alpha + \frac{1}{2} + 2i\mu)|^2}{\Gamma(2\alpha + 1)} \phi_{2\lambda}^{(\alpha, -\frac{1}{2})}(t) \phi_{2\mu}^{(\alpha, -\frac{1}{2})}(t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{2\nu}^{(\alpha, -\frac{1}{2})}(t) \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 d\nu, \tag{6.2}$$

where $\alpha > -\frac{1}{2}$ and $t, \lambda, \mu \in \mathbb{R}$. Note that the integral in (6.2) converges absolutely by (2.24) and by estimates for the Gamma quotient using [17, (5.5.5) and (5.11.12)] and, from [17, (5.5.3)],

$$|\Gamma(\frac{1}{2} + i\nu)|^2 = \frac{\pi}{\cosh(\pi\nu)}.$$

The cases $\alpha = 0$ and $\frac{1}{2}$ of (6.2) were earlier given by Mizony [16].

We recognize the weight function in the integrand of (6.2) as the weight function in the orthogonality relation (2.27) for Wilson polynomials with parameters $\pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}$. The case $t = 0$ of (6.2) coincides with the case $m = n = 0$ of (2.27).

Similarly as the sum (4.4) is suggested by formula (4.2), we are led by formula (6.2) to try to evaluate the integral

$$I_n^\alpha(\lambda, \mu) := \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{2\nu}^{(\alpha, -\frac{1}{2})}(t) W_n(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}) \times \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 d\nu. \tag{6.3}$$

Here and further in this section we will only work formally. We will not bother about convergence, moving of integration contours and justification of Fourier–Wilson inversion. Certainly this should be repaired later.

By (2.28) and by shifting integration contours we get

$$I_n^\alpha(\lambda, \mu) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\phi_{2\nu-i}^{(\alpha, -\frac{1}{2})}(t) - \phi_{2\nu+i}^{(\alpha, -\frac{1}{2})}(t)}{2i\nu} \times W_{n-1}(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}(\alpha + 1) + \frac{1}{4})$$

$$\times \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}(\alpha + 1) + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 d\nu = \frac{\sinh^2 t}{\alpha + 1} I_{n-1}^{\alpha+1}(\lambda, \mu),$$

where the last equality follows by (2.25). Iteration gives

$$I_n^\alpha(\lambda, \mu) = \frac{(\sinh t)^{2n}}{(\alpha + 1)_n} I_0^{\alpha+n}(\lambda, \mu).$$

Hence, by (6.3) and (6.2),

$$\begin{aligned} & \frac{1}{4\pi} \int_{-\infty}^{\infty} \phi_{2\nu}^{(\alpha, -\frac{1}{2})}(t) W_n(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}) \\ & \times \left| \frac{\Gamma(i\nu \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4})}{\Gamma(2i\nu)} \right|^2 d\nu \\ & = \frac{\Gamma(\alpha + \frac{1}{2})^2 |\Gamma(n + \alpha + \frac{1}{2} + 2i\lambda)|^2 |\Gamma(n + \alpha + \frac{1}{2} + 2i\mu)|^2}{\Gamma(2n + 2\alpha + 1)} \\ & \times \frac{(\sinh t)^{2n}}{(\alpha + 1)_n} \phi_{2\lambda}^{(\alpha+n, -\frac{1}{2})}(t) \phi_{2\mu}^{(\alpha+n, -\frac{1}{2})}(t). \end{aligned} \tag{6.4}$$

By (2.27) there corresponds, at least formally, to (6.4) the orthogonal expansion

$$\begin{aligned} \phi_{2\nu}^{(\alpha, -\frac{1}{2})}(t) & = \sum_{k=0}^{\infty} \frac{(\sinh t)^{2k}}{(\alpha + 1)_k (k + 2\alpha)_k k!} \phi_{2\lambda}^{(\alpha+k, -\frac{1}{2})}(t) \phi_{2\mu}^{(\alpha+k, -\frac{1}{2})}(t) \\ & \times W_k(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}). \end{aligned}$$

Equivalently, by (2.21), we can write what we call the *dual addition formula for Gegenbauer functions*:

$$\begin{aligned} \phi_{4\nu}^{(\alpha, \alpha)}(t) & = \sum_{k=0}^{\infty} \frac{(\sinh 2t)^{2k}}{(\alpha + 1)_k (k + 2\alpha)_k k!} \phi_{4\lambda}^{(\alpha+k, \alpha+k)}(t) \phi_{4\mu}^{(\alpha+k, \alpha+k)}(t) \\ & \times W_k(\nu^2; \pm i\lambda \pm i\mu + \frac{1}{2}\alpha + \frac{1}{4}). \end{aligned} \tag{6.5}$$

7. Further Perspective

The results of this paper suggest much further work. The author will only discuss here the polynomial case. A very obvious thing to do is to imitate the approach of Section 4 for q -ultraspherical polynomials, starting with their linearization formula [2, (10.11.10)], which goes back to Rogers (1895). Quite probably, the q -Racah polynomials will pop up there. The results

should be compared with the Rahman–Verma product and addition formula [19] for q -ultraspherical polynomials. It would be very interesting to find a dual addition formula for the addition formula for Jacobi polynomials [13], starting with the linearization formula [18].

It would be quite challenging to search for an addition formula on a higher level (for $(q-)$ Racah polynomials) which has both the addition formula and the dual addition formula for Gegenbauer polynomials as a limit case. The recent formula for the linearization coefficients of Askey–Wilson polynomials by Foupouagnigni *et al.* [6, Theorem 21], which involves four summations, does not give much hope for a quick answer to this problem.

Very important will also be to give a group theoretic interpretation for the dual addition formula for Gegenbauer polynomials, for instance when $\alpha = 0$. Possibly this can be done in the context of tensor algebras associated with the group $SU(2)$.

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