Combining strategies efficiently: high-quality decisions from conflicting advice
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Chapter 4

Freezing & Sleeping
Abstract  A problem posed by Freund is how to efficiently track a small pool of experts out of a much larger set. This problem was solved when Bousquet and Warmuth introduced their mixing past posteriors (MPP) algorithm in 2001.

In Freund’s problem the experts would normally be considered black boxes. However, in this chapter we re-examine Freund’s problem in case the experts have internal structure that enables them to learn. In this case the problem has two possible interpretations: should the experts learn from all data or only from the subsequence on which they are being tracked? The MPP algorithm solves the first case. Our contribution is to generalise MPP to address the second option. The results we obtain apply to any expert structure that can be formalised using (expert) hidden Markov models. Curiously enough, for our interpretation there are two natural reference schemes: freezing and sleeping. For each scheme, we provide an efficient prediction strategy and prove the relevant loss bound.
4.1 Introduction

Freund’s problem arises in the context of prediction with expert advice [25]. In this setting a sequence of outcomes needs to be predicted, one outcome at a time. Thus, prediction proceeds in rounds: in each round we first consult a set of experts, who give us their predictions. Then we make our own prediction and incur some loss based on the discrepancy between this prediction and the actual outcome. The goal is to minimise the difference between our cumulative loss and some reference scheme. For this reference there are several options; we may, for example, compare ourselves to the cumulative loss of the best expert in hindsight. A more ambitious reference scheme was proposed by Yoav Freund in 2000.

**Freund’s Problem**  Freund asked for an efficient prediction strategy that suffers small additional loss compared to the following reference scheme:

(a) Partition the data into several subsequences.
(b) Select an expert for each subsequence.
(c) Sum the loss of the selected experts on their subsequences.

In 2001, Freund’s problem was addressed by Bousquet and Warmuth, who developed the efficient algorithm called mixing past posteriors (MPP) [19]. MPP’s loss is bounded by the loss of Freund’s scheme plus some overhead that depends on the number of bits required to encode the partition of the data, and it has found successful application in [70]. Problem solved. Or is it?

4.1.1 Three Reference Schemes

In this paper we take another look at Freund’s reference scheme for learning experts and ask: if an expert is selected for some segment, then should the expert learn from all data or only from the data in that segment?

We may assume that the experts do not know the segmentation chosen in step a of the reference scheme. (Otherwise, why wouldn’t we just ask them?) Hence if we treat the experts as black boxes and only ask for their prediction at each time step as in [19], it is natural that they
learn from all data. We call this interpretation of Freund’s problem the full reference scheme.

However, as the following example will illustrate, it may be beneficial if experts learn only from the segment for which they are selected, because they may get confused by data in other segments that follow a different pattern. As a slight complication, it will turn out that we have a further choice: whether to tell a learning expert the timing of its segment or not, which generally makes a difference. When segment timing is preserved, we obtain the sleeping reference scheme; when segment timing is not preserved we obtain the freezing reference scheme. The next intuitive example demonstrates that the full, freezing and sleeping reference schemes are fundamentally different, and that the latter two can be dramatically more appropriate for prediction with learning experts.

4.1.1.1 Motivating Example: Drifting Mean

In applications one would usually build up complicated prediction strategies from simpler ones in a hierarchical fashion. Following that fashion, we first define simple constant experts, parametrised by $\mu \in \mathbb{R}$, which predict according to a normal distribution with mean $\mu$ and unit variance in each round.

Learning Experts Now define a learning expert $\text{DM}[\theta]$, as displayed in Figure 4.1, that has a stochastic model for the (unobservable) drift of $\mu$ over time. This drifting mean learning expert predicts according to a hidden Markov model in which the hidden state at time $t$ is $\mu_t$ and the production probability of an outcome given $\mu_t$ is determined by the simple expert with parameter $\mu_t$. Initially, $\mu_1 = 0$ with probability one. Then $\mu_{t+1} = \mu_t + 1$ with probability $\theta$ and $\mu_{t+1} = \mu_t$ with probability $1 - \theta$ for some fixed parameter $\theta$.
4.1. Introduction

The expert DM[$\theta$] may be said to be learning, because its posterior distribution of $\mu_t$ given outcomes $x_1, \ldots, x_{t-1}$ indicates how much credibility the expert assigns to each value of $\mu_t$: high weight on, say, $\mu_t = 3$ indicates that DM[$\theta$] considers it likely for $\mu_t = 3$ to give the best prediction for $x_t$.

Data Consider the two artificial data sets displayed in Figures 4.2a and 4.2b. These data sets were obtained as follows. First, we generated two straight-line data sets, with outcomes increasing at a rate of 0.1 and 0.3 per trial respectively. Then we divided both data sets in segments of 100 outcomes each. The data in Figure 4.2a were obtained by interleaving 10 segments from the 0.1 and 0.3 data sets, whereas the data in Figure 4.2b were obtained by alternating 10 segments from the 0.1 and 0.3 data sets. By construction, the freezing reference scheme is suited for the data in Figure 4.2a, while the sleeping reference scheme is appropriate for the data in Figure 4.2b.

Prediction Task We now evaluate the performance of the three reference schemes on the two data sets. In each case we consider two experts: DM[0.1] and DM[0.3], and split the data into two subsequences (step (a), according to the true rate, either 0.1 or 0.3. We predict all outcomes for which the actual rate was $\theta \in \{0.1, 0.3\}$ using the expert DM[$\theta$].

The difference between the three schemes lies in which data is used by both experts to learn from. In the full reference scheme DM[0.1] and DM[0.3] are shown all the data, even those samples they do not predict. In the two other reference schemes, on the other hand, DM[0.1] only sees the data for which it is selected, that is, the data with true rate 0.1. Similarly, DM[0.3] only sees the data with true rate 0.3. For freezing DM[$\theta$] predicts as if the data it has observed are the only data, thus the original timing of the samples is lost. For sleeping the original timing of the samples is preserved, and DM[$\theta$] has to predict with uncertainty about the intermediate unobserved samples.

Posterior Figures 4.2c and 4.2d show the posterior distribution of the expert DM[0.1] on states after 200 trials for each reference scheme. These posterior distributions can be interpreted as the belief of the learning expert DM[0.1] about the unobserved drifting mean after 200 trials.
Figure 4.2 The difference between the full, freezing and sleeping reference schemes. Note the logarithmic scale of the y-axis in (e) and (f)!

(a) Suitable freezing data  
(b) Suitable sleeping data

(c) Belief of DM[0.1] after 200 trials of (a)  
(d) Belief of DM[0.1] after 200 trials of (b)

(e) Cumulative loss on data (a)  
(f) Cumulative loss on data (b)
4.1. Introduction

We see in Figure 4.2c that, for the freezing data, the expert posterior obtained by the freezing reference scheme matches the 201st outcome (which is 10 in the freezing data set) best. Recall that this posterior is obtained by first showing $\text{DM}[0.1]$ outcomes 1 through 100, and then asking it to predict outcome 201 as if it was the next outcome in the sequence.

We also see in Figure 4.2d that, for the sleeping data, the expert posterior obtained by the sleeping reference scheme matches the 201st outcome (which is 20 in the sleeping data set) best. Recall that this posterior is obtained by first showing $\text{DM}[0.1]$ outcomes 1 through 100, and then asking it to predict outcome 201 with all intermediate outcomes unobserved.

Finally, we see that in both cases, the expert posterior obtained by the full reference scheme, which shows all outcomes to $\text{DM}[0.1]$, overshoots: the expert is confused by observing the intermediate outcomes.

**Loss** These snapshots of the expert’s posteriors provide an intuitive understanding of what the reference schemes do and which one is appropriate. We now quantify the predictive performance by looking at the resulting cumulative loss. Figures 4.2e and 4.2f show the cumulative log(arithmetic) loss for all three reference schemes. Note that the difference between the schemes is so large that their losses had to be plotted on a logarithmic scale.

We see in Figure 4.2f that for the sleeping data the sleeping reference scheme has much smaller loss than the other two schemes. And for the freezing data the freezing reference scheme has the smallest loss by far, as shown in Figure 4.2e. (Mind the logarithmic scale of the y-axis, which puts the loss of sleeping deceptively close to the loss of freezing in Figure 4.2e: a constant offset indicates a fixed multiplicative overhead.) In both cases the reason for the large differences between the reference schemes is that both experts $\text{DM}[0.1]$ and $\text{DM}[0.3]$ get confused if they learn from the wrong data.

Note that for this synthetic example, we knew which partitioning into subsequences to choose, since we constructed the data ourselves. For real data a partitioning is not readily available. The challenge addressed in this chapter is to learn the best partition of the data online.
In this chapter, we solve Freund’s problem under the interpretation that experts only observe the subsequence on which they are evaluated. Of course, for arbitrary experts, this is impossible. For in the setting of prediction with expert advice (see [25]), the expert predictions that we receive each round are always in the context of all data. We have no access to the experts’ predictions in the context of any subsequence, and these predictions may differ drastically from those on the whole data.

Often however, experts have internal structure. For example, in [108, 80, 180, 181] adaptive prediction strategies (i.e. learning experts) are explicitly constructed from basic experts. To represent such structured experts, we use the general framework called expert hidden Markov models (EHMMs), that was introduced in Chapter 3. EHMMs are hidden Markov models in which the production probabilities are determined by expert advice. A structured expert in EHMM form provides sufficient information about its predictions on any isolated subsequence.

Many strategies for prediction with expert advice (i.e. learning experts) can be rendered as EHMMs. For example all adaptive strategies in the papers above (see Chapter 3). But there are also strategies that cannot be brought into EHMM form, like e.g. follow the perturbed leader [73] and variable share [80].

Our approach may also be of interest to machine learning with regular hidden Markov models (HMMs) [146]. Although existing approaches to shift between multiple HMMs [65, 66, 104] usually focus on change-point detection, prediction seems a highly related issue.

4.1.2 Overview

After preliminaries we start by reviewing the main existing loss bound for mixing past posteriors in Section 4.3. Then, in Section 4.4, we review EHMMs as a way to represent structured experts.

The next section, Section 4.5, contains our results for Freund’s problem when structured experts are evaluated on isolated subsequences. We formalise sleeping and freezing as two different ways of presenting a subsequence of the data to an EHMM, and present the evolving past posteriors (EPP) algorithm that takes an EHMM as input. The EPP algorithm has two variants, which both generalise the mixing past pos-
4.2 Preliminaries

Prediction With Expert Advice Each round $t$, we first receive advice from each expert $e \in \mathcal{E}$ in the form of an action $a^e_t \in \mathcal{A}$. Then we distill our own action $a^{alg}_t \in \mathcal{A}$ from the expert advice. Finally, the actual outcome $x_t \in \mathcal{X}$ is observed, and everybody suffers loss as specified.
by a fixed loss function \( \ell : A \times \mathcal{X} \to [0, \infty] \). Thus, the performance of
a sequence of actions \( a_1 \cdots a_T \) upon data \( x_1 \cdots x_T \) is measured by the
cumulative loss \( \sum_{t=1}^T \ell (a_t, x_t) \).

**Log Loss** For log loss the actions \( A \) are probability distributions on \( \mathcal{X} \)
and \( \ell (p, x) = - \log p(x) \), where \( \log \) denotes the natural logarithm. It is
important to notice that minimising log loss is equivalent to maximising
the predicted probability of outcome \( x \). We write \( p^e_t \) for the prediction
of expert \( e \) at time \( t \) and denote these predictions jointly by \( p^e_{1:T} \).

**Subsequences** For \( m \leq n \), we abbreviate \( \{m, \ldots, n\} \) to \( m:n \). For com-
pleteness, we set \( m:n = \emptyset \) for \( m > n \). For any sequence \( y_1, y_2, \ldots \)
and any set of indices \( C = \{i_1, i_2, \ldots\} \) we write \( y_C \) for the subsequence
\( \langle y_i \rangle_{i \in C} \). For example, \( x_C = \langle x_i \rangle_{i \in C} \) and \( p^e_{1:T} = p^e_{1}, \ldots, p^e_{T} \). If members of
a family \( C = \{C_1, C_2, \ldots\} \) are pairwise disjoint and together cover \( 1:T \)
(\( \bigcup C = 1:T \)), then we call \( C \) a partition of \( 1:T \), and its members cells.

### 4.3 Mixing Past Posteriors

Mixing past posteriors (MPP) is a strategy for prediction with expert
advice. It operates by maintaining a table of so-called posterior distribu-
tions on the set of experts. Each round, we first compute the predic-
tive distribution on experts by mixing all the posteriors in the table.
Then the next outcome is predicted by mixing the expert predictions
according to this distribution. Finally, the next outcome is observed.
The predictive distribution on experts is conditioned on this outcome,
and the posterior distribution thus obtained is appended to the table of
posteriors. Note the recursive construction of the distributions in the
table; they are not Bayesian posteriors, but conditioned mixtures of all
earlier distributions from that same table.

We will not formally introduce MPP here, but recover it as a special
case of both the freezing and sleeping algorithms in Section 4.5.4. Here
we state the classical loss bound \([19, \text{Theorem 7}]\), introducing our nota-
tion along the way. This loss bound relates the loss of MPP to Freund’s
full reference scheme, where we choose a partition of the data (step a)
and select an expert for each partition cell (step b). We measure expert
performance (step c) using the predictions issued in the context of all data, i.e. the full interpretation of Freund’s scheme.

4.3. Mixing Past Posteriors

4.3.1 Loss Bound

We bound the overhead of MPP over the full reference scheme in terms of the complexity of the reference partition. We first state the theorem, and then explain the ingredients. We write $P^\text{MPP}_w(x_{1:T})$ for the probability that MPP assigns to data $x_{1:T}$ (so $-\log(P^\text{MPP}_w(x_{1:T}))$ is MPP’s cumulative log loss).

4.3.1. Theorem ([19, Theorem 7]). For any mixing scheme $\beta$, Bayesian joint distribution $P^B$ with prior distribution $w$ on experts, partition $C$ of $1:T$, data $x_{1:T}$ and expert predictions $p^e_{1:T}$

$$P^\text{MPP}_w(x_{1:T}) \geq \beta(C)P^B_C(x_{1:T}). \quad (4.1)$$

A mixing scheme $\beta$ is a sequence $\beta_1, \beta_2, \ldots$ of distributions, where $\beta_{i+1}$ is a probability distribution on $0:i$. In [19] several mixing schemes are listed, e.g. Uniform Past and Decaying Past. A mixing scheme is turned into a distribution on partitions as follows. Let $C$ be a partition of $1:T$, and let $i \in 1:T$. The cell of $i$, denoted $C(i)$, is the unique $C \in C$ such that $i \in C$. We write $\text{prev}^C(i)$ for the predecessor of $i$, defined as the largest element in $C(i) \cup \{0\}$ that is smaller than $i$. Using this notation, the distribution on partitions is given by

$$\beta(C) := \prod_{t \in 1:T} \beta_t(\text{prev}^C(t)).$$

Note that this distribution is potentially defective; two elements $i < j$ cannot share the same nonzero predecessor, but $\beta_i$ may assign nonzero probability to $\text{prev}^C(j)$ nonetheless.

Now that we have seen how the loss bound encodes partition, we turn to $P^B_C(x_{1:T})$, the probability of the data $x_{1:T}$ given a particular partition $C$. To compute it, we treat the cells independently (4.2), and per cell we use the Bayesian mixture with prior $w$ on experts (4.3), thus
mixing the predictions the experts issued in the context of all data (4.4).

\[ P^B_C(x_{1:T}) := \prod_{C \in \mathcal{C}} P^B_C(x_C), \text{ where} \]

\[ P^B_C(x_C) := \sum_{e \in \mathcal{E}} w(e) P^e_C(x_C) \]

and

\[ P^e_C(x_C) := \prod_{i \in C} p^e_i(x_i). \]

A second bounding step allows us to relate the performance of MPP directly to Freund’s full scheme. Let \( w \) be the uniform prior over a finite set of experts \( \mathcal{E} \), and select an expert \( e^C \) for each partition cell \( C \in \mathcal{C} \). Then bound each sum (4.3) from below by one of its terms to obtain

\[ P^MPP_w(x_{1:T}) \geq \beta(C) |\mathcal{E}|^{-|C|} \prod_{C \in \mathcal{C}} P^e_C(x_C). \]

Thus the log-loss overhead of MPP over the full reference scheme is bounded by \(- \log \beta(C) + |C| \log |\mathcal{E}|\), which can be related to the number of bits to encode the chosen partition and the selected experts for each cell [19].

Convex Combinations In [19], the authors make a point of selecting a convex combination of experts for each subsequence, where the loss of a convex combination of experts is the weighted average loss of the experts. The loss of such a convex combination is therefore always higher than the loss of its best expert. Uniform bounds in terms of arbitrary experts, like Corollary 4.3.2, apply in particular to the best expert, and hence to any convex combination. Therefore, without loss of generality, we do not discuss convex combinations any further.

Interpreting Freund’s Problem The loss bound Theorem 4.3.1 shows that MPP solves the black-box-experts interpretation of Freund’s problem. This can be seen clearly in (4.4). To predict the subsequence \( x_C \), it uses predictions \( p^e_C \) which were issued in the context of all data. This means that the experts observe the entire history \( x_{1:i} \) before predicting the next outcome \( X_{i+1} \).

Switching between learning experts that observe all data is useful when the data are homogeneous, and the experts learn its global pattern at different speeds. In such cases we want to train each expert on
all observations, for then by switching at the right time, we can predict each outcome using the expert that has learned most until then. This scenario is analysed in [178], where experts are parameter estimators for a series of statistical models of increasing complexity.

On the other hand, if the data have local patterns then our new interpretation of Freund’s problem applies, and we want to train each expert on the subsequence on which it is evaluated, so that it can exploit its local patterns. To solve Freund’s problem for such learning experts, we need to know about its internal structure.

4.4 Structured Experts

Assume there is only a single expert and fix a reference partition. Suppose we want to predict as if the expert is restarted on each cell of the partition, when in reality the expert just makes her predictions as if all the data were in a single cell. Then clearly this is impossible if we treat this expert completely as a black box: if we do not know what the expert’s predictions would have been if a certain outcome were, say, the start of a new cell, then we cannot match these predictions.

The expert therefore needs to reveal to us some of her internal state. To this end, we will represent the parts of her internal state that will not be revealed to us by lower level experts that we will treat as black boxes, and assume our main expert combines the predictions of these base experts using an expert hidden Markov model (EHMM).

4.4.1 EHMMs

Expert Hidden Markov Models (EHMMs) were introduced in Chapter 3 as a language to specify strategies for prediction with expert advice. We briefly review them here. An EHMM $A$ is a probability distribution that is constructed according to the Bayesian network in Figure 4.4. It is used to sequentially predict outcomes $X_1, X_2, \ldots$ which take values in outcome space $\mathcal{X}$. At each time $t$, the distribution of $X_t$ depends on a hidden state $Q_t$, which determines mixing weights for the experts’ predictions. Formally, the production function $p^d_{q_t}$ determines the interpretation of a state: it maps any state $q_t \in Q$ to a distribution $p^d_{q_t}$ on the identity $E_t$ of the expert that should be used to predict $X_t$. Then given $E_t = e$, the distribution of $X_t$ is base expert $e$’s prediction $p^e_t$. It remains
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Figure 4.4 Bayesian network specification of an EHMM

![Bayesian network specification](image)

Figure 4.5 Hidden state transitions in slot machine HMM

![Hidden state transitions](image)

to define the distribution of the hidden states. The starting state $Q_1$ has *initial distribution* $p_\circ$, and the state evolves according to the *transition function* $p_\rightarrow$, which maps any state $q_t$ to a distribution $p_{q_t}$ on states.

An EHMM $A$ defines a prediction strategy as follows; after observing $x_{1:t}$, predict outcome $X_{t+1}$ using the marginal $A(X_{t+1}|x_{1:t})$, which is a *mixture* of the expert’s predictions $p_{t+1}^e$.

**4.4.1. Example (Any Ordinary HMM).** To illustrate how ordinary hidden Markov models are a special case of EHMMs, consider the following naive gambler’s HMM model of an old-fashioned slot machine: in each round the gambler inserts one nickel into the slot machine and then the machine pays out a certain number of nickels depending on its hidden internal state: in state *Cold* it pays out nothing; in state *Hot* it pays out an amount between one and five nickels, uniformly at random; and then there’s *Jackpot* in which it always pays out ten nickels. The machine always starts in state *Cold* and the state transitions are as in Figure 4.5.

To make an EHMM out of this HMM, we just identify experts with states: $Q = E = \{Cold, Hot, Jackpot\}$, $p_1^e(e) = 1$, and each expert predicts according to the corresponding payout scheme. The distributions on states follow the original HMM: $p_\circ(Cold) = 1$ and $p_\rightarrow$ as in Figure 4.5.
4.4.2. Example (Bayes on base experts). We identify the Bayesian distribution with prior \( w \) on base experts \( E \) and the EHMM with \( Q = E, p_o = w \), and \( p^e_\circ(e) = p^e_i(e) = 1 \), since their marginals coincide. Despite its deceptive simplicity, this EHMM learns: its marginal distribution on the next outcome is a mixture of the expert’s predictions according to the Bayesian posterior.

4.4.3. Example (Bayes on EHMMs). Fix EHMMs \( A^1, \ldots, A^n \) with disjoint state spaces and the same basic experts, and let \( w \) be a prior distribution on 1: \( n \). The Bayesian mixture EHMM has state space \( Q = \bigcup_i Q^i \), and for any two states \( q, q' \in Q^i \) belonging to the same original EHMM, \( p_o(q) = w(i) p^i_o(q), p^q_\circ(q') = p^{i,q}_i(q') \) and \( p^q_i(e) = p^{i,q}_i(e) \). Again, this EHMM learns which of the given EHMMs is the best predictor.

4.4.2 The Forward Algorithm

Sequential predictions for EHMMs can be computed efficiently using the forward algorithm (see Algorithm 3.1 on page 77), which maintains a posterior distribution over states, and predicts each outcome with a mixture of the experts’ predictions. Given a posterior \( \lambda_t(Q_t) = A(Q_t|x_{1:t-1}) \) for the hidden state at time \( t \), the forward algorithm predicts \( x_t \) using the marginal of \( A(Q_t, E_t, X_t|x_{1:t-1}) \). Then, after observing outcome \( x_t \), it updates its posterior \( \lambda_t \) for \( Q_t \) to a posterior \( \lambda_{t+1} \) for \( Q_{t+1} \).

For finite \( Q, E \) and \( X \), the running time of the algorithm is determined by this last posterior update step, which in general may require \( O(|Q|^2) \) computation steps for each round \( t \). On \( T \) outcomes, this gives a total running time of \( O(|Q|^2 \cdot T) \). In Appendix 4.A we provide a more careful analysis.

4.5 Freezing & Sleeping

Let \( x_{1:T} = x_1, \ldots, x_T \) be a sequence of data and suppose that a reference partition \( C \) of \( 1:T \) is given in advance. We are interested in the performance of a structured expert \( A_C \), which for each cell \( C \in C \) runs a separate instance of the structured expert \( A \) on the subsequence \( x_C \). This leaves unspecified, however, whether the original timing of \( x_C \) should be preserved when \( x_C \) is presented to \( A \). This is a modelling choice,
which depends on the application at hand. We therefore treat both the case where the timing is preserved, which we call *sleeping*, and the case where the timing is not preserved, which we call *freezing*. (See also Figure 4.2 in the introduction.)

**Sleeping** We say that the instance of $A$ that is used to predict cell $C$ is sleeping if it does notice the passing of time during outcomes outside of $C$, even though it does not observe them. We write $A_{C}^{sl}$ for the resulting EHMM, which is shown in Figure 4.6a for the example $C = \{2, 4, 5, \ldots\}$. Notice that $A_{C}^{sl}$ contains all five states $Q_{1:5}$, even though it does not observe $x_1$ or $x_3$. This has the effect that state transitions from e.g. $Q_2$ to $Q_4$ are composed of two transition steps according to $p_{\circ}$. The distributions on individual cells combine into the following distribution on all data $x_{1:T}$:

$$A_{C}^{sl}(x_{1:T}) := \prod_{C \in C} A_{C}^{sl}(x_C).$$

To memorise the nature of sleeping, one may think of the way television channels get interleaved as you zap between them: a channel not being
watched is not paused, but instead continues broadcasting even when its content is not observed.

**Freezing** In freezing, the instance of $\mathcal{A}$ that is used to predict cell $C \in \mathcal{C}$ is frozen when outcomes outside of $C$ occur: its internal state should not change based on those outcomes. (Of course we have no control over the base experts on which $\mathcal{A}$ is based, so they may do whatever they please with such data. We therefore do have to preserve the timing of the base experts’ predictions.) The resulting EHMM $\mathcal{A}^f_C$ is shown for the example $C = \{2, 4, 5, \ldots\}$ in Figure 4.6b. Note that $Q_2$, $Q_4$ and $Q_5$ are the first, second and third state of $\mathcal{A}^f_C$; state transitions between them consist of a single transition step according to $p_-$. The resulting distribution on all data is defined by

$$\mathcal{A}^f_C(x_{1:T}) := \prod_{C \in \mathcal{C}} \mathcal{A}^f_C(x_C).$$

One might associate freezing with the way different e-mail conversations get interleaved in your inbox (if it is sorted by order of message arrival): a conversation about your latest research is paused (remains frozen) regardless of how much spam you receive in between.

### 4.5.1 An Infeasible Solution

The freezing or sleeping distributions can be computed if the reference partition $\mathcal{C}$ is given in advance. The problem we are addressing, however, is that we do not assume $\mathcal{C}$ to be known. An easy (but impractical) solution to this problem is to predict according to the Bayesian mixture of all possible partitions: let $w$ be a prior on the set of all possible partitions and predict such that the joint distribution on all data is given by

$$\mathcal{B}(x) := \sum_C w(C) \mathcal{A}^{f/s}_C(x),$$

where $f/s$ denotes either $f$ for freezing or $s$ for sleeping. Lower bounding the sum by the term for the reference partition $C$ directly gives an upper bound on the log loss:

$$- \log \mathcal{B}(x) \leq - \log w(C) - \log \mathcal{A}^{f/s}_C(x).$$
To predict according to \( \mathcal{B} \) in general would require an exponential amount of state to keep track of all possible partitions, which is completely impractical. In the following section we therefore present generalisations to both sleeping and freezing of the mixing past posteriors algorithm and show that their running time is comparable to that of the forward algorithm on \( \mathcal{A} \) itself. Then in section Section 4.5.3 we prove bounds that relate the additional loss to the encoding cost of the reference partition \( \mathcal{C} \).

### 4.5.2 The EPP Algorithm

Here we present a generalisation of the mixing past posteriors (MPP) algorithm, which we call evolving past posteriors (EPP). It is based on the view that MPP internally uses the Bayesian mixture of base experts, which is a standard EHMM. Given this perspective and after making the distinction between sleeping and freezing, the generalisation to other EHMMs is straightforward. We will discuss the connections between MPP and EPP in more detail in Section 4.5.4.

The EPP algorithm has variants for sleeping and freezing, which are both given in Algorithm 4.1. It takes an EHMM \( \mathcal{A} \) and mixing scheme \( \beta \) (see Section 4.3.1) as input. Given a distribution \( \lambda_t \) on the hidden state \( Q_t \) at time \( t \), the EPP algorithm predicts \( X_t \) exactly like the forward algorithm. It differs from the forward algorithm, however, in the way it computes \( \lambda_t \). Whereas in the forward algorithm \( \lambda_t \) may be interpreted as the posterior distribution on \( Q_t \), in the EPP algorithm \( \lambda_t \) is a \( \beta \)-mixture of the algorithm’s own past posteriors. This recursive nature of EPP, which it inherits from the MPP algorithm, makes it hard to analyse.

We denote by \( P_{f}^{fr} \) and \( P_{s}^{sl} \) the probability distributions on random variables \( \langle Q_t, E_t, X_t \rangle_{t \in \mathbb{N}} \) defined by EPP-FREEZING and EPP-_SLEEPING on EHMM \( \mathcal{A} \) and mixing scheme \( \beta \). For both \( f/s \in \{sl, fr\} \)

\[
P_{A}^{f/s}(q_{1:T}, e_{1:T}, x_{1:T}) = \prod_{t \in 1:T} p_{alg}^{f/s}(q_t, e_t, x_t).
\]

### 4.5.2.1 Representation Invariance

Let \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) be EHMMs that are based on the same set of experts \( \mathcal{E} \), but have different state spaces. We call \( \mathcal{A}^1 \) and \( \mathcal{A}^2 \) equivalent if
Algorithm 4.1 Evolving past posteriors (EPP)

Input:
- An EHMM $A$ with components $p_\circ$, $p_\rightarrow$ and $p_\leftarrow$ (see Section 4.4)
- A mixing scheme $\beta_1, \beta_2, \ldots$ (see Section 4.3.1 and Section 4.5.2.2)
- Expert predictions $p_1^E, p_2^E, \ldots$ and data $x_1, x_2, \ldots$

Output: Predictions $p_1^{\text{alg}}, p_2^{\text{alg}}, \ldots$

Storage: Past posteriors $\pi_1, \pi_2, \ldots$ on $Q$, the states of $A$

Algorithm
1: Set the first posterior to the initial distribution of $A$
   $$\pi_1(q_1) \leftarrow p_\circ(q_1)$$
2: for $t = 1, 2, \ldots$ do
3:   Form $\lambda_t$, the current configuration, as the $\beta_t$-mixture of past posteriors:
   $$\lambda_t(q_t) \leftarrow \sum_{0 \leq j < t} \beta_t(j) \pi_{j+1}(q_t).$$
4:   Compute $p_t^{\text{alg}}$, the joint distribution on states, experts and outcomes:
   $$p_t^{\text{alg}}(q_t, e_t, x_t) \leftarrow \lambda_t(q_t) \ p_t^q(q_t)p_t^e(e_t)p_t^x(x_t).$$
5:   Predict $x_t$ using the marginal $p_t^{\text{alg}}(x_t),$
6:   Observe $x_t$. Suffer log loss
   $$\ell_t^{\text{alg}} \leftarrow - \log(p_t^{\text{alg}}(x_t)).$$
7:   Perform loss update and state evolution to obtain the next posterior
   $$\pi_{t+1}(q_{t+1}) \leftarrow \sum_{q_t \in Q} p_t^{\text{alg}}(q_t|x_t) \ p_{q_t}^\beta(q_{t+1}).$$
8: Only for sleeping: perform state evolution for all past posteriors ($1 \leq j \leq t$)
   $$\pi_j(q_{t+1}) \leftarrow \sum_{q_t \in Q} \pi_j(q_t) \ p_{q_t}^\beta(q_{t+1}).$$
9: end for
Chapter 4. Freezing & Sleeping

Table 4.1 Mixing schemes

<table>
<thead>
<tr>
<th>Mixing scheme</th>
<th>$\beta_{t+1}(t)$</th>
<th>$\beta_{t+1}(j)$ for $0 \leq j &lt; t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yesterday</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Fixed Share($\alpha$)</td>
<td>$1 - \alpha$</td>
<td>$\alpha$ if $j = 0$ and 0 o.w.</td>
</tr>
<tr>
<td>Uniform past($\alpha$)</td>
<td>$1 - \alpha$</td>
<td>$\alpha / t$</td>
</tr>
<tr>
<td>Decaying past($\alpha, \gamma$)</td>
<td>$1 - \alpha$</td>
<td>$\alpha(t - j)^{-\gamma} / Z_t$</td>
</tr>
</tbody>
</table>

$A^1(e_{1:T}) = A^2(e_{1:T})$ for all $e_{1:T}$. Consequently, equivalent EHMMs assign the same probability $A^1(x_{1:T}) = A^2(x_{1:T})$ to all data $x_{1:T}$, hence the difference between $A^1$ and $A^2$ is merely a matter of representation. As an important sanity check, we need to verify that EPP on either EHMM issues the same predictions.

4.5.1. Theorem (Invariance). Let $f/s$ denote either $fr$ or $sl$. Fix equivalent EHMMs $A^1$ and $A^2$. Then for all data $x_{1:T}$

$$P^f/s_{A^1}(x_{1:T}) = P^f/s_{A^2}(x_{1:T}).$$

Proof. Given in Appendix 4.C. \qed

Thus, from the perspective of predictive performance, the difference between $A^1$ and $A^2$ is irrelevant. Of course, it does matter for the computational cost of EPP, see Section 4.5.2.3.

4.5.2.2 Mixing Schemes

Bousquet and Warmuth [19] provide an extensive discussion of possible mixing schemes. Their loss bounds for various schemes carry over directly to our setting. It is interesting, however, to analyse the running times of the Fixed-Share to uniform past and to decaying past mixing schemes for EPP. For further information we refer the reader to [19].

Both schemes (see Table 4.1) depend on a switching rate $\alpha \in [0, 1]$, which determines whether to continue with yesterday’s posterior or switch back to an earlier one: $\beta_{t+1}(t) = 1 - \alpha$ and $\sum_{0 \leq j < t} \beta_{t+1}(j) = \alpha$.

**Uniform Past** Given the choice to switch back, the uniform past mixing scheme gives equal weights to the entire past: $\beta_{t+1}(j) = \alpha / t$ for $0 \leq j < t$. 
Decaying Past The decaying past scheme assigns larger weight to the recent past: 
\[ \beta_{t+1}(j) = \alpha(t-j)^{-\gamma}/Z_t \text{ for } 0 \leq j < t, \]
where 
\[ Z_t = \sum_{0 \leq j < t}(t-j)^{-\gamma} \]
is a normalising constant and \( \gamma \geq 0 \) is a parameter that determines the rate of decay.

4.5.2.3 Running Times

Appendix 4.A provides a detailed comparison of the running times and space requirements of EPP and the forward algorithm. The upshot is that for the uniform past mixing scheme the sleeping variant of EPP is as efficient as the forward algorithm, in terms of both running time and space requirements; the freezing variant is equally efficient if the set of hidden states \( Q \) is finite, but may be a factor \( O(T) \) less efficient on \( T \) outcomes for countably infinite \( Q \). The decaying past mixing scheme is a factor \( O(T) \) less efficient (for both time and space) than uniform past in all cases, but may be approximated by a scheme described in [19] that reduces this factor to \( O(\log T) \).

4.5.3 Loss Bound

We relate the performance of EPP-FREEZING and EPP-SLEEPING (defined in Algorithm 4.1) to that of \( A^f_C \) and \( A^s_C \) for all partitions \( C \) jointly.

4.5.2. Theorem (EPP Loss Bounds). For both \( f/s \in \{fr, sl\} \) and any mixing scheme \( \beta \), data \( x_{1:T} \) and expert predictions \( P^f_{1:T} \)

\[ P^f_{A}(x_{1:T}) \geq \sum_{C} \beta(C) A^f_C(x_{1:T}). \]  

(4.5)

Proof. Given in Appendix 4.B. \( \square \)

Using this bound, we can relate the predictive performance of EPP-SLEEPING and EPP-FREEZING to that of \( A^s_C \) and \( A^f_C \) for any reference partition \( C \).

4.5.3. Corollary. \( P^s_{A}(x_{1:T}) \geq \beta(C) A^s_C(x_{1:T}). \)

From the brutal way in which Corollary 4.5.3 was obtained, we may expect to often do much better in practice; many partitions may contribute significantly to (4.5).
4.5.4 Recovering MPP

We now substantiate our claim that EPP generalises MPP by proving that MPP results from running EPP-FREEZING or EPP-SLEEPING on the Bayesian EHMM (Example 4.4.2).

4.5.4. Theorem. Let $A$ be the Bayesian EHMM with initial distribution $w$, and let $P^MPP_w$ denote the probability distribution defined by MPP with prior $w$. Then for all data $x_{1:T}$

$$P^A_{fr}(x_{1:T}) = P^A_{sl}(x_{1:T}) = P^MPP_w(x_{1:T}).$$

Proof. The difference between freezing and sleeping (line 8) evaporates since state evolution is the identity operation. By identifying states and experts the MPP algorithm [19, Figure 1] remains. $\square$

The theorem does not require the set of experts $E$ to be finite. If $E$ is infinite (or too large), MPP is intractable. Still, a small EHMM may exist that implements Bayes (say with the uniform prior) on $E$, and we can use EPP-SLEEPING (which is faster than EPP-FREEZING) for sequential prediction. For example, we may implement MPP on the infinite set of Bernoulli experts efficiently, in time $O(T^2)$, using EPP-SLEEPING on the universal element-wise mixture EHMM of [100, §4.1].

4.5.4.1 Improved MPP Loss Bound

[19, Theorem 7] (our Theorem 4.3.1) bounds the overhead of MPP over Freund’s full scheme in terms of $\beta(C)$, the complexity of the reference partition $C$ according to the mixing scheme $\beta$. A more general bound follows directly from Theorems 4.5.2 and 4.5.4:

4.5.5. Corollary. $P^MPP_w(x_{1:T}) \geq \sum C \beta(C)P^\beta_C(x_{1:T}).$

Even with a fixed reference partition $C$ in mind, we get a better bound by considering small modifications of $C$, e.g. finer partitions or partitions that disagree about a single round.

Adversarial Experts  For each number of rounds $T$ one can construct a set of $T$ base experts and data $x_{1:T}$ such that the loss of Freund’s full scheme is infinite for all partitions except the finest one. We simply
have expert $t$ suffer infinite loss in all rounds other than $t$. In this pathological case the bounds in Theorem 4.3.1 for that partition and Corollary 4.5.5 are equal and tight.

### 4.5.4.2 Is EPP strictly more general than MPP?

A natural question is whether either EPP-Sleeping or EPP-Freezing can be implemented using MPP on a rich set of meta-experts. To preclude the trivial answer that regards either algorithm as a single meta-expert, we ask for a fixed construction that works for all mixing schemes.

**Sleeping** For any EHMM $A$, EPP-Sleeping can be reduced to MPP on meta-experts. Let the set of meta-experts be $Q^\infty$, the set of paths through the hidden states of $A$. Each meta-expert $q_N$ predicts $x_t$ using the $p_{q_t}$-mixture of base expert predictions. We set the prior $w$ in MPP equal to the marginal probability measure of $A$ on paths (as determined by $p_o$ and $p_\omega$). We omit the proof that the predictions made by MPP on these meta-experts with prior $w$ are equal to those made by EPP on $A$.

**Freezing** The next example shows that EPP-Freezing really is more general than MPP. Fix two experts $E = \{a, b\}$. Consider the EHMM $A$ that predicts the first outcome using expert $a$, and the second outcome using expert $b$, i.e. $Q = E$, and $p_o(a) = p^a_1(b) = p^q_2(q) = 1$. Running EPP-Freezing on $A$ results in $\pi_2(b) = \pi_1(a) = 1$, so that the first outcome is predicted using expert $a$, and the second outcome is predicted using the $\beta_2$-mixture of experts. Thus any candidate meta-expert must predict the first outcome using base expert $a$. But that means that for MPP with prior $w$ on meta-experts, the loss update has no effect, so that $w = \pi_1 = \pi_2 = \lambda_2$. Hence the second outcome will be predicted according to the prior mixture of experts. Since $\beta_2$ is arbitrary and $w$ is fixed, there can be no general scheme to reduce EPP-Freezing to MPP.

### 4.6 Other Loss Functions

We will now show how the EPP algorithm for logarithmic loss can be directly translated into an algorithm with corresponding loss bound for any other mixable loss function. The same construction works for any
logarithmic loss algorithm that predicts according to a mixture of the experts’ predictions at each trial and whose predictions only depend on the experts’ past losses on outcomes that actually occurred.

**Mixability** A loss function $\ell: A \times X \to [0, \infty]$ is called $\eta$-mixable for $\eta > 0$ if any distribution $p$ on experts $E$ can be mapped to a single action $\text{Pred}(p) \in A$ in a way that guarantees that
\[
\ell(\text{Pred}(p), x) \leq -\frac{1}{\eta} \log \mathbb{E}_{e \sim p} \left[ \exp(-\eta \ell(a^e, x)) \right]
\] (4.6)
for all outcomes $x \in X$ and expert predictions $a^e$. It is called mixable if it is $\eta$-mixable for some $\eta > 0$ [25]. Mixability ensures that expert predictions for $\ell$ loss can be mixed in essentially the same way as for log loss.

For example, logarithmic loss itself is 1-mixable. And for $A = [0, 1]$ and $X = \{0, 1\}$ the square loss $\ell(a, x) := (a - x)^2$ is 2-mixable and the Hellinger loss $\ell(a, x) := ((\sqrt{1-x} - \sqrt{1-a})^2 + (\sqrt{x} - \sqrt{a})) / 2$ is $\sqrt{2}$-mixable.[75, 25]

**The Benefits of Lying** Given data $x_{1:t}$ and expert predictions $a^e_{1:t}$, let $\ell^e_{1:t} := \ell(a^e_1, x_1), \ldots, \ell(a^e_t, x_t)$ denote the sequence of losses of expert $e$, and let $\ell_{1:t}$ denote these losses jointly for all experts. In the special case that $\ell$ is the logarithmic loss we write $\ell_{1:t}$ and $\ell^e_{1:t}$, respectively.

Suppose $\text{ALG}$ is an algorithm for log loss that predicts each outcome $x_t$ by mixing the experts’ predictions $p^e_t$ according to the distribution $p^e_t[x_{<t}, \ell^e_{<t}]$ on experts. The square-bracket expression indicates that $p^e_t$ may depend on the past outcomes $x_{1:t-1}$ and the losses of the experts on these outcomes, but not on the experts’ past or current predictions in any other way. Following this convention, the algorithm predicts $x_t$ using:
\[
p^e_t[x_{<t}, \ell^e_{<t}](x_t) := \sum_e p^e_t[x_{<t}, \ell^e_{<t}](e)p^e_t(x_t).
\]

Now for any game with $\eta$-mixable loss $\ell$ and the same set of experts $E$, we can derive from $\text{ALG}$ an algorithm $\text{ALG}^{\eta}_t$ that predicts $x_t$ according to
\[
a^e_t := \text{Pred} \left( p^e_t[x_{<t}, \eta \ell^e_{<t}] \right).
\]
4.6. Other Loss Functions

Note that $\text{ALG}_\ell^\eta$ is lying to $\text{ALG}$: while $\text{ALG}$ thinks it is playing a game for log loss in which experts have incurred log losses $\eta E_{t \leq t}$, in reality $\text{ALG}_\ell^\eta$ is playing a game for loss $\ell$ and is feeding $\text{ALG}$ fake inputs and redirecting $\text{ALG}$’s outputs. Let us now analyse the loss of the derived algorithm $\text{ALG}_\ell^\eta$.

4.6.1. Lemma (Other Loss Functions). Suppose $\text{ALG}$ is an algorithm for logarithmic loss that predicts according to $p_{t \leq t}^a$ at each time $t$, $\ell$ is an $\eta$-mixable loss function, and $f(x_{1:T}, E_{1:T})$ is an arbitrary function that maps outcomes and expert losses to real numbers. Then any log loss bound for $\text{ALG}$ of the form

$$-\log p_{1:T}^a(x_{1:T}) \leq f(x_{1:T}, E_{1:T})$$

for all $p_{1:T}^a$, (4.7)

directly implies the $\ell$ loss bound for $\text{ALG}_\ell^\eta$:

$$\ell(a_{1:T}^\ell, x_{1:T}) \leq \frac{1}{\eta} f(x_{1:T}, E_{1:T})$$

for all $a_{1:T}^\ell$. (4.8)

Proof. Construct a log loss game in which at any time $t$ each expert $e$ predicts according to a distribution $p_t^e$ such that $p_t^e(x_t) = \exp(-\eta E_t^e)$ for the actual outcome $x_t$ and $p_t^e$ is arbitrary on other outcomes such that $\sum_p p_t^e(x_t) = 1$. In this game the log loss of $\text{ALG}$ is

$$-\log p_{1:T}^a(x_{1:T}) = \sum_{t \in 1:T} -\log p_t^a(x_{t \leq t}, \eta E_{t \leq t})(x_t).$$

By $\eta$-mixability of $\ell$

$$\ell(a_{1:T}^\ell, x_{1:T}) = \sum_{t \in 1:T} \ell\left(\text{Pred}(p_t^a(x_{t \leq t}, \eta E_{t \leq t})), x_t\right)$$

$$\leq \frac{1}{\eta} \sum_{t \in 1:T} -\log p_t^a(x_{t \leq t}, \eta E_{t \leq t})(x_t).$$ (4.9)

Combining with (4.7) and (4.9) completes the proof.

Algorithms that satisfy the requirements of the lemma include Bayes, follow the (perturbed) leader, the forward algorithm, MPP and EPP. An algorithm that does not satisfy them is the last-step minimax algorithm [172], because it takes into account the experts’ predictions on outcomes that do not occur.
In the literature it is common to construct algorithms for arbitrary mixable losses and point out their probabilistic interpretation for the special case of log loss \([75, 80, 19]\). Instead, we have proceeded the other way around: first we derived results for log loss and then we showed that they generalise to other losses. This allowed us to draw on concepts and results from probability theory like conditional probabilities, HMMs and the forward algorithm, without reproving them in a more general setting.

Lemma 4.6.1 generalises results by Vovk [183], who shows that the most important loss bounds for Bayes with logarithmic loss can actually also be derived for arbitrary mixable losses. Our algorithm \(\text{Alg}\) plays a role similar to his APA algorithm.

## 4.7 Discussion

**Relearning vs Continuing to Learn** Corollary 4.5.3 bounds the regret of EPP with respect to a reference partition \(C\) by \(-\log \beta(C)\). Consider the asymptotic behaviour of this bound if \(C\) has infinitely many shifts. (A shift occurs when \(\text{prev}^C(t + 1) \neq t\).) For both decaying past with \(\gamma \leq 1\) (e.g. following recommendations in \([19]\)) and uniform past (see Table 4.1) \(\max_{0 \leq j < t} \beta_{t+1}(j)\) goes to zero as a function of \(t\). Thus, the cost per shift (be it to continue an earlier cell or to start a new one) grows without bound. On the other hand for fixed share \(\beta_{t+1}(0) = \alpha\) for all \(t\), hence fixed share can start a new cell at fixed cost. It depends on the structured expert whether continuing previously selected cells at increasing cost is advantageous over relearning from scratch after each shift at fixed cost. For EHMM experts with a finite state space \(Q\) (including Bayes), relearning from scratch will cost at most a factor \(|Q|\) over learning on. This factor is constant, so that fixed share will eventually win.

## 4.8 Conclusion

We revisited Freund’s problem, which asks for a strategy for prediction with expert advice that suffers small additional loss compared to Freund’s reference scheme. We discussed the solution by Bousquet and Warmuth, which interprets the experts as black boxes. We proposed
a new interpretation of Freund’s scheme which is natural for learning experts, namely to train experts on the subsequence on which they are evaluated. This allows the reference scheme to exploit local patterns in the data, and thus makes the problem harder.

We solved Freund’s problem for structured experts that are represented as EHMMs, building on the work of Bousquet and Warmuth. We showed that our prediction strategies are efficient, and have desirable loss bounds that apply to all mixable losses.

4.A Running Times

We compare the running times on $T$ outcomes of EPP and the forward algorithm, with respect to an arbitrary EHMM $A$ with a countable set of hidden states $Q$. For simplicity we assume that the sets of experts $E$ and outcomes $X$ are finite.

Let $Q_t$ denote the hidden state of $A$ at time $t$, and let $p_o$, $p_\omega$ and $p_\perp$ denote $A$’s other components. Both algorithms base their predictions on a distribution $\lambda_t$ on $Q_t$ at time $t$, but differ in how they update $\lambda_t$ after observing $x_t$. As the number of computations for this step depends on the size of the support of $\lambda_t$ and on $p_\omega$, we will need the following concepts. For any probability distribution $p$ on $Q$, let $\text{Sp}(p) = \{q \in Q \mid p(q) > 0\}$ denote its support. We recursively define $Q_t$, the set of states reachable in exactly $t$ steps, and $Q_{\leq t}$, the set of states reachable in at most $t$ steps, by

$$Q_1 := \text{Sp}(p_o), \quad Q_{t+1} := \bigcup_{q \in Q_t} \text{Sp}(p_{\omega q}), \quad Q_{\leq t} := \bigcup_{i \in 1:t} Q_i.$$  

Obviously, $Q_t \subseteq Q_{\leq t} \subseteq Q$ holds for all $t$. Let $g(S) := \sum_{q \in S} |\text{Sp}(p_q)|$ be the number of outgoing transitions from any set of states $S \subseteq Q$.

4.A.1 Forward

The forward algorithm computes $\lambda_{t+1}$ by conditioning $\lambda_t$ on $x_t$ and applying the transition function $p_\omega$. As $\lambda_t$ has support $Q_t$, the forward algorithm requires $O(g(Q_t))$ work per time step, and $O(|Q_t| + |Q_{t+1}|)$ space. Notice that, for finite $Q$, the number of transitions is bounded by $g(S) \leq |Q|^2$ for any $S$. A rough upper bound on the total running
time of forward on \( T \) outcomes is therefore \( O(|Q|^2 T) \), which is linear in \( T \).

**4.A.2 EPP**

The EPP algorithm comes in two variants: one for sleeping and one for freezing. For sleeping the order of the running time is determined by the evolution of past posteriors (line 8 in Algorithm 4.1); for freezing, which skips line 8, either computation of \( \lambda_t \) (line 3) or of the next posterior (line 7) is the dominant step. The main difference for the running times of the two variants, however, is that in sleeping \( \pi_j \) has support \( Q_t \) at any time \( t \), whereas for freezing \( \pi_j \) has support \( Q_{\leq t} \).

**4.A.2.1 Uniform Past**

For the uniform past mixing scheme, one can keep track of \( \sum_{j=0}^{t} \pi_j(q_t) \) to speed up computation of \( \lambda_{t+1} \).

**Sleeping** This even works for sleeping, because applying the state evolution to this sum in line 8 of the algorithm is equivalent to applying it to the individual \( \pi_j \) and then summing. Consequently, sleeping requires \( O(g(Q_t)) \) work and \( O(|Q_t| + |Q_{t+1}|) \) space per time step, which makes it as efficient as the forward algorithm.

**Freezing** For freezing, computing the next posterior (line 7) determines the running time. It requires \( O(g(Q_{\leq t})) \) work and \( O(|Q_{\leq t+1}|) \) space per time step. Depending on the EHMM \( A \), this may be significantly slower than the forward algorithm. First, for finite \( Q \), each of \( Q_t \), \( Q_{\leq t} \) and \( Q \) have size \( O(1) \) in \( t \), and freezing runs in time \( O(T) \), just like the forward algorithm. Second, for infinite \( Q \), \( Q_{\leq t} \) may be unbounded as a function of \( t \). Still, on \( T \) outcomes

\[
\sum_{t \in 1:T} g(Q_{\leq t}) \leq T g(Q_{\leq T}) \leq T \sum_{t \in 1:T} g(Q_t),
\]

which implies that freezing is no more than a factor \( T \) slower than the forward algorithm.
4.A.2.2 Decaying Past

For the decaying past scheme the relative mixing weights of any two past posteriors change from $\beta_t$ to $\beta_{t+1}$, which prevents us from summing them as for uniform past. Implementing decaying past therefore slows down both the evolution of past posteriors and computation of $\lambda_t$ by a factor of $O(t)$, and increases the required space by the same factor. Fortunately, however, the decaying past scheme can be approximated using a logarithmic number of uniform blocks, as described in Appendix C of [19]. This reduces the slowdown factor from $O(t)$ to $O(\log t)$.

Thus, both for sleeping and for freezing, approximated decaying past is only a factor $O(\log T)$ slower than uniform past on $T$ outcomes, and requires only a factor $O(\log T)$ more space.

4.B Loss Bounds

We identify $\lambda_t$ with the EHMM on $\langle Q_i, E_i, X_i \rangle_{i \geq t}$ with initial distribution $\lambda_t$, and with the transition and production functions of $\mathcal{A}$. So in particular $\lambda_1 = \mathcal{A}$. For convenience, we shorten $(\lambda_t)_{C}(x_C)$ to $\lambda_{t}^{fr}(x_{C})$ and $(\lambda_t)_{C}(x_C)$ to $\lambda_{t}^{sl}(x_{C})$. Thus, among others, $\lambda_t(x_t) = \lambda_{t}^{sl}(x_{t}) = \lambda_{t}^{fr}(x_{t})$.

4.B.1. Lemma. For any $C \subseteq t:T$, interpreting $\lambda_0(\cdot|x_0)$ as $\lambda_1$,

$$\lambda_{t}^{fr}(x_{C}) = \sum_{j \in 0:t-1} \beta_t(j)\lambda_{j}^{fr}(x_{C}|x_{j}).$$

Proof. Let $\pi_t^j$ denote the past posterior $\pi_j$ at the beginning of round $t$. Thus for freezing $\pi_t^j = \pi_j$, and for sleeping $\pi_t^j$ is $\pi_j$ evolved $t-j$ steps. Then by definition $\lambda_t(x_C) = \sum_{j=0}^{t-1} \beta_t(j)\pi_{t+j}(x_C)$). The operations $(\cdot)^{fr}$ and $(\cdot)^{sl}$ distribute over taking mixtures. The lemma follows from the fact that $(\pi_t^j)^{sl}(x_C) = \pi_{t}^{sl}(x_{C})$ and $(\pi_t^j)^{fr}(x_C) = \pi_{t}^{fr}(x_{C})$.

Proof of Theorem 4.5.2. For any $t$, we view the mixing scheme $\beta_t$ as defining the distribution of a randomised choice $j_t \in 0:(t - 1)$ for the predecessor of the $t$th outcome. Let $j_{k+1:T} := j_{k+1:T} = (j_{k+1}, \ldots, j_T)$ denote a

---

1In [19] it is suggested to weight each block of posteriors $\pi_{[j_1,j_2]}$ by $(j_2 - j_1)\beta_t(j_1)$. It seems that a marginal improvement is possible by weighting by $\sum_{j_1 \leq j < j_2} \beta_t(j)$ instead, which can be implemented equally efficiently for decaying past.
vector of the choices beyond turn $k$. Unfortunately, some choices of $j > k$ are inconsistent with any partition, because an element can only have one successor in a partition. Thus $j > k$ is inconsistent with any partition if $j_m = j_n > 0$ for $k < m \neq n \leq T$. Let the predicate $I(j > k)$ be true iff $j > k$ is consistent with some partition.

Some elements of $j > k$ may indicate the start of a new cell of the partition. Let $S(j > k)$ denote the set of times when $j > k$ prescribes to start a new cell, i.e. $S(j > k) := \{ t \in k + 1: T | j_t = 0 \}$. For an example, consult Figure 4.7.

Consistent values of $j > k$ specify the last part of a partition. For any $1 \leq t \leq k$, we may ask which of the times $k + 1: T$ will be put in the same cell as $t$. Let $R_t(j > k)$ denote this set, including $t$. For convenience, we abbreviate

$$
\beta(j > k) := \prod_{t \in k + 1: T} \beta_t(j_t),
$$

$$
W(j > k) := \prod_{i \in S(j > k)} \lambda_1^{t/s}(x_{R_i(j > k)}),
$$

and

$$
U_l(j > k) := \prod_{i \in 1: l} \lambda_i^{t/s}(x_{R_i(j > k)})
$$

for all $l \leq k$,

to name the intermediate debris arising from the incremental reduction of $P_{A}(x_{1: T})$. $W$-terms deal with cells that are completely specified by $j > k$, while $U$-terms keep track of the remaining partially specified cells. The proof proceeds by downward induction on $k$, with induction hypothesis

$$
\prod_{i \in 1: T} \lambda_i(x_i) \geq \sum_{j > k: I(j > k)} \beta(j > k)W(j > k)U_k(j > k). \quad (4.10)
$$

For the base case $k = T$ the hypothesis holds with equality, and for $k = 0$ the hypothesis is equivalent to the desired result (4.5). It remains
4.C. Invariance

to verify that it holds for \( k - 1 \geq 0 \) if it holds for \( k \). To this end, fix \( k \geq 1 \). To prove (4.10), it suffices to show that for consistent \( j > k \)

\[
W(j > k) U_k(j > k) \geq \sum_{j_k : l(j > k)} \beta_k(j_k) W(j > k) U_{k-1}(j > k),
\]

where \( j > k \) denotes \( j_k : t \), i.e. \( j_k \) followed by \( j > k \). We expand the last factor of \( U_k(j > k) \) using Lemma 4.B.1, and bound

\[
U_k(j > k) = \sum_{j_k \in 0:k-1} \beta_k(j_k) \lambda_{j_k}^{t/s}(x_R_t(j > k) | x_{j_k}) U_{k-1}(j > k)
\]

\[
\geq \sum_{j_k : l(j > k)} \beta_k(j_k) \lambda_{j_k}^{t/s}(x_R_t(j > k) | x_{j_k}) U_{k-1}(j > k).
\]

Observe that \( R_t(j > k) = R_t(j > k) \) for all \( 1 \leq t < k \) except \( t = j_k \). There are two cases. If \( j_k = 0 \), then

\[
U_{k-1}(j > k) = U_{k-1}(j > k) \quad \text{and} \quad W(j > k) \lambda_{j_k}^{t/s}(x_R_t(j > k)) = W(j > k).
\]

On the other hand if \( j_k > 0 \) then \( W(j > k) = W(j > k) \). For consistent \( j \geq k \), \( U_{k-1}(j > k) \) contains the factor \( \lambda_{j_k}^{t/s}(x_{j_k}) \), which implies that

\[
\lambda_{j_k}^{t/s}(x_R_t(j > k) | x_{j_k}) U_{k-1}(j > k) = U_{k-1}(j > k).
\]

\[
\square
\]

4.C Invariance

Proof of Theorem 4.5.1. Let \( \mu^1 \) and \( \mu^2 \) be distributions on \( Q^1 \) and \( Q^2 \). We overload notation, and write \( \mu^1 \) and \( \mu^2 \) for the EHMMs \( A^1 \) and \( A^2 \) with initial distribution replaced by \( \mu^1 \) and \( \mu^2 \). Recall that \( \mu^1 \) and \( \mu^2 \) are equivalent if \( \mu^1(e_{1:T}) = \mu^2(e_{1:T}) \) for all \( e_{1:T} \). Thus, \( A^1 \) and \( A^2 \) are equivalent iff \( p^1_{c} \) and \( p^2_{c} \) are equivalent.

To prove the theorem, we need to prove that equivalence is preserved by all the operations that EPP performs, i.e. taking mixtures, performing loss update and performing state evolution. Mixtures of equivalent distributions are equivalent, since mixing and marginalisation commute. For loss update, note that \( p^1_c(x_1) = \mu^1(x_1 | e_{1:T}) = \mu^2(x_1 | e_{1:T}) \) for all \( p^1_c \) and all \( e_{1:T} \). Finally, for state evolution, the claim follows from \( (\mathrm{p}_\rightarrow \circ \mu)(e_{1:T}) = \mu(E_{2:T+1} = e_{1:T}) \). \[ \square \]