Logic, algebra and topology: investigations into canonical extensions, duality theory and point-free topology
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In this chapter, we provide a perspective on the toolkit of canonical extensions of lattices and order-preserving maps with an eye to connections with domain theory. This leads to new results and new insights into the reasons why and when canonical extensions work.

Canonical extensions were introduced by Jónsson and Tarski in 1951 [58, 59], as part of the representation theory for Boolean algebras with operators (BAOs), which play an important role in algebraic logic. Examples of BAOs considered by Jónsson & Tarski were e.g. relation algebras, cylindric algebras and closure algebras. Another example, which was not immediately recognized, is the class of modal algebras, which are used to study modal logic via algebra. The connection between BAOs and modal logic, or rather the connection between BAOs and Kripke semantics for modal logic via Stone duality became more widely recognized in the 1970s via the work of Thomason [87] and Goldblatt [46]; see [48] for historical context. The algebraic approach to BAO representation theory via the purely algebraic theory of canonical extensions was revived in the 1990s and 2000s in the work of Jónsson, Gehrke and Harding [57, 38, 34, 39] and De Rijke and Venema [31], and this approach has remained active since. Moreover canonical extensions have been generalized from a representation theory for Boolean algebras with operators to a more general representation theory toolkit for lattice-based algebras. We will wait with defining canonical extensions of lattice-based algebras until Chapter 3. In this chapter, we will introduce the reader to two important parts of the canonical extensions toolkit: canonical extensions of lattices and canonical extensions of order-preserving maps between lattices. Moreover, while doing so we will demonstrate the utility of methods from domain theory in relation to canonical extensions.

Domain theory was pioneered by D.S. Scott, with the aim to “give a mathematical semantics for high-level computer languages” [80]. It has been used to study subjects as diverse as recursive equations, semantics for untyped λ-calculus,
computability and partial information (see [1]). In this chapter we will borrow several techniques from the study of domain theory and continuous lattices when studying canonical extensions:

- Order topologies. Order topologies are used heavily in domain theory, and they occur very naturally when studying canonical extensions [39]. We offer the most extensive treatment of the topological properties of canonical extensions to date, and we present two results in §2.1.3 which make clear that the topologies on canonical extensions are both central, even defining, and natural.

- Filter and ideal completions. Many results about canonical extensions can be understood by looking at an intermediate level of filters and ideals, rather than only at the canonical extensions themselves [44]. In fact, the filter and ideal completion functors play a central role in the theory of canonical extensions of order-preserving maps — a fact which was foreshadowed in [41]. We revisit and expand upon the results of [34], showing how the filter completion and the ideal completion play an important role ‘under the hood’.

- Dcpo presentations. Directed complete partial orders are central in domain theory. We already indicated that the canonical extension is intimately connected with the filter completion and the ideal completion. In fact, we can present the canonical extension of a lattice $L$ as a dcpo generated by the filter completion $\mathcal{F}L$.

This chapter is organized as follows. In §2.1, we introduce canonical extensions of lattices, both classically and using a new topological characterization in §2.1.3. In §2.2, we develop the theory of canonical extensions of order-preserving maps with an emphasis on the role of the filter and ideal completion. Finally, in §2.3 we present an alternative characterization of canonical extensions using dcpo presentations. Furthermore, at the end of each section we provide a discussion of the contributions in that section and suggestions for further work.

### 2.1 Canonical extension via filters, ideals and topology

In this section we want to introduce the canonical extension of a bounded lattice, which is a well-studied lattice completion, together with an improved topological perspective on this completion. The key to this topological perspective lies in understanding the role that the ideal and the filter completion play with respect to the canonical extension.

In §2.1.1, we introduce the classical definition of canonical extensions of lattices. In §2.1.2, we will introduce several important topologies on partial orders and
lattice completions. In §2.1.3, we will present the two main new results of this section, which characterize canonical extensions of lattices topologically, and which identify the characterizing topologies of canonical extensions. Finally, in §2.1.4, we present assorted additional properties of canonical extensions of lattices that will be of use later on.

2.1.1 The canonical extension of a lattice

In this subsection, we will introduce the canonical extension $L^\delta$ of a lattice $L$. We will define a concrete construction on a given lattice, using filters and ideals, and we will then state a uniqueness result which tells us that any completion of $L$ satisfying certain abstract order-theoretic properties is in fact isomorphic to $L^\delta$. From that point on, we will no longer concern ourselves with the actual concrete construction of $L^\delta$; rather, we will show how one can understand $L^\delta$ through its abstract characterization.

Overlapping sets, filters and ideals

Before we go ahead and introduce the canonical extension, we would like to introduce a very elegant concept from constructive mathematics. We will often want to talk about sets $U, V$ which have a non-empty intersection, i.e. $U \cap V \neq \emptyset$. We would like to think of this relation on sets as a positive property.

2.1.1. Definition. Given two sets $U, V \subseteq X$, we write $U \not\not V$ ($U$ and $V$ overlap) if $U \cap V \neq \emptyset$.

What makes the overlapping relation interesting is that it interacts nicely with several operations and relations on sets.

2.1.2. Lemma. Let $X$ be a set and let $U, V \subseteq X$ such that $U \not\not V$.

1. If $U', V' \subseteq X$ such that $U \subseteq U'$ and $V \subseteq V'$, then also $U' \not\not V'$.

2. If $f : X \to Y$ is a function to another set $Y$, then also $f[U] \not\not f[V]$.

Consequently, if $f : P \to Q$ is an order-preserving map and if $F \in \mathcal{F}P$, $I \in \mathcal{I}P$ such that $F \not\not I$, then also $\mathcal{F}f(F) \not\not \mathcal{I}f(I)$.

Proof Parts (1) and (2) are elementary. For the last part it suffices to recall that $\mathcal{F}f(F) := \uparrow f[F]$ and $\mathcal{I}f(F) := \downarrow f[F]$.

The following lemma provides a further indication that the overlapping relation also interacts in a nice way with filters and ideals. Recall from §A.5.1 that given a poset $P$, $\mathcal{I}P := \langle \text{Idl } P, \subseteq \rangle$ is the ideal completion of $P$, and dually that $\mathcal{F}P := \langle \text{Filt } P, \supseteq \rangle$ is the filter completion of $P$. 

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2.1.3. Lemma. Let $\mathcal{P}$ be a poset. If $F \in \mathcal{F}\mathcal{P}$ and $I \in \mathcal{I}\mathcal{P}$ such that $F \not\triangleright I$, then $\downarrow (F \cap I) = I$ and $\uparrow (F \cap I) = F$.

Proof We only show the first case, since the other follows by order duality. Take $y \in I$, we will show that there exists $z \in F \cap I$ such that $y \leq z$. Since $F \not\triangleright I$, there exists $x \in F \cap I$. Because $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$. Because $x \in F$ and $F$ is an upper set, $z \in F$. But then $y \leq z \in F \cap I$, so that $I \subseteq \downarrow (F \cap I)$. The other inclusion follows immediately from the fact that $F \cap I \subseteq I$ and $I$ is a lower set. 

Existence and uniqueness of the canonical extension

We will now present the canonical extension first as a concrete construction on lattices, and later as a completion of lattices which is unique up to isomorphism. The particular concrete construction we have chosen, which seems to go back to [44], is not the only possible one. In light of the uniqueness theorem however, the concrete construction of the canonical extension we choose now is not particularly important. Our construction will be a two-stage construction on a given lattice $\mathbb{L}$. The first stage in the construction consists of creating a pre-order. The order relation we use goes back to [44].

2.1.4. Definition. Let $\mathbb{L}$ be a lattice. We define a structure $\text{Int} \mathbb{L} := (\mathcal{F}\mathbb{L} \cup \mathcal{I}\mathbb{L}, \sqsubseteq)$, where for all $F, F' \in \mathcal{F}\mathbb{L}$ and $I, I' \in \mathcal{I}\mathbb{L},$

\[
\begin{align*}
F \subseteq I & \quad \text{if } F \not\triangleright I; \\
F \subseteq F' & \quad \text{if } F \supseteq F'; \\
I \subseteq I' & \quad \text{if } I \subseteq I'; \\
I \subseteq F & \quad \text{if } I \times F \subseteq \leq \mathbb{L},
\end{align*}
\]

i.e. $I \subseteq F$ iff for all $a \in I$ and for all $b \in F$, we have $a \leq b$.

The following fact is well-known, cf. [44, p. 11].

2.1.5. Lemma. Let $\mathbb{L}$ be a lattice. Then $\text{Int} \mathbb{L}$ is a pre-order.

Proof It suffices to show that $\sqsubseteq$ is transitive. For this, we need to make a case distinction. Let $F, F' \in \mathcal{F}\mathbb{L}$ and $I, I' \in \mathcal{I}\mathbb{L},$

\[
\begin{align*}
F \subseteq F' \subseteq I & \Rightarrow F \subseteq I & \quad \text{by Lemma 2.1.2}; \\
F \subseteq I \subseteq I' & \Rightarrow F \subseteq I' & \quad \text{idem}; \\
I \subseteq I' \subseteq F & \Rightarrow I \subseteq F & \quad \text{easy},
\end{align*}
\]
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since if $I \subseteq I'$ and $I \times F \subseteq \leq_L$ then $I \times F \subseteq I' \times F \subseteq \leq_L$;

$I \subseteq F \subseteq F' \Rightarrow I \subseteq F'$ 

$\text{idem;}$

$I \subseteq F \subseteq I' \Rightarrow I \subseteq I'$ 

$\text{see below;}$

$F \subseteq I \subseteq F' \Rightarrow F \subseteq F'$ 

$\text{see below.}$

The last two cases have essentially the same proof; we only discuss the latter. If $F \subseteq I \subseteq F'$, i.e. if $F \nmid I$ and $I \times F' \subseteq \leq_L$, then we need to show that also $F \subseteq F'$, i.e. that $F \supseteq F'$. Let $a \in F'$. Since $F \nmid I$, there exists $b \in F \cap I$. Since $I \times F' \subseteq \leq_L$, we have $b \leq a$. But then $a \in F'$; it follows that $F \supseteq F'$.

Note that in other places where $\text{Int}\ L$ is introduced [44, 32, 41], one also takes a quotient of the pre-order to make it into a partial order. We do not bother with this because the pre-order is flattened into a partial order in the second stage of constructing the canonical extension anyway. This second stage consists of taking the MacNeille completion (see §A.5.2) of $\text{Int}\ L$.

2.1.6. Definition. We define the canonical extension of $L$ to be $L^\delta := \overline{\text{Int}\ L}$, with $i : \text{Int}\ L \to L^\delta$ the embedding of the MacNeille completion. We map $L$ to $L^\delta$ by setting $e_L : a \mapsto i(\downarrow a)$.

We now have a way to construct a complete lattice $L^\delta$, given a lattice $L$, and a function $e : L \to C$. We see below that $e_L$ is in fact a lattice completion, and that we can characterize it up to isomorphism of completions.

2.1.7. Definition. Given a (bounded) lattice $L$ and a complete lattice $C$, we call a lattice embedding $e : L \to C$ a completion of $L$. We say two completions of $e : L \to C$ and $e' : L \to C'$ are isomorphic if there exists an isomorphism $h : C \to C'$ such that $he = e'$.

Before we state the basic uniqueness result concerning canonical extensions, we would like to point out that any completion $e : L \to C$ induces two auxiliary maps $e^\mathcal{F} : \mathcal{F}L \to C$ and $e^\mathcal{I} : \mathcal{I}L \to C$, if we exploit the fact that a complete lattice $C$ is simultaneously a dcpo and a co-dcpo.

2.1.8. Definition. Given a lattice completion $e : L \to C$, we define $e^\mathcal{I} : \mathcal{I}L \to C$ by $I \mapsto \bigvee I e[I]$. It is easy to see that $e^\mathcal{I} : \mathcal{I}L \to C$ is the unique Scott-continuous extension of $e : L \to C$, the existence of which is stipulated by Fact A.5.3.
Dually, we define $e^F : \mathcal{F} \mathbb{L} \to \mathbb{C}$ by $F \mapsto \bigwedge \{ e[F] \}$. We emphasize that $e^F(\downarrow x) = e^F(\uparrow x) = e(x)$ for all $x \in \mathbb{L}$. We refer to $\{ e^F(F) \mid F \in \mathcal{F} \mathbb{L} \}$ and $\{ e^I(I) \mid I \in \mathcal{I} \mathbb{L} \}$ as the filter and ideal elements of $\mathbb{C}$, respectively.\footnote{This is an abuse of language, since strictly speaking they are the filter and ideal elements of $e : \mathbb{L} \to \mathbb{C}$.}

Filter and ideal elements play a crucial role in defining and understanding the canonical extension.

**2.1.9. Fact ([41], Proposition 3.6).** Let $\mathbb{L}$ be a lattice, then the map $e_\mathbb{L} : \mathbb{L} \to \mathbb{L}^\delta$ defined above is a lattice embedding. Moreover, if $e : \mathbb{L} \to \mathbb{C}$ is a completion of $\mathbb{L}$ such that

1. for all $x \in \mathbb{C}$,
   $$x = \bigvee \{ e^F(F) \mid e^F(F) \leq x, F \in \mathcal{F} \mathbb{L} \} = \bigwedge \{ e^I(I) \mid e^I(I) \geq x, I \in \mathcal{I} \mathbb{L} \};$$

2. for all $F \in \mathcal{F} \mathbb{L}$, $I \in \mathcal{I} \mathbb{L}$, if $e^F(F) \leq e^I(I)$ then $F \nmid I$;

then there exists a (unique) isomorphism of completions $h : \mathbb{C} \to \mathbb{L}^\delta$, i.e. such that $he = e_\mathbb{L}$.

We will refer to a lattice completion $e : \mathbb{L} \to \mathbb{C}$ satisfying the conditions above as a canonical extension of $\mathbb{L}$. The first condition of Fact 2.1.9, which is traditionally called density, states that given a canonical extension $e : \mathbb{L} \to \mathbb{C}$,

- the filter elements of $\mathbb{C}$ are join-dense, and
- the ideal elements of $\mathbb{C}$ are meet-dense.

The second condition (traditionally known as compactness) could also have been stated as:

- for all $F \in \mathcal{F} \mathbb{L}$, $I \in \mathcal{I} \mathbb{L}$, $e^F(F) \leq e^I(I)$ iff $F \nmid I$,

since $F \nmid I$ implies that there is $a \in F \cap I$, so that $\bigwedge e[F] \leq e(a) \leq \bigvee e[I]$. From here on, we will simply refer to the density and compactness properties of the canonical extension instead of explicitly referring to Fact 2.1.9.

We conclude this subsection with a class of examples of canonical extensions which is both important and trivial. Recall that a poset $\mathbb{P}$ is said to satisfy the ascending chain condition (ACC) if for every countable chain $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_i \leq \cdots$, there exists a $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $x_k \leq x_n$.

**2.1.10. Fact.** Let $\mathbb{L}$ be a (bounded) lattice. The identity embedding $id_\mathbb{L} : \mathbb{L} \to \mathbb{L}$ is a canonical extension of $\mathbb{L}$ iff $\mathbb{L}$ satisfies both the ascending chain condition (ACC) and the descending chain condition (DCC).
2.1.11. Example. Examples of lattices satisfying ACC and DCC include finite lattices, and lattices such as $\mathbb{M}_\infty$, where

$$\mathbb{M}_\infty := \langle \{0, 1\} \cup \{a_n \mid n \in \mathbb{N}\}, \leq \rangle,$$

and $x \leq y$ iff $x = 0$ or $y = 1$, see Figure 2.1.

![Figure 2.1: The lattice $\mathbb{M}_\infty$ is a fixed point of the canonical extension](image)

2.1.12. Convention. If $L$ is a finite lattice, then we define $L^\delta := L$ in light of Fact 2.1.10.

2.1.2 Topologies on posets and completions

We will now introduce two families of topologies which are defined on posets, and one which is defined on completions. The Scott topologies and the interval topologies are defined on any poset. The $\delta$ topologies are defined on any lattice completion. All three families come in three kinds: one where every open set is an upper set, one where every open set is a lower set, and the join of these upper and lower topologies, where every basic open is an intersection of an open upper set and an open lower set.

2.1.13. Definition. Let $\mathbb{P} = \langle P, \leq \rangle$ be a poset.

- By $\iota^\uparrow(\mathbb{P}) := \langle \{P \setminus \downarrow x \mid x \in P\} \rangle$ we denote the upper interval topology of $\mathbb{P}$, and $\iota^\downarrow(\mathbb{P}) := \iota^\uparrow(\mathbb{P}^\text{op}).$

- By $\sigma^\uparrow(\mathbb{P})$ we denote the Scott topology on $\mathbb{P}$: $U \subseteq P$ is $\sigma^\uparrow$-open if $U$ is an upper set which is inaccessible by directed joins, or equivalently if $P \setminus U$ is a lower set closed under all existing directed joins. By $\sigma^\downarrow(\mathbb{P})$ we denote $\sigma^\downarrow(\mathbb{P}^\text{op}).$
Let $e : L \to C$ be a lattice completion. We define two topologies on $C$:

- $\delta^\uparrow(C) := \langle \{ \uparrow e^F | F \in \mathcal{F} L \} \rangle$ and $\delta^\downarrow(C) := \langle \{ \downarrow e^I | I \in \mathcal{I} L \} \rangle$.

Recall from §A.7 that if $\tau$ and $\tau'$ are topologies on a set $X$, then $\tau \vee \tau' := \langle \{ U \cap V | U \in \tau, V \in \tau' \} \rangle$ is the least topology on $X$ containing $\tau$ and $\tau'$. We define $\sigma(P) := \sigma^\uparrow(P) \vee \sigma^\downarrow(P)$ (the bi-Scott topology) and $\delta(C) := \delta^\uparrow(C) \vee \delta^\downarrow(C)$.

Below, if e.g. $f : C \to M$ is some map, we will say that $f$ is $(\delta^\uparrow, \sigma^\uparrow)$-continuous if $f : \langle C, \delta^\uparrow(C) \rangle \to \langle M, \sigma^\uparrow(M) \rangle$ is a continuous function.

**2.1.14. Remark.** Observe that given a completion $e : L \hookrightarrow C$, the filter and ideal elements of $C$ are often referred to as the ‘closed’ and ‘open’ elements of $C$ in the canonical extension literature (cf. [34, Lemma 3.3]). This makes our definitions of the $\delta^\uparrow$ and $\delta^\downarrow$ topologies equivalent with those in [39], where they are called $\sigma^\uparrow$ and $\sigma^\downarrow$ respectively.

Although we will not use it, it is worth noting that $\iota := \iota^\uparrow \vee \iota^\downarrow$ is the usual interval topology. For example, $\iota(\mathbb{R})$ is the topology generated by

$$\\{ \{ z \in \mathbb{R} | x < z < y \} | x, y \in \mathbb{R} \},$$

i.e. the usual topology on the real line.

**2.1.15. Lemma.** Let $e : L \to C$ be a lattice completion.

1. The following set is a base for $\delta^\uparrow(C)$:

$$\{ \uparrow \bigvee_{F \in S} e^F | S \subseteq \mathcal{F} L \text{ finite} \}. $$

2. The following set is a base for $\delta^\downarrow(C)$:

$$\{ \downarrow \bigwedge_{I \in T} e^I | T \subseteq \mathcal{I} L \text{ finite} \}. $$

3. The following set is a base for $\delta(C)$:

$$\{ \uparrow \bigvee_{F \in S} e^F \cap \downarrow \bigwedge_{I \in T} e^I | S \subseteq \mathcal{F} L, T \subseteq \mathcal{I} L \text{ finite} \}. $$

**Proof** It suffices to show that (1) holds. First, observe that if $S \subseteq \mathcal{F} L$, then since $C$ is complete, it follows by order theory that

$$\bigcap_{F \in S} \uparrow e^F = \uparrow \bigvee_{F \in S} e^F. \quad (2.1)$$

Next, observe that since $\{ \uparrow e^F | F \in \mathcal{F} L \}$ is a subbase for $\delta^\uparrow(C)$ by definition, it follows by general topology that

$$\{ \bigcap_{F \in S} \uparrow e^F | S \subseteq \mathcal{F} L \text{ finite} \}$$

is a base for $\delta^\uparrow(C)$. It now follows by (2.1) that (1) holds.

$\blacksquare$
2.1. Canonical extension via filters, ideals and topology

We are considering topologies defined on ordered sets, and the order plays an important role in defining these topologies. Below we will establish some elementary facts about the interaction between our topologies and the order.

2.1.16. Lemma. Let \( \mathbb{P} \) be a poset. If \( O \subseteq P \) is \( \sigma \)-open, then \( P \setminus O \) is closed under all existing directed joins and codirected meets.

Proof We only show the case of directed joins. Towards a contradiction, let \( S \subseteq P \setminus O \) be a directed set such that \( \bigvee S \in O \). Then by definition of \( \sigma \), there must exist a \( \sigma^\dagger \)-open \( U \subseteq \mathbb{P} \) and a \( \sigma^\dagger \)-open \( V \subseteq \mathbb{P} \) such that \( \bigvee S \in U \cap V \subseteq O \). Since \( U \) is Scott-open and \( \bigvee S \in U \), there is some \( x \in S \cap U \). Since \( V \) is a lower set, we also get \( x \in V \), which is a contradiction. \( \blacksquare \)

2.1.17. Lemma. Let \( \mathbb{P} \) be a poset and let \( e \colon \mathbb{L} \to \mathbb{C} \) be a lattice completion.

1. \( \iota^\uparrow(\mathbb{P}) \subseteq \sigma^\uparrow(\mathbb{P}) \) and \( \iota^\downarrow(\mathbb{P}) \subseteq \sigma^\downarrow(\mathbb{P}) \);
2. \( \{U \in \sigma(\mathbb{P}) \mid U \text{ is an upper set}\} = \sigma^\uparrow(\mathbb{P}) \);
3. \( \{U \in \delta(\mathbb{C}) \mid U \text{ is an upper set}\} = \delta^\uparrow(\mathbb{C}) \).

Consequently, an order-preserving map \( f \colon \mathbb{P} \to \mathbb{Q} \) is \( (\sigma, \sigma) \)-continuous iff it is both \( (\sigma^\dagger, \sigma^\dagger) \)-continuous and \( (\sigma^\dagger, \sigma^\dagger) \)-continuous.

Proof (1). This is easy to see: any subbasic \( \iota^\downarrow \)-open \( \downarrow x \) is also \( \sigma^\downarrow \)-open, since obviously \( \downarrow x \) is a lower set closed under directed joins.

(2). Suppose that \( U \) is a \( \sigma \)-open set such that \( U \) is an upper set. Then all we have to do is to show that \( U \) is \( \sigma^\dagger \)-open, is to show that \( P \setminus U \) is closed under directed joins. But this follows immediately from Lemma 2.1.16.

(3). Suppose that \( U \subseteq \mathbb{C} \) is a \( \delta \)-open upper set. To show that \( U \) is \( \delta^\dagger \)-open, it suffices to show that for every \( x \in U \), there exists a \( \delta^\dagger \)-open \( U' \subseteq U \) such that \( x \in U' \). Take \( x \in U \). Since \( U \) is \( \delta \)-open, by Lemma 2.1.15(3), there exist finite sets \( S \subseteq \mathbb{F}L \) and \( T \subseteq \mathbb{I}L \) such that
\[
x \in \uparrow \bigvee_{F \in S} e^F(F) \cap \bigwedge_{I \in T} e^I(I) \subseteq U.
\]
Then \( \bigvee_{F \in S} e^F(F) \in U \), so since \( U \) is an upper set,
\[
U' := \uparrow \bigvee_{F \in S} e^F(F) \subseteq U.
\]
By Lemma 2.1.15(1), \( U' \) is \( \delta^\dagger \)-open; since \( x \in U' \subseteq U \) was arbitrary it follows that \( U \) is \( \delta \)-open.

For the last claim of the lemma, suppose that \( f : \mathbb{P} \to \mathbb{Q} \) is order preserving. If \( f \) is both \( (\sigma^\dagger, \sigma^\dagger) \)-continuous and \( (\sigma^\dagger, \sigma^\dagger) \)-continuous, then it follows by general topology (Lemma A.7.3) that \( f \) is \( (\sigma^\dagger \vee \sigma^\dagger, \sigma^\dagger \vee \sigma^\dagger) \)-continuous, so since \( \sigma := \sigma^\dagger \vee \sigma^\dagger \), \( f \) is \( (\sigma, \sigma) \)-continuous. Conversely, if \( f \) is \( (\sigma, \sigma) \)-continuous and \( U \subseteq \mathbb{Q} \) is an upper set, then \( f^{-1}(U) \) is also an upper set since \( f \) is order-preserving. Now by part (2) above, \( f^{-1}(U) \) is \( \sigma^\dagger \)-open; since \( U \subseteq \mathbb{Q} \) was arbitrary, it follows that \( f \) is \( (\sigma^\dagger, \sigma^\dagger) \)-continuous. The argument for \( (\sigma^\dagger, \sigma^\dagger) \)-continuity is analogous. \( \blacksquare \)
When forming, say, the topology $\sigma = \sigma^1 \lor \sigma^1$ on a poset $\mathbb{P}$, we are creating many new open sets. A priori, it is possible that $\sigma$ contains new open upper sets which are not $\sigma^1$-open. The lemma above tells us that in the case of the $\sigma$ and $\delta$ topologies, this does not happen.

Recall that the $\sigma$ and $\nu$-topologies were defined using order duality: $\nu^1(\mathbb{P}) := \nu^1(\mathbb{P}^{op})$. The following lemma states that we could have done the same with the $\delta$ topologies.

\textbf{2.1.18. Lemma.} Let $e \colon L \to C$ be a lattice completion. Then the following topologies on $C$ coincide: $\delta^1(L) = \delta^1(L^{op})$. Consequently, $\delta(L) = \delta(L^{op})$.

\textbf{Proof} This follows easily from Fact A.5.2.

Of course, the above lemma also holds, by definition, for the Scott topology. At this point we come to a property which the Scott topology notoriously lacks.

\textbf{2.1.19. Lemma.} Let $e_1 \colon L_1 \to C_1$ and $e_2 \colon L_2 \to C_2$ be two lattice completions. Then the following topologies on $C_1 \times C_2$ coincide: $\delta^1(C_1) \times \delta^1(C_2) = \delta^1(C_1 \times C_2)$ and $\delta^1(C_1) \times \delta^1(C_2) = \delta^1(C_1 \times C_2)$.

\textbf{Proof} We will show that $\delta^1(C_1) \times \delta^1(C_2) = \delta^1(C_1 \times C_2)$. Recall that

$$
\delta^1(C_1 \times C_2) = \langle \{\uparrow_{C_1 \times C_2}(e_1 \times e_2)\rangle \times \langle \{\downarrow_{C_1 \times C_2}(e_1 \times e_2)\rangle | F \in \mathcal{F}(L_1 \times L_2) \rangle
$$

where the last equality follows from fact that $\mathcal{F}$ commutes with products (Fact A.5.4). Moreover, if $(F_1, F_2) \in \mathcal{F}(L_1 \times \mathcal{F}L_2)$ so that $(e_1^F(F_1), e_2^F(F_2)) \in C_1 \times C_2$, then again by Fact A.5.4 (applied to $C_1 \times C_2$), we see that

$$
\uparrow_{C_1 \times C_2}(e_1^F(F_1), e_2^F(F_2)) = \uparrow_{C_1}(e_1^F(F_1)) \times \uparrow_{C_2}(e_2^F(F_2))
$$

But we know from general topology that

$$
\delta^1(C_1 \times C_2) = \langle \{\uparrow_{C_1}(e_1^F(F_1)) \times \uparrow_{C_2}(e_2^F(F_2)) | F_1 \in \mathcal{F}L_1, F_2 \in \mathcal{F}L_2 \rangle
$$

so it follows that $\delta^1(C_1) \times \delta^1(C_2) = \delta^1(C_1 \times C_2)$. 

The above lemma is not true of the Scott topology: there exist lattices $L$, $M$ such that $\sigma^1(L \times M) \neq \sigma^1(L) \times \sigma^1(M)$ [45, Thm. II-4.11].

\textbf{2.1.3 Characterizing the canonical extension via the $\delta$-topology}

In this subsectoin, we present the two main new results of this section. Firstly, we prove a new, topological characterization theorem for the canonical extension. Secondly, we show how the $\delta$-topologies can be given a natural description as subspace topologies with respect to the Scott and co-Scott topology on superstructures of $\mathbb{L}^1$. 
The interaction between the $\delta$-topologies and the maps $e^{\mathcal{F}}$ and $e^{\mathcal{I}}$

We start with a lemma which tells us that the compactness property of canonical extensions is in fact a topological property.

2.1.20. Lemma. Let $e: \mathbb{L} \hookrightarrow \mathbb{C}$ be a completion of $\mathbb{L}$. Then the following are equivalent:

1. for all $F \in \mathcal{F}\mathbb{L}$ and $I \in \mathcal{I}\mathbb{L}$, $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$ iff $F \not\subseteq I$.

2. $e^{\mathcal{I}}: \mathcal{I}\mathbb{L} \rightarrow \mathbb{C}$ is $(\sigma^1, \delta^1)$-continuous and $e^{\mathcal{F}}: \mathcal{F}\mathbb{L} \rightarrow \mathbb{C}$ is $(\sigma^1, \delta^1)$-continuous.

Proof Assume (1) holds. Let $\uparrow e^{\mathcal{F}}(F)$ be a subbasic open set of $\delta^1$ (for some $F \in \mathcal{F}\mathbb{L}$). We will show that $U := (e^{\mathcal{I}})^{-1}(\uparrow e^{\mathcal{F}}(F))$ is Scott-open in $\mathcal{I}\mathbb{L}$. Since $e^{\mathcal{I}}$ is order-preserving, we see that $U$ must be an upper set. Now let $S \subseteq \mathcal{I}\mathbb{L}$ be directed; then $\bigvee S = \bigcup S$. If $\bigvee S \in U$, then

$$e^{\mathcal{I}}(\bigvee S) = e^{\mathcal{I}}(\bigcup S) \in \uparrow e^{\mathcal{F}}(F),$$

i.e. $e^{\mathcal{I}}(\bigcup S) \geq e^{\mathcal{F}}(F)$. It follows by (1) that there is $a \in F \cap \bigcup S$, i.e. there is an $I \in S$ such that $a \in F \cap I$. But then $F \not\subseteq I$, so by (1), it follows that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$, so that $I \in (e^{\mathcal{I}})^{-1}(\uparrow e^{\mathcal{F}}(F)) = U$. Since $S$ was arbitrary it follows that $U$ is Scott-open. The proof of the statement for $e^{\mathcal{F}}$ is the order dual of the above; it follows that (2) holds.

Conversely, assume that (2) holds and let $F \in \mathcal{F}\mathbb{L}$, $I \in \mathcal{I}\mathbb{L}$. If $F \not\subseteq I$ then there is $a \in F \cap I$, so we see that

$$e^{\mathcal{F}}(F) = \bigwedge e[F] \leq e(a) \leq \bigvee e[I] = e^{\mathcal{I}}(I).$$

Now suppose that $e^{\mathcal{F}}(F) \leq e^{\mathcal{I}}(I)$, so that $I \in U := (e^{\mathcal{I}})^{-1}(\uparrow e^{\mathcal{F}}(F))$. By (2), $U$ is Scott-open. Now since $I = \bigvee \downarrow a \in I$ is a directed join, it follows that there is some $a \in I$ such that $\downarrow a \in U$, i.e. $e^{\mathcal{I}}(\downarrow a) \in \uparrow e^{\mathcal{F}}(F)$. Equivalently, $e^{\mathcal{F}}(F) \in \downarrow e^{\mathcal{I}}(\downarrow a)$, so by an argument analogous to that above, there is some $b \in F$ such that $e^{\mathcal{F}}(\downarrow b) \leq e^{\mathcal{I}}(\downarrow a)$. But now

$$e(b) = e^{\mathcal{F}}(\downarrow b) \leq e^{\mathcal{I}}(\downarrow a) = e(a),$$

so that $b \leq a$. Since $a \in I$ and $I$ is a lower set, we also get $b \in I$, so that $F \not\subseteq I$. It follows that (1) holds.

The above lemma effectively translates one of the defining properties (compactness) of the canonical extension into a topological property of lattice completions. The following lemmas establish some topological properties of $e^{\mathcal{F}}$ and $e^{\mathcal{I}}$ which will be of use later. First, we show that $e^{\mathcal{I}}: \mathcal{I}\mathbb{L} \rightarrow \mathbb{C}$ is a Scott-continuous lattice homomorphism under certain topological assumptions.
2.1.21. **Lemma.** Let $\mathbb{L}$ be a bounded lattice and let $e: \mathbb{L} \hookrightarrow \mathbb{C}$ be a completion of $\mathbb{L}$.

1. $e^T: \mathcal{T}\mathbb{L} \rightarrow \mathbb{C}$ preserves all joins (including 0), and the top element 1.

2. If $e^T: \mathcal{T}\mathbb{L} \rightarrow \mathbb{C}$ is $(\sigma^1, \delta^1)$-continuous and $\delta^1$ is $T_0$, then $e^T: \mathcal{T}\mathbb{L} \rightarrow \mathbb{C}$ preserves finite meets.

**Proof** (1). Preservation of 0 and 1 follows from the fact that $e: \mathbb{L} \rightarrow \mathbb{C}$ is a bounded lattice embedding; if we look at e.g. the top element of $\mathcal{T}\mathbb{L}$, i.e. $\downarrow 1$, then $e^T(\downarrow 1) = \bigvee e[\downarrow 1] = e(1) = 1$. The argument for 0 is identical. Now for binary joins, recall that the join of two ideals $I_1, I_2 \in \mathcal{T}\mathbb{L}$ is $I_1 \vee I_2 := \downarrow \{a_1 \vee a_2 \mid a_1 \in I_1, a_2 \in I_2\}$. Now

$$e^T(I_1 \vee I_2) = \bigvee e[I_1 \vee I_2] = \bigvee \{e(a_1 \vee a_2) \mid a_1 \in I_1, a_2 \in I_2\} =$$

$$\bigvee \{e(a_1) \vee e(a_2) \mid a_1 \in I_1, a_2 \in I_2\} = \bigvee e[I_1] \vee \bigvee e[I_2] = e^T(I_1) \vee e^T(I_2),$$

where the penultimate equality follows from the fact that $I_1$ and $I_2$ are directed. Since $e^T$ is Scott-continuous by Fact A.5.3, it follows that $e^T$ preserves all joins.

(2). We will only consider binary meets. Assume that $\delta^1$ is $T_0$ and let $I_1, I_2 \in \mathcal{T}\mathbb{L}$. Since $e^T$ is order-preserving, we only need to show that $e^T(I_1) \land e^T(I_2) \leq e^T(I_1 \land I_2)$. Suppose not, then since $\delta^1$ is $T_0$ there must be some $\delta^1$-open $U$ such that $e^T(I_1) \land e^T(I_2) \notin U$ and $e^T(I_1 \land I_2) \notin U$. We see that also $e^T(I_1) \in U$ and $e^T(I_2) \in U$; moreover, without loss of generality we may assume $U = \bigcap_{1 \leq i \leq n} e^T(F_i)$ for some finite set $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}\mathbb{L}$. Since $e^T: \mathcal{T}\mathbb{L} \rightarrow \mathbb{C}$ is $(\sigma^1, \delta^1)$-continuous, $(e^T)^{-1}(U)$ is Scott-open. Now since $I_1 = \bigvee_{g \in I_1} \downarrow a \in (e^T)^{-1}(U)$ is a directed join, there must be some $a_1 \in I_1$ such that $e^T(\downarrow a_1) = e(a_1) \in U$, and similarly there must be some $a_2 \in I_2$ such that $e(a_2) \in U$. Since $U = \bigcap_{1 \leq i \leq n} e^T(F_i)$, it follows that for all $1 \leq i \leq n$, we have that $e(a_1), e(a_2) \in e^T(F_i)$, so also $e(a_1 \land a_2) = e(a_1 \land a_2) \in e^T(F_i)$. Since $i$ was arbitrary, it follows that $e(a_1 \land a_2) \in U$. Since we also have that $a_1 \land a_2 \in I_1 \land I_2$, it follows that

$$e^T(I_1 \land I_2) = \bigvee e[I_1 \land I_2] \geq e(a_1 \land a_2) \in U,$$

which is a contradiction since we assumed $U$ is an upper set not containing $e^T(I_1 \land I_2)$. It follows that $e^T(I_1 \land I_2) = e^T(I_1) \land e^T(I_2)$. \]

Next, we show that under the assumptions of Lemma 2.1.21, the $\delta^1$-topology on a completion $e: \mathbb{L} \rightarrow \mathbb{C}$ has a base of principal lower sets.

2.1.22. **Corollary.** Under the assumptions of Lemma 2.1.21, $\{\downarrow e^T(I) \mid I \in \mathcal{T}\mathbb{L}\}$ is not only a subbase for $\delta^1$, but in fact a base.

**Proof** We will show that $\{\downarrow e^T(I) \mid I \in \mathcal{T}\mathbb{L}\}$ is closed under finite intersections. Let $I_1, I_2 \in \mathcal{T}\mathbb{L}$, then

$$\downarrow e^T(I_1) \cap \downarrow e^T(I_2) = \downarrow (e^T(I_1) \land e^T(I_2)) = \downarrow e^T(I_1 \land I_2).$$
The $\delta$-topology determines the canonical extension

Now that we have proved the required technical results about the $\delta$-topologies, we can present the first main result of this section: we show how the $\delta$-topologies, which arise naturally on canonical extensions, in fact characterize it.

2.1.23. Theorem. Let $L$ be a bounded lattice and let $\epsilon : L \hookrightarrow C$ be a completion of $L$. Then $\epsilon : L \hookrightarrow C$ is a canonical extension of $L$ iff

1. $\epsilon^\uparrow : \mathcal{I}L \rightarrow C$ is $(\sigma^\uparrow, \delta^\uparrow)$-continuous and $\epsilon^\downarrow : \mathcal{F}L \rightarrow C$ is $(\sigma^\downarrow, \delta^\downarrow)$-continuous,

2. $\delta^\uparrow$ and $\delta^\downarrow$ are both $T_0$.

Proof First assume that $\epsilon : L \rightarrow C$ is a canonical extension of $L$. It follows from Lemma 2.1.20 that (1) holds. Moreover, if $x, y \in C$ and $x \nless y$, then there must be some $F \in \mathcal{F}L$ and $I \in \mathcal{I}L$ such that $\epsilon^\uparrow(F) \leq x$ and $\epsilon^\uparrow(F) \nless y$. It follows that $x \in \uparrow \epsilon^\uparrow(F)$ and $y \notin \uparrow \epsilon^\uparrow(F)$, so $\delta^\uparrow$ is $T_0$. The proof for $\delta^\downarrow$ is analogous; it follows that (2) holds.

Conversely, assume that (1) and (2) hold. It follows from Lemma 2.1.20 that condition (2) of Fact 2.1.9 holds. Moreover, if $x, y \in C$ and $x \nless y$, then since $\delta^\downarrow$ is $T_0$, there must exist some finite $S \subseteq \mathcal{F}L$ such that

$$x \in \bigcup_{F \in S} \epsilon^\uparrow(F) \nless y.$$ 

It follows that there must be some $F \in S$ such that $y \notin \uparrow \epsilon^\uparrow(F)$; now we see that $\epsilon^\uparrow(F) \leq x$ and $\epsilon^\uparrow(F) \nless y$. An analogous argument shows that since $\delta^\downarrow$ is $T_0$, there must be some $I \in \mathcal{I}L$ such that $y \leq \epsilon^\downarrow(I)$ and $x \nless \epsilon^\downarrow(I)$. Since $x, y \in C$ were arbitrary, it follows that (1) of Fact 2.1.9 holds, so $\epsilon : L \rightarrow C$ is a canonical extension of $L$.

Explaining the $\delta$-topology

We now present the second main result of this section, which sheds a different light on the definition of the $\delta$-topology. In particular, we will focus on the $\delta^\uparrow$-topology. We will show that there is a natural way to embed $L^\delta$ in $\mathcal{I}\mathcal{F}L$, and that the $\delta^\uparrow$-topology on $L^\delta$ is simply the Scott topology on $\mathcal{I}\mathcal{F}L$, restricted to $L^\delta$.

So how do we embed $L^\delta$ in $\mathcal{I}\mathcal{F}L$? The key insight is that since $\epsilon^\uparrow : \mathcal{F}L \rightarrow L^\delta$ preserves all finite joins (by Lemma 2.1.24), the map $\mathcal{I}\epsilon^\uparrow : \mathcal{I}\mathcal{F}L \rightarrow \mathcal{I}L^\delta$ has a left adjoint, namely $(\epsilon^\uparrow)^{-1} : \mathcal{I}L^\delta \rightarrow \mathcal{I}\mathcal{F}L$.

2.1.24. Lemma. Let $L, M$ be lattices and let $f : L \rightarrow M$ be a map preserving $\vee$ and $0$.

1. The inverse image function $f^{-1}$ maps ideals of $M$ to ideals of $L$;
2. \( \mathcal{I} f \downarrow f^{-1} \), i.e. for all \( I \in \mathcal{I}\mathbb{L} \), \( J \in \mathcal{I}\mathbb{M} \), we have
\[ \mathcal{I} f(I) \subseteq J \text{ iff } I \subseteq f^{-1}(J); \]

**Proof** (1). If \( J \in \mathcal{I}\mathbb{M} \), then \( f^{-1}(J) \) is a lower set since \( f \) is order-preserving. Moreover, if \( a, b \in f^{-1}(J) \), then \( f(a), f(b) \in J \), so \( f(a) \lor f(b) = f(a \lor b) \in J \), so that \( a \lor b \in f^{-1}(J) \). Finally, since \( 0 = f(0) \in J \), it follows that \( 0 \in f^{-1}(J) \), so that \( f^{-1}(J) \) is non-empty.

(2). If \( \mathcal{I} f(I) = \downarrow f[I] \subseteq J \), then for every \( a \in I \), \( f(a) \in J \), i.e. \( a \in f^{-1}(J) \). Conversely, if \( I \subseteq f^{-1}(J) \), then for every \( a \in I \), \( f(a) \in J \), i.e. \( f[I] \subseteq J \). Since \( J \) is a lower set, we also get \( \mathcal{I} f(I) = \downarrow f[I] \subseteq J \).

We now define \( g : \mathbb{L}^\delta \to \mathcal{I}\mathcal{F}\mathbb{L} \) as \( g := \left( e^\mathcal{F}\right)^{-1} \circ \downarrow_{\mathbb{L}^\delta} \). This map \( g \) will be the embedding that shows that \( \mathbb{L}^\delta \) is isomorphic to a subposet of \( \mathcal{I}\mathcal{F}\mathbb{L} \).

**2.1.25. Theorem.** Let \( e : \mathbb{L} \to \mathbb{L}^\delta \) be a canonical extension.

1. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{I}\mathcal{F}\mathbb{L} & \xrightarrow{\mathcal{I}e^\mathcal{F}} & \mathcal{I}\mathbb{L}^\delta \\
\downarrow_{\mathcal{F}\mathbb{L}} & & \downarrow_{\mathbb{L}^\delta} \\
\mathcal{F}\mathbb{L} & \xrightarrow{e^\mathcal{F}} & \mathbb{L}^\delta
\end{array}
\]

2. The composite \( g := \left( e^\mathcal{F}\right)^{-1} \circ \downarrow_{\mathbb{L}^\delta} \) is a \((\delta^1, \sigma^1)\)-continuous homeomorphic embedding.

**Proof** (1). We need to show that
\[ \mathcal{I} e^\mathcal{F} \circ \downarrow_{\mathcal{F}\mathbb{L}} = \downarrow_{\mathbb{L}^\delta} \circ e^\mathcal{F} \tag{2.2} \]
and that
\[ \left( e^\mathcal{F}\right)^{-1} \circ \downarrow_{\mathbb{L}^\delta} \circ e^\mathcal{F} = \downarrow_{\mathcal{F}\mathbb{L}}. \tag{2.3} \]

The validity of (2.2) follows from the fact that \( \downarrow \) is a natural transformation by Fact A.5.3(1). To see why (2.3) holds, first observe that since \( e^\mathcal{F} : \mathcal{F}\mathbb{L} \to \mathbb{L}^\delta \) preserves all finite joins by Lemma 2.1.21, it follows by Lemma 2.1.24(2) that \( \left( e^\mathcal{F}\right)^{-1} : \mathcal{I}\mathbb{L}^\delta \to \mathcal{I}\mathcal{F}\mathbb{L} \) is right adjoint to \( \mathcal{I} e^\mathcal{F} \). Since \( \mathcal{I} e^\mathcal{F} \) is an order-embedding by Fact A.5.3(5), it follows by Fact A.3.3(2) that
\[ \left( e^\mathcal{F}\right)^{-1} \circ \mathcal{I} e^\mathcal{F} = \text{id}_{\mathcal{I}\mathcal{F}\mathbb{L}}. \tag{2.4} \]

Now we see that
\[
\begin{align*}
\left( e^\mathcal{F}\right)^{-1} \circ \downarrow_{\mathbb{L}^\delta} \circ e^\mathcal{F} &= \left( e^\mathcal{F}\right)^{-1} \circ \mathcal{I} e^\mathcal{F} \circ \downarrow_{\mathcal{F}\mathbb{L}} \\
&= \downarrow_{\mathcal{F}\mathbb{L}} \quad \text{by (2.2)}, \quad \text{by (2.4)}.
\end{align*}
\]
We now define $g := (e^F)^{-1} \circ \downarrow_{\mathbb{L}^\delta}$; observe that
\[
g(x) = (e^F)^{-1}(\downarrow_{\mathbb{L}^\delta} x) = \{ F \in \mathcal{F} \mathbb{L} \mid e^F(F) \leq x \},
\] (2.5)
which is an ideal of $\mathcal{F} \mathbb{L}$. This map will be the embedding of $\mathbb{L}^\delta$ into $\mathcal{I} \mathcal{F} \mathbb{L}$. Conversely, there is a natural map from $\mathcal{I} \mathcal{F} \mathbb{L}$ to $\mathbb{L}^\delta$: since $e^F : \mathcal{F} \mathbb{L} \rightarrow \mathbb{L}^\delta$ is an order preserving map from $\mathcal{F} \mathbb{L}$ to a dcpo $\mathbb{L}^\delta$, the universal property of ideal completion tells us that there exists a unique Scott-continuous map $(e^F)^I : \mathcal{I} \mathcal{F} \mathbb{L} \rightarrow \mathbb{L}^\delta$ such that $(e^F)^I \circ \downarrow_{\mathbb{L}^\delta} = e^F$; namely
\[
(e^F)^I : I \mapsto \bigvee \{ e^F(F) \mid F \in I \},
\] (2.6)
where $I \in \mathcal{I} \mathcal{F} \mathbb{L}$.

Now observe that $(e^F)^I \circ g = \text{id}_{\mathbb{L}^\delta}$: take $x \in \mathbb{L}^\delta$, then
\[
(e^F)^I \circ g(x) = \bigvee \{ e^F(F) \mid F \in g(x) \} \quad \text{by (2.6)},
\]
\[
= \bigvee \{ e^F(F) \mid e^F(F) \leq x \} \quad \text{by (2.5)},
\]
\[
= x \quad \text{by join-density of filter elements}.
\]

Now it is easy to see that $g$ is an order embedding:
\[
g(x) \leq g(y) \quad \Rightarrow \quad (e^F)^I \circ g(x) \leq (e^F)^I \circ g(y)
\]
\[
\Rightarrow \quad x \leq y
\]
\[
\text{since } (e^F)^I \text{ is order-preserving},
\]
\[
\text{since } (e^F)^I \circ g = \text{id}_{\mathbb{L}^\delta}.
\]

We will now show that $g : \mathbb{L}^\delta \rightarrow \mathcal{I} \mathcal{F} \mathbb{L}$ is a $(\delta^1, \sigma^1)$-continuous homeomorphic embedding, meaning that for every $\delta^1$-open $U \subseteq \mathbb{L}^\delta$ there exists a $\sigma^1$-open $U' \subseteq \mathcal{I} \mathcal{F} \mathbb{L}$ such that $U = g^{-1}(U')$. It suffices to show this for the case that $U$ is a basic open, i.e. for the case that $U = \uparrow_{\mathbb{L}^\delta} e^F(F)$ for an arbitrary $F \in \mathcal{F} \mathbb{L}$. Since $F \in \mathcal{F} \mathbb{L}$, we know that $\downarrow_{\mathcal{F} \mathbb{L}} F$ is a compact element of $\mathcal{I} \mathcal{F} \mathbb{L}$, so that $\uparrow_{\mathcal{I} \mathcal{F} \mathbb{L}}(\downarrow_{\mathbb{L}^\delta} F)$ is $\sigma^1$-open. Now
\[
g^{-1}(\uparrow_{\mathcal{I} \mathcal{F} \mathbb{L}} \downarrow_{\mathbb{L}^\delta} F)
\]
\[
= g^{-1}(\uparrow_{\mathcal{I} \mathcal{F} \mathbb{L}} g \circ e^F(F))
\]
\[
= \{ y \in \mathbb{L}^\delta \mid g \circ e^F(x) \leq g(y) \}
\]
\[
= \{ y \in \mathbb{L}^\delta \mid e^F(F) \leq y \}
\]
\[
= \uparrow_{\mathbb{L}^\delta} e^F(F).
\]

It follows that $g$ is a homeomorphic embedding. \qed
The fact that \( g : \mathbb{L}^\delta \to \mathcal{I} \mathcal{F} \mathbb{L} \) is a \((\delta^!, \sigma^!\))-homeomorphic embedding tells us two things. Firstly, that \( \mathbb{L}^\delta \) is isomorphic to a subposet of \( \mathcal{I} \mathcal{F} \mathbb{L} \). Secondly, this tells us that the \( \delta^! \)-topology is precisely the topology that \( \mathbb{L}^\delta \) inherits as a subspace of \( \mathcal{I} \mathcal{F} \mathbb{L} \), where the latter is endowed with the \( \sigma^!\)-topology. Naturally, this result dualizes if we view \( \mathbb{L}^\delta \) as a subposet of \( \mathcal{F} \mathcal{I} \mathbb{L} \).

### 2.1.4 Basic properties of the canonical extension

In this section, we will introduce assorted basic properties of canonical extensions which will be of use later on. The first observations concern the interaction with products and the operation of taking the order dual of a lattice. After that, we will prove certain topological and order-theoretical properties of the maps \( e^\mathcal{I} \) and \( e^{\mathcal{F}} \) and of the \( \delta \)-topologies. In particular, we will see that canonical extensions satisfy two distributive laws with respect to joins of filter elements and meets of ideal elements. We conclude the subsection with a result about the internal structure of the lattices that arise as canonical extensions.

We begin by showing that canonical extensions commute with finite products and order duals of lattices.

#### 2.1.26. Lemma

Let \( e_1 : \mathbb{L}_1 \to \mathbb{L}_1^\delta \) and \( e_2 : \mathbb{L}_2 \to \mathbb{L}_2^\delta \) be canonical extensions.

1. \( e_1 \times e_2 : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \) is a canonical extension of \( \mathbb{L}_1 \times \mathbb{L}_2 \).

2. \( e_1^{op} : \mathbb{L}_1^{op} \to (\mathbb{L}_1^\delta)^{op} \) is a canonical extension of \( \mathbb{L}_1^{op} \).

**Proof** (1). We will verify that \( e_1 \times e_2 : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \) satisfies the topological conditions of Theorem 2.1.23. Since the product of two \( T_0 \) spaces is again \( T_0 \), it follows from Lemma 2.1.19 that both \( \delta^!(\mathbb{L}_1^\delta \times \mathbb{L}_2^\delta) \) and \( \delta^!(\mathbb{L}_1 \times \mathbb{L}_2) \) are \( T_0 \).

Now since \( e_i^\mathcal{I} : \mathcal{I} \mathbb{L}_i \to \mathbb{L}_i^\delta \) is \((\sigma^!, \delta^!\))-continuous for \( i = 1, 2 \), it follows from Fact A.5.4, Lemma 2.1.19 and general topology that

\[
e_1^\mathcal{I} \times e_2^\mathcal{I} : \mathcal{I} \mathbb{L}_1 \times \mathcal{I} \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \text{ is } (\sigma^!, \delta^!\text{-continuous)}.
\]

Let \( h : \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \to \mathcal{I} \mathbb{L}_1 \times \mathcal{I} \mathbb{L}_2 \) be the order-isomorphism witnessing that \( \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \simeq \mathcal{I} \mathbb{L}_1 \times \mathcal{I} \mathbb{L}_2 \); observe that \( h \) is \((\sigma^!, \delta^!)\)-continuous. Now consider the diagram in Figure 2.2. The upper left triangle commutes by Fact A.5.4, the upper right triangle commutes by the universal property of \( \mathcal{I}_1 \times \mathcal{L}_2 : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathcal{I}(\mathbb{L}_1 \times \mathbb{L}_2) \) (Fact A.5.3), and the lower triangle commutes by definition of \((e_1 \times e_2)^\mathcal{I}\). Now since \((e_1^\mathcal{I} \times e_2^\mathcal{I}) \circ h \) is \((\sigma^!, \delta^!)\)-continuous, so is \((e_1 \times e_2)^\mathcal{I}\), which is what we needed to show. The argument showing that \((e_1 \times e_2)^\mathcal{F} : \mathcal{F}(\mathbb{L}_1 \times \mathbb{L}_2) \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \) is \((\sigma^!, \delta^!)\)-continuous is identical; it follows by Theorem 2.1.23 that \( e_1 \times e_2 : \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \) is a canonical extension of \( \mathbb{L}_1 \times \mathbb{L}_2 \).

(2). This follows readily from the order duality between filters and ideals and Lemma 2.1.18.
The fact that canonical extension commutes with taking finite products is in fact an instance of the more powerful result that canonical extensions commute with Boolean products (see §A.6). An alternative reference for the following result is [43, Theorem 5.1].

2.1.27. Fact ([34]). Let \( L \) be a lattice and let \((p_x: L \to M_x)_{x \in X}\) be a Boolean product decomposition of \( L \). Then \( L^\delta \cong \prod_X M_x^\delta \).

Parts (1) and (2) of the following lemma can be found in [41]. Part (3) however, which we will use very frequently, is new. In light of Theorem 2.1.25, part (3) below is perhaps not that surprising.

2.1.28. Lemma. Let \( e: L \to L^\delta \) be the canonical extension of a bounded lattice \( L \).

1. (a) \( e^\mathcal{F}: \mathcal{F}L \to L^\delta \) is an \( \vee, \wedge \)-embedding;
   
   (b) \( e^\mathcal{I}: \mathcal{I}L \to L \) is an \( \wedge, \vee \)-embedding;

2. for all \( x \in L^\delta \),
   
   (a) \( \{e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\} \) is directed;
   
   (b) \( \{e^\mathcal{I}(I) \mid x \leq e^\mathcal{I}(I)\} \) is co-directed;

3. (a) \( \sigma^\uparrow(L^\delta) \subseteq \delta^\uparrow(L^\delta) \);
   
   (b) \( \sigma^\downarrow(L^\delta) \subseteq \delta^\downarrow(L^\delta) \);
   
   (c) \( \sigma(L^\delta) \subseteq \delta(L^\delta) \);

4. \( e[L] \) is dense in \( \langle L^\delta, \delta(L^\delta) \rangle \) and in \( \langle L^\delta, \sigma(L^\delta) \rangle \).

Proof (1). We will only prove part (a). It follows from (the order dual of) Lemma 2.1.21 that \( e^\mathcal{F} \) is a \( \vee, \wedge \)-homomorphism. To show that it is an embedding, we will show that \( e^\mathcal{F} \) is order-reflecting. Assume that \( F \nsubseteq F' \); we will show that \( e^\mathcal{F}(F) \nsubseteq e^\mathcal{F}(F') \). By assumption, there exists \( a \in F' \setminus F \), so that \( F \cap \downarrow a = \emptyset \).
and $F' \upharpoonright a$. It follows by the compactness property of canonical extension that $e^\delta(F) \not\leq e^\delta(\downarrow a)$ and $e^\delta(F') \leq e^\delta(\downarrow a)$. Now by the meet-density of ideal elements, it follows that $e^\delta(F) \not\leq e^\delta(F')$.

(2) We only show part (a). This follows from (1), since $\downarrow x$ is an ideal. \[ \{e^\delta(F) \mid e^\delta(F) \leq x\} \]

(3) Suppose $U \subseteq \mathbb{L}^\delta$ is Scott-open and that $x \in U$. Then $x = \bigvee \{e^\delta(F) \mid e^\delta(F) \leq x\}$ is a directed join by (1), so there must be some $F$ such that $e^\delta(F) \leq x$ and $e^\delta(F) \in U$. It follows that $x \in \uparrow e^\delta(F) \subseteq U$; hence $U$ is $\delta$-open.

(4) Consider a non-empty basic open $U$ of $(\mathbb{L}^\delta, \sigma(\mathbb{L}^\delta))$, i.e. there are $F \in \mathcal{F} \mathbb{L}$, $I \in \mathcal{I} \mathbb{L}$ such that $U = \{x \in \mathbb{L}^\delta \mid e^\delta(F) \leq x \leq e^\delta(I)\}$. Since $U \neq \emptyset$, it must be that $e^\delta(F) \leq e^\delta(I)$, so that $F \not\uparrow I$. Take $a \in F \cap I$, then $\bigwedge \{e[F] \leq e(a) \leq \bigvee \{e[I]\}$ so that $e(a) \in U$. It follows that $e[L]$ is dense in $(\mathbb{L}^\delta, \delta(\mathbb{L}^\delta))$. Since $\sigma \subseteq \delta$ by (3), it is also the case that $e[L]$ is dense in $(\mathbb{L}^\delta, \sigma(\mathbb{L}^\delta))$. 

The distributive law below, which is similar to Lemma 3.2 of [34], is a very powerful result. We will in fact use it to characterize the canonical extension as a dcpo in §2.3.

**2.1.29. Lemma.** Let $e: \mathbb{L} \to \mathbb{L}^\delta$ be the canonical extension of a lattice $\mathbb{L}$. Then for all $S \subseteq \mathcal{F} \mathbb{L}$ and $S' \subseteq \mathcal{I} \mathbb{L}$,

\[ \bigvee \{e^\delta(F) \mid F \in S\} = \bigwedge \{e^\delta(I) \mid \forall F \in S, F \not\uparrow I\} \]

and

\[ \bigwedge \{e^\delta(I) \mid I \in S'\} = \bigvee \{e^\delta(F) \mid \forall I \in S', F \not\uparrow I\}. \]

**Proof** We only prove the first statement. Since the ideal elements of $\mathbb{L}^\delta$ are meet-dense, we see that

\[ \bigvee \{e^\delta(F) \mid F \in S\} = \bigwedge \{e^\delta(I) \mid \bigvee_{F \in S} e^\delta(F) \leq e^\delta(I)\}. \]

Now observe that $\bigvee_{F \in S} e^\delta(F) \leq e^\delta(I)$ iff for all $F \in S$, $e^\delta(F) \leq e^\delta(I)$ iff for all $F \in S, F \not\uparrow I$. The (first) statement of the lemma follows.

We conclude this subsection with a well-known result from the canonical extension literature, which plays an important role in the duality theory for canonical extensions. Let $\mathbb{L}$ be a complete lattice. Recall that an element $p \in \mathbb{L}$ is called completely join-irreducible if for all $S \subseteq \mathbb{L}$ such that $p = \bigvee S$, there exists $a \in S$ such that $p = a$. We denote the set of completely join-irreducible elements of $\mathbb{L}$ by $J^\infty(\mathbb{L})$. Completely meet-irreducible elements are defined dually; we denote them by $M^\infty(\mathbb{L})$. The following fact, which is related to Stone duality, requires the Axiom of Choice.

**2.1.30. Fact ([34], Lemma 3.4).** Let $\mathbb{L}$ be a lattice. Then $\mathbb{L}^\delta$ is join-generated by $J^\infty(\mathbb{L}^\delta)$ and meet-generated by $M^\infty(\mathbb{L}^\delta)$. 
2.1.5 Conclusions and further work

This section served two purposes: to make the reader familiar with the classical definition and well-known properties of the canonical extension, and to add a number of fundamental topological results on canonical extensions. The work of Gehrke and Jónsson [39] has shown that the topological perspective on canonical extensions is worthwhile, however the basic topological properties of canonical extensions in the general setting of (not necessarily distributive) lattices have not previously been studied. One fundamental difference between the distributive and the non-distributive settings is that the Scott topology and the topology generated by principal up-sets of completely join irreducibles no longer coincide.

All the results in §2.1.1 are known from the work of Gehrke and Harding [34], save perhaps the small technical results concerning the overlap relation. It should be noted however that the emphasis on the auxiliary maps \( e^F : F L \to L^\delta \) and \( e^I : I L \to L^\delta \) is a departure from the view on canonical extensions furthered in [34, 39].

In §2.1.3 we presented results which were first reported at TACL 2009 in Amsterdam. Both the topological characterization theorem (Theorem 2.1.23) and the result which casts the \( \delta \)-topologies as subspace topologies (Theorem 2.1.25) were previously unknown. The topological characterization theorem was inspired by the work of Theunissen and Venema [86] on MacNeille completions of lattice-based algebras.

Further work

- It would be very interesting to see if Fact 2.1.27, which deals with canonical extensions of Boolean products of lattices, can be given a new proof reducing it to a statement about the ideal completion and filter completion of lattices.

- It would also be interesting to see if there is a version of the topological characterization theorem (Theorem 2.1.23) which does not rely on \( e^F \) and \( e^I \), while still giving a topological characterization of \( L^\delta \).

- An alternative approach to canonical extensions using filters and ideals has been developed by Gehrke, Jansana & Palmigiano [37], using logical filters rather than order filters. It would be interesting to see if their approach also admits a topological characterization.

2.2 Canonical extensions of maps I: order-preserving maps

In the previous section, we have studied the canonical extension as a construction on lattices to some length. We will now turn to the subject of canonical extensions
of maps $f: L \rightarrow M$ between lattices, which is probably an even richer subject. In this dissertation, we will consider two approaches to obtaining a canonical extension of a map $f: L \rightarrow M$ between lattices.

- The first approach is to assume that $f$ is order-preserving, so that we may first extend $f: L \rightarrow F M$ to maps $F f: F L \rightarrow M$ and $I f: I L \rightarrow I M$ acting on filters and ideals, respectively, and then work from there. This is the approach we will take in the current section.

- The second approach is to assume that $M^{\delta}$ has nice topological properties, in which case we can develop a substantial part of the basic theory of extensions of maps without making any assumptions about the map $f: L \rightarrow M$. We will pursue this approach in §3.2.

It is an interesting open question whether the two approaches above form a dichotomy of some sort. We will return to this question in Remark 3.2.22.

In §2.2.1, we will first give the basic definition of the lower ($f^\triangleright$) and the upper ($f^\triangleleft$) canonical extension of an order-preserving map $f: L \rightarrow M$, and we will characterize these extensions as a largest and smallest continuous extension of $f$ to a map $L^\delta \rightarrow M^\delta$, respectively (Theorem 2.2.4). We will then show in §2.2.2 that if we assume that $f$ preserves joins (or dually, meets), even in only one coordinate, then this vastly improves the behaviour of $f^\triangleright$ and $f^\triangleleft$ (Theorem 2.2.18). Finally in §2.2.3 we will put this good behaviour to use by showing that canonical extensions of lattice homomorphisms are particularly well-behaved, so well in fact that canonical extension is a functor on the category of lattices and lattice homomorphisms (Theorem 2.2.24). All along, we will see that almost every result we prove about $f^\triangleright$ and $f^\triangleleft$ reduces to statements about filters and ideals.

### 2.2.1 The lower and upper extensions of an order-preserving map

In this subsection we will discuss the two canonical [34] ways to extend an order-preserving map $f: L \rightarrow M$ to a map $f': L^\delta \rightarrow M^\delta$, namely the lower and the upper canonical extension. We will then prove some of the basic facts that hold true for any order-preserving map. We conclude the subsection with a topological characterization theorem, which tells us that $f^\triangleright$ and $f^\triangleleft$ can be seen as the largest and smallest continuous extension of $f$, respectively.

Consider the following diagram, where $e_L: L \rightarrow L^\delta$ and $e_M: M \rightarrow M^\delta$ are the canonical extensions of $L$ and $M$, respectively:

$$
\begin{array}{cccc}
L & \xrightarrow{1_L} & F L & \xrightarrow{e_L^F} & L^\delta \\
| & f & \downarrow & \uparrow & f^\triangleright \\
M & \xrightarrow{1_M} & F M & \xrightarrow{e_M^F} & M^\delta
\end{array}
$$
(Observe that we factor $e_L: L \to L^\delta$ as $e(a) = e_L^F(\uparrow a)$.) We would like to find a function that can take the place of the ‘?’ in the diagram. To do this, we use the fact that any $x \in L^\delta$ is approximated from below by $S_x := \{ F \in \mathcal{F}L \mid e_L^F(F) \leq x \}$. Now each $F \in S_x$ can be mapped into $L^\delta$ via the assignment $F \mapsto e_M \circ \mathcal{F} f(F)$.

2.2.1. Definition. Let $f: L \to M$ be an order-preserving map between lattices. Then we define $f^\triangledown: L^\delta \to M^\delta$, the lower extension of $f$, as follows:

$$f^\triangledown: x \mapsto \bigvee \{ e_M^T \circ \mathcal{F} f(F) \mid e_L^T(F) \leq x \}.$$  

Dually, we define $f^\triangledown: L^\delta \to M^\delta$, the upper extension of $f$, as follows:

$$f^\triangledown: x \mapsto \bigwedge \{ e_M^T \circ \mathcal{I} f(I) \mid x \leq e_L^T(I) \}.$$  

2.2.2. Remark. There is a slight discrepancy between our definition of $f^\triangledown$ and $f^\triangledown$ when we compare it with the working definition found in e.g. [34, Lemma 4.3]. Using our notation, the working definition of [34] amounts to

$$f^\triangledown: x \mapsto \bigvee \{ \bigwedge e_M \circ f[F] \mid e_L^F(F) \leq x \}.$$  

The difference is that we use $\mathcal{F} f(F)$ rather than $f[F]$; however this difference is inconsequential. If $F \in \mathcal{F}L$, then $\mathcal{F} f(F) = \uparrow f[F] \supseteq f[F]$, so

$$e_M^T(\mathcal{F} f(F)) = \bigwedge e_M[f[F]] \quad \text{by def. of } e_M^T,$$

$$\leq \bigwedge e_M \circ f[F] \quad \text{since } \mathcal{F} f(F) \supseteq f[F].$$  

Conversely, since $e_M$ is order-preserving, we know by Fact A.3.1 that for all $U \subseteq M$, $\uparrow e_M[U] \supseteq e_M[\uparrow U]$. Now we see that

$$\bigwedge e_M \circ f[F] = \bigwedge \uparrow e_M \circ f[F] \quad \text{by order theory},$$

$$\leq \bigwedge e_M[\uparrow f[F]] \quad \text{since } \uparrow e_M \circ f[F] \supseteq e_M[\uparrow f[F]].$$  

In the following lemma we see that $f^\triangledown$ and $f^\triangledown$ mingle well with the auxilliary maps induced by $e_L: L \to L^\delta$ and $e_M: M \to M^\delta$, or alternatively, that $f^\triangledown$ and $f^\triangledown$ behave well on filter and ideal elements.

2.2.3. Lemma ([34]). Let $f: L \to M$ be order-preserving. Then $f^\triangledown: L^\delta \to M^\delta$ and $f^\triangledown: L^\delta \to M^\delta$ are order-preserving maps, which satisfy the following additional properties:

1. $f^\triangledown \circ e_L^F = e_M^T \circ \mathcal{F} f$;

2. $f^\triangledown \circ e_L^T = e_M^T \circ \mathcal{I} f$;
3. $f^\triangledown \leq f^\wedge$;

4. $f^\triangledown \circ e^\triangledown_L = f^\wedge \circ e^\wedge_L$;

5. $f^\wedge \circ e^\wedge_L = f^\triangledown \circ e^\triangledown_L$.

**Proof** It is easy to see why $f^\triangledown$ and $f^\wedge$ are order-preserving. Take $x, y \in \mathbb{L}^\delta$ such that $x \leq y$ and consider $f^\triangledown$. Then

$$\{F \in \mathcal{F} \mathbb{L} \mid e^\triangledown_L(F) \leq x\} \subseteq \{F \in \mathcal{F} \mathbb{L} \mid e^\triangledown_L(F) \leq y\},$$

so also

$$\bigvee \{e^\triangledown_M \circ \mathcal{F} f(F) \in \mathcal{F} \mathbb{L} \mid e^\triangledown_L(F) \leq x\} \subseteq \bigvee \{e^\triangledown_M \circ \mathcal{F} f(F) \in \mathcal{F} \mathbb{L} \mid e^\triangledown_L(F) \leq y\},$$

i.e. $f^\triangledown(x) \leq f^\triangledown(y)$. Below, we will only prove statements (1), (3) and (4), since (2) and (5) are order duals of (1) and (4).

(1). Let $F \in \mathcal{F} \mathbb{L}$; the set $\{F' \in \mathcal{F} \mathbb{L} \mid e^\triangledown_L(F') \leq e^\triangledown_L(F)\}$ has a maximal element, viz. $F$. It follows that

$$f^\triangledown(e^\triangledown_L(F)) = \bigvee \{e^\triangledown_M \circ \mathcal{F} f(F') \mid e^\triangledown_L(F') \leq e^\triangledown_L(F)\} \quad \text{by definition of } f^\triangledown,$$

$$= e^\triangledown_M \circ \mathcal{F} f(F) \quad \text{since } e^\triangledown_M \circ \mathcal{F} f \text{ is order-preserving.}$$

(3). Let $x \in \mathbb{L}^\delta$. Recall that $f^\triangledown(x) = \bigvee \{e^\triangledown_M \circ \mathcal{F} f(F) \mid e^\triangledown_L(F) \leq x\}$ and $f^\wedge(x) = \bigwedge \{e^\wedge_M \circ \mathcal{I} f(I) \mid x \leq e^\wedge_L(I)\}$; we now have to show that

$$\bigvee \{e^\triangledown_M \circ \mathcal{F} f(F) \mid e^\triangledown_L(F) \leq x\} \leq \bigwedge \{e^\wedge_M \circ \mathcal{I} f(I) \mid x \leq e^\wedge_L(I)\}. \quad (2.7)$$

Take $F \in \mathcal{F} \mathbb{L}$, $I \in \mathcal{I} \mathbb{L}$ such that $e^\triangledown_L(F) \leq x \leq e^\wedge_L(I)$, then it follows by compactness that $F \not\triangleleft I$. Now by Lemma 2.1.2, we get that $\mathcal{F} f(F) \not\triangleleft \mathcal{I} f(I)$, so that also $e^\triangledown_M \circ \mathcal{F} f(F) \leq e^\triangledown_M \circ \mathcal{I} f(I)$. It follows that (2.7) holds.

(4). Let $I \in \mathcal{I} \mathbb{L}$. We claim that

$$\forall J \in \mathcal{I} \mathbb{M}, [\forall F \in \mathcal{F} \mathbb{L}, F \not\triangleleft I \Rightarrow \mathcal{F} f(F) \not\triangleleft J] \iff \mathcal{I} f(I) \subseteq J. \quad (2.8)$$

Take $J \in \mathcal{I} \mathbb{M}$ and suppose that the left-hand side of (2.8) holds. Take $a \in I$, then $\uparrow a \not\triangleleft I$, so by our assumption regarding $J$, $\uparrow f(a) \not\triangleleft J$, i.e. $f(a) \in J$. It follows that $f[I] \subseteq J$ and thus $\mathcal{I} f(I) = \downarrow f[I] \subseteq J$. Conversely, suppose that $\mathcal{I} f(I) \subseteq J$ and that $F \in \mathcal{F} \mathbb{L}$ such that $F \not\triangleleft I$. Then there is some $a \in F \cap I$. Since $a \in F$, we also get $f(a) \in f[F] \subseteq \downarrow f[I] = \mathcal{F} f(I)$. Since $a \in I$, we get that $f(a) \in f[I] \subseteq \downarrow f[I] = \mathcal{I} f(I)$. Since we assumed that $\mathcal{I} f(I) \subseteq J$, it follows that $f(a) \in J$. But then $\mathcal{F} f(F) \not\triangleleft J$. It follows that (2.8) holds.
We can now see that
\[
 f^\circ(e^L_\ell(I)) = \bigvee \{ e^\circ_M \circ F f(F) \mid e^\circ_L(F) \leq e^L_\ell(I) \} \quad \text{by def. of } f^\circ,
\]
\[
 = \bigvee \{ e^\circ_M \circ F f(F) \mid F \not\subseteq I \} \quad \text{by compactness},
\]
\[
 = \bigwedge \{ e^\circ_M(J) \mid \forall F[F \not\subseteq I \Rightarrow F f(F) \not\subseteq J] \} \quad \text{by Lemma 2.1.29},
\]
\[
 = \bigwedge \{ e^\circ_M(J) \mid I f(I) \subseteq J \} \quad \text{by (2.8)},
\]
\[
 = e^\circ_M \circ I f(I) \quad \text{by order theory},
\]
\[
 = f^\delta(e^L_\ell(I)) \quad \text{by (2)}.
\]

It is known that the lower extension of an arbitrary map \( f : L \to M \) between distributive lattices \( L, M \) can be characterized as the largest continuous extension of \( f \) [39, Theorem 2.15]. The result below, which is new, tells us that we can say the same about maps between non-distributive lattices, if we assume that \( f \) is order-preserving. We will return to this issue in §3.2.2.

2.2.4. THEOREM. Let \( f : L \to M \) be an order-preserving map between lattices.

1. (a) The map \( f^\circ : L^\delta \to M^\delta \) is \((\delta^1, \sigma^1)\)-continuous and \( f^\circ \circ e_L = e_M \circ f \).

   (b) The map \( f^\delta : L^\delta \to M^\delta \) is \((\delta^1, \sigma^1)\)-continuous and \( f^\delta \circ e_L = e_M \circ f \).

2. Let \( f^\prime : L^\delta \to M^\delta \) be an order-preserving extension of \( f : L \to M \), i.e. assume that \( f^\prime \circ e_L = e_M \circ f \).

   (a) If \( f^\prime \) is \((\delta^1, \iota^1)\)-continuous, then \( f^\prime \leq f^\circ \).

   (b) If \( f^\prime \) is \((\delta^1, \iota^1)\)-continuous, then \( f^\delta \leq f^\prime \).

**Proof** We will only prove the statements concerning \( f^\circ \), since the proofs for those concerning \( f^\delta \) are identical modulo order duality.

(1). Let \( x \in L^\delta \); we will first show that \( f^\circ \) is locally \((\delta^1, \sigma^1)\)-continuous at \( x \). Suppose that \( f^\circ(x) \in U \), where \( U \subseteq M^\delta \) is a \( \sigma^1 \)-open set. Recall that
\[
f^\circ(x) = \bigvee \{ e^\circ_M \circ F f(F) \mid e^\circ_L(F) \leq x \},
\]
and observe that this join is directed by Lemma 2.1.28(1). Since \( f^\circ(x) \in U \), it follows by definition of the \( \sigma^1 \)-topology that some element of the join above lies in \( U \), i.e. that there must be some \( F \in F L \) such that \( e^\circ_L(F) \leq x \) and \( e^\circ_M \circ F f(F) \in U \). Now \( \uparrow e^\circ_L(F) \) is a \( \delta^1 \)-open neighborhood of \( x \); it remains to be shown that \( f^\circ[\uparrow e^\circ_L(F)] \subseteq U \). But this is easy to see: since \( f^\circ \) is order-preserving, it follows by Fact A.3.1 that \( f^\circ[\uparrow e^\circ_L(F)] \subseteq \uparrow f^\circ(e^\circ_L(F)) \). Now by Lemma 2.2.3(1),
\[
f^\circ(e^\circ_L(F)) = e^\circ_M(F) \circ F f(F); \text{ since we assumed that } e^\circ_M(F) \circ F f(F) \in U \text{ and we know that } U \text{ is an upper set, it follows that } \uparrow f^\circ(e^\circ_L(F)) \subseteq U. \text{ Since } U \text{ was arbitrary, it follows that } f^\circ \text{ is locally } (\delta^1, \sigma^1)\text{-continuous at } x; \text{ since } x \in L^\delta \text{ was arbitrary, it follows that } f^\circ \text{ is } (\delta^1, \sigma^1)\text{-continuous.}
To see that $f^\triangledown \circ e_L = e_M \circ f$, observe that the right square below commutes by Lemma 2.2.3(2) and the left square commutes by the universal property of $\mathcal{F}$.

\[
\begin{array}{ccc}
L & \xrightarrow{f} & \mathcal{F}L \\
\downarrow & & \downarrow \mathcal{F}f \\
M & \xrightarrow{e_M} & \mathcal{M}L
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L \delta & \xrightarrow{\mathcal{F}} & \mathcal{F}L \delta \\
\downarrow & & \downarrow \mathcal{F}f \\
M \delta & \xrightarrow{e_M} & \mathcal{M}L \delta
\end{array}
\]

(2). Let $f': \mathbb{L}^\delta \to \mathbb{M}^\delta$ be an order-preserving extension of $f: \mathbb{L} \to \mathbb{M}$. We will show something stronger than statement (2)(a): we will show that for all $x \in \mathbb{L}^\delta$, if $f'$ is locally $(\delta^1, \iota^1)$-continuous at $x$, then $f'(x) \leq f^\triangledown(x)$. Suppose towards a contradiction that $f'$ is locally continuous but that we have $x \in \mathbb{L}^\delta$ such that $f'(x) \not\leq f^\triangledown(x)$. Now since $\downarrow f^\triangledown(x)$ is $\iota^1$-closed, by local continuity there must be some $\delta^1$-open set $U \subseteq \mathbb{L}^\delta$ such that $x \in U$ and $f'[U] \subseteq \mathbb{M}^\delta \setminus \downarrow f^\triangledown(x)$. We may assume that $U$ is a basic open set, i.e. that $U = \uparrow e^\mathcal{F}_L(F)$ for some $F \in \mathcal{F}L$, so it must be the case that $e^\mathcal{F}_L(F) \leq x$ and $f'(e^\mathcal{F}_L(F)) \not\leq \downarrow f^\triangledown(x)$. But now we can use the fact that $f'$ is order-preserving to see that

\[
f'(e^\mathcal{F}_L(F)) = f'(\Lambda e_L[F]) \leq \Lambda f' \circ e_L[F] = \Lambda e_M \circ f[F] \leq e^\mathcal{F}_M \circ \mathcal{F} f(F) \leq f^\triangledown(x)
\]

This is a contradiction because we also assumed that $f'(e^\mathcal{F}_L(F)) \not\leq \downarrow f^\triangledown(x)$; it follows that indeed, local continuity of $f'$ at $x$ implies that $f'(x) \leq f^\triangledown(x)$. Thus, if $f'$ is continuous at every $x \in \mathbb{L}^\delta$, it follows that $f' \leq f^\triangledown$.

Even though $f^\triangledown$ is not Scott-continuous in general, it is when we restrict it to ideal elements. The following result is known from the study of duality theory for Heyting algebras and modal algebras, but it holds in fact at the level of generality of lattices and order-preserving maps.

**2.2.5. Corollary (Esakia’s Lemma).** Let $f: \mathbb{L} \to \mathbb{M}$ be an order-preserving map between lattices $\mathbb{L}, \mathbb{M}$. Then $f^\triangledown \circ e^\mathcal{F}_L = f^\wedge \circ e^\mathcal{F}_L$ is Scott-continuous and $f^\wedge \circ e^\mathcal{F}_L = f^\triangledown \circ e^\mathcal{F}_L$ is co-Scott continuous.

**Proof** We only consider the first statement. The equality follows by Lemma 2.2.3; the continuity follows since $e^\mathcal{F}_L$ is $(\sigma^1, \delta^1)$-continuous by Lemma 2.1.20 and $f^\triangledown$ is $(\delta^1, \sigma^1)$-continuous by Theorem 2.2.4.

As another application of Lemma 2.2.3, we will show that the canonical extension of an order embedding is again an order embedding.
2.2. Canonical extensions of maps I: order-preserving maps

2.2.6. **Lemma.** Let \( \mathbb{L}, \mathbb{M} \) be lattices. If \( f : \mathbb{L} \to \mathbb{M} \) is an order embedding, then so are \( f^\triangleright \) and \( f^\triangleleft \).

**Proof** Let \( f : \mathbb{L} \to \mathbb{M} \) be an order embedding. We will only prove part (1); part (2) follows by order duality. Let \( x, y \in \mathbb{L}^\delta \) and suppose that \( f^\triangleright (x) \leq f^\triangleright (y) \). We will show that

\[
\forall F \in \mathcal{F} \mathbb{L}, \forall I \in \mathcal{I} \mathbb{L}, \text{ if } e^F_L (F) \leq x \text{ and } y \leq e^I_L (I) \text{, then } e^F_L (F) \leq e^I_L (I). \quad (2.9)
\]

This is not hard to see. Take \( F \in \mathcal{F} \mathbb{L} \) and \( I \in \mathcal{I} \mathbb{L} \) such that \( e^F_L (F) \leq x \) and \( y \leq e^I_L (I) \). Then

\[
e^F_M \circ \mathcal{F} f (F) = f^\triangleright (e^F_L (F)) \quad \text{ by Lemma 2.2.3 (1)},
\]

\[
\leq f^\triangleright (x) \quad \text{ since } e^F_L (F) \leq x,
\]

\[
\leq f^\triangleright (y) \quad \text{ by assumption},
\]

\[
\leq f^\triangleright (e^I_L (I)) \quad \text{ since } y \leq e^I_L (I),
\]

\[
e^I_M \circ \mathcal{I} f (I) \quad \text{ by Lemma 2.2.3 (2)}.
\]

Consequently, \( \mathcal{F} f (F) \upharpoonright \mathcal{I} f (I) \), i.e. \( \uparrow f[F] \upharpoonright \downarrow f[I] \). It follows that there must exist \( a \in F \) and \( b \in I \) such that \( f(a) \leq f(b) \). Since \( f \) is an order embedding, \( a \leq b \), so \( F \upharpoonright I \). It follows that \( e^F_L (F) \leq e^I_L (I) \), concluding our proof of (2.9). It now follows from the density of filter and ideal elements that \( x \leq y \), concluding our proof.

2.2.2 Operators and join-preserving maps

So far we have seen how to take an order-preserving map \( f : \mathbb{L} \to \mathbb{M} \) and extend it covariantly to maps \( \mathcal{F} f : \mathcal{F} \mathbb{L} \to \mathcal{F} \mathbb{M} \) and \( \mathcal{I} f : \mathcal{I} \mathbb{L} \to \mathcal{I} \mathbb{M} \) at the intermediate level of filters and ideals, and from there to maps \( f^\triangleright : \mathbb{L}^\delta \to \mathbb{M}^\delta \) and \( f^\triangleleft : \mathbb{L}^\delta \to \mathbb{M}^\delta \). In this section we will see that if \( f : \mathbb{L} \to \mathbb{M} \) preserves binary joins, then the set-theoretic inverse function \( f^{-1} : \mathcal{P} (\mathbb{L}) \to \mathcal{P} (\mathbb{M}) \) becomes a partial function from \( \mathcal{I} \mathbb{M} \) to \( \mathcal{I} \mathbb{L} \). This contravariant partial extension of \( f \) at the intermediate level will then tell us a lot about the topological properties of \( f^\triangleright : \mathbb{L}^\delta \to \mathbb{M}^\delta \).

**Applying join-preserving maps to filters and ideals**

We begin by making the observation that when an order-preserving map \( g : \mathbb{L} \to \mathbb{M} \) preserves binary joins, we not only get a map \( \mathcal{I} g : \mathcal{I} \mathbb{L} \to \mathcal{I} \mathbb{M} \), defined as \( \mathcal{I} g (I) := \downarrow g[I] \), but also a well-behaved partial map \( g^{-1} : \mathcal{I} \mathbb{M} \to \mathcal{I} \mathbb{L} \).

2.2.7. **Lemma.** Let \( \mathbb{L}, \mathbb{M} \) be lattices and let \( g : \mathbb{L} \to \mathbb{M} \) be a map preserving binary joins. Then

1. \( \forall J \in \mathcal{I} \mathbb{M}, \; g^{-1} (J) \in \mathcal{I} \mathbb{L} \text{ iff } g^{-1} (J) \neq \emptyset ; \)
2. \( \forall F \in \mathcal{F} \mathbb{L}, \forall J \in \mathcal{I} \mathbb{M}, \mathcal{F} g(F) \uparrow J \iff g^{-1}(J) \in \mathcal{I} \mathbb{L} \) and \( F \uparrow g^{-1}(J) \).

**Proof** (1). Let \( J \in \mathcal{I} \mathbb{M} \). Since ideals are non-empty by definition, the left-to-right implication is immediate. For the converse, first observe that \( g^{-1}(J) \) is non-empty by assumption. Moreover, \( g^{-1} \) is order-preserving, so by Fact A.3.1(3), \( g^{-1}(J) \) is a lower set. Moreover, if \( a, b \in g^{-1}(J) \), then \( g(a), g(b) \in J \), so since \( J \) is an ideal we get \( g(a) \lor g(b) \in J \). Since \( g \) preserves binary joins, we see that \( g(a \lor b) = g(a) \lor g(b) \in J \), so that \( a \lor b \in g^{-1}(J) \). It follows that \( g^{-1}(J) \) is an ideal of \( \mathbb{L} \).

(2). Let \( F \in \mathcal{F} \mathbb{L} \) and \( J \in \mathcal{I} \mathbb{M} \). If \( \mathcal{F} g(F) \uparrow J \), then \( \uparrow g[F] \uparrow J \), i.e. there must be some \( b \in (\uparrow g[F]) \cap J \). Since \( b \in \uparrow g[F] \), there is some \( a \in F \) such that \( g(a) \leq b \). Since \( b \in J \) and \( J \) is a lower set, it follows that \( g(a) \in J \). But then \( a \in F \cap g^{-1}(J) \), so that \( F \uparrow g^{-1}(J) \). Moreover, since \( g^{-1}(J) \neq \emptyset \), it follows by (1) that \( g^{-1}(J) \in \mathcal{I} \mathbb{L} \). Conversely, if \( F \uparrow g^{-1}(J) \) then there must exist \( a \in F \cap g^{-1}(J) \), i.e. \( a \in F \) and \( g(a) \in J \). Since \( g(a) \in g[F] \subseteq \uparrow g[F] \), it follows that \( \uparrow g[F] \uparrow J \).

Up to this point, we have only considered canonical extensions of unary maps \( f: \mathbb{L} \to \mathbb{M} \). We would now like to state several more detailed results, concerning canonical extensions of \( n \)-ary maps. This is non-problematic since canonical extensions commute with finite products; however, we would like to ignore the technical difference between, say, \((\mathbb{L}_1 \times \mathbb{L}_2) \delta\) and \(\mathbb{L}_1^\delta \times \mathbb{L}_2^\delta\) whenever possible.

**2.2.8. Convention.** If \( f: \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \to \mathbb{M} \) is an \( n \)-ary order-preserving map, then we regard \( \mathcal{F} f \) as a map \( \mathcal{F} \mathbb{L}_1 \times \cdots \times \mathcal{F} \mathbb{L}_n \to \mathcal{F} \mathbb{M} \) and \( f^\uparrow \) as a map \( \mathbb{L}_1^\delta \times \cdots \times \mathbb{L}_n^\delta \to \mathbb{M}^\delta \), rather than as maps \( \mathcal{F}(\mathbb{L}_1 \times \cdots \times \mathbb{L}_n) \to \mathcal{F} \mathbb{M} \) and \( (\mathbb{L}_1 \times \cdots \times \mathbb{L}_n)^\delta \to \mathbb{M}^\delta \). This is justified by Fact A.5.4 and Lemma 2.1.26. Using this convention, \( f^\uparrow \) is calculated as follows for \((x_1, \ldots, x_n) \in \mathbb{L}_1^\delta \times \cdots \times \mathbb{L}_n^\delta\):

\[
 f^\uparrow(x_1, \ldots, x_n) = \bigvee \{ e^F_m \circ \mathcal{F} f(F_1, \ldots, F_n) \mid e^F_{L_1}(F_1) \leq x_1, \ldots, e^F_{L_n}(F_n) \leq x_n \},
\]

and \( f^\downarrow(x_1, \ldots, x_n) \) is calculated similarly.

We will now work towards the main technical lemma of this section, which concerns canonical extensions of binary maps \( f: \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M} \) which preserve joins in only one coordinate. We will first prove a result about the extension \( \mathcal{F} f: \mathcal{F} \mathbb{L}_1 \times \mathcal{F} \mathbb{L}_2 \to \mathcal{F} \mathbb{M} \) of such a map \( f \), which will also be of use to us in §2.3.3.

**2.2.9. Lemma.** Let \( f: \mathbb{L}_1 \times \mathbb{L}_2 \to \mathbb{M} \) be an order-preserving map between lattices \( \mathbb{L}_1, \mathbb{L}_2 \) and \( \mathbb{M} \) which preserves binary joins in its first coordinate. Let \( S \cup \{ F' \} \subseteq \mathcal{F} \mathbb{L}_1 \) and \( G \in \mathcal{F} \mathbb{L}_2 \). If \( S \neq \emptyset \) and

\[
 \forall I \in \mathcal{I} \mathbb{L}_1, \{ \forall F \in S, F \uparrow I \} \Rightarrow F' \uparrow I,
\]

(2.10)
We need to show that $\forall J \in \mathcal{I} \mathcal{M}, [\forall F \in S, \mathcal{F} f(F,G) \uparrow J] \Rightarrow \mathcal{F} f(F',G) \uparrow J. \quad (2.11)$

**Proof** For each $b \in \mathbb{L}_2$, we define $f_b: \mathbb{L}_1 \rightarrow \mathcal{M}$ as $f_b: a \mapsto f(a,b)$. It follows from our assumptions above that for each $b \in \mathbb{L}_2$, $f_b: \mathbb{L}_1 \rightarrow \mathcal{M}$ is a map preserving binary joins. Now we claim that

$$\forall b \in G, \forall F \in \mathcal{F} \mathbb{L}_1, \mathcal{F} f(F,G) \supseteq \mathcal{F} f_b(F). \quad (2.12)$$

After all,

$$\begin{align*}
\mathcal{F} f(F,G) &= \uparrow f[F,G] & \text{by definition of } \mathcal{F} f, \\
\supseteq &\uparrow f[F,\{b\}] & \text{since } G \supseteq \{b\}, \\
= &\uparrow f_b[F] & \text{by definition of } f_b, \\
= &\mathcal{F} f_b(F). & \text{by definition of } \mathcal{F} f_b.
\end{align*}$$

Now suppose that (2.10) holds and that $J \in \mathcal{I} \mathcal{M}$ such that $\forall F \in S, \mathcal{F} f(F,G) \uparrow J$. We need to show that $\mathcal{F} f(F',G) \uparrow J$. We define $I_1 := \bigcup_{b \in G} f^{-1}_b(J)$. We claim that

$$I_1 \in \mathcal{I} \mathbb{L}_1, \text{ i.e. } I_1 \text{ is an ideal of } \mathbb{L}_1. \quad (2.13)$$

To establish this we need to show three things: we want that $I_1$ is a lower set, that $I_1$ is directed and that $I_1$ is non-empty. Since $f_b$ is order-preserving for each $b \in G$, it follows by Fact A.3.1(3) that $I_1$ is a union of lower sets and hence, itself a lower set. To see that $I_1$ is directed, first observe that

for all $b, b' \in G$, if $b \geq b'$ then $f^{-1}_b(J) \subseteq f^{-1}_{b'}(J). \quad (2.14)$

After all, if $a \in f^{-1}_b(J)$, then $f(a,b) \in J$. Since $f$ is order-preserving and $b' \leq b$, we see that $f(a,b') \leq f(a,b) \in J$. Since $J$ is a lower set, it follows that $f(a,b') \in J$, so that $a \in f_{b'}^{-1}$; since $a \in f_b^{-1}$ was arbitrary, it follows that (2.14) holds. Now since $G$ is a filter, it is co-directed; consequently, $I_1 := \bigcup_{b \in G} f^{-1}_b(J)$ is a directed union. To see why $I_1$ is a directed subset of $\mathbb{L}_1$, consider $a, a' \in I_1$. Because $I_1$ is a directed union, there must exist some $b \in G$ such that $a, a' \in f_b^{-1}(J)$. Now $f_b^{-1}(J)$ is non-empty, so it is an ideal by Lemma 2.2.7(1); consequently, there must exist some $c \in f_b^{-1}(J)$ such that $a, a' \leq c$. Since $f_b^{-1}(J) \subseteq I_1$, it follows that $I_1$ is directed. Finally, to see that $I_1$ is non-empty, observe that since $S$ is non-empty, there is some $F \in S$ such that $\mathcal{F} f(F,G) \uparrow J$. Since $\mathcal{F} f(F,G) = \uparrow f[F,G]$, this means that there is some $c \in (\uparrow f[F,G]) \cap J$. Since $c \in \uparrow f[F,G]$, there must be $a \in F$ and $b \in G$ such that $f(a,b) \leq c$. Since $J$ is a lower set and $c \in J$, it follows that $f(a,b) \in J$. But then also $f_b(a) \in J$, so that $a \in f_b^{-1}(J)$; since $f_b^{-1}(J) \subseteq I_1$, it follows that $I_1 \neq \emptyset$. We conclude that (2.13) holds. Next, we observe that

$$\text{If } \forall F \in S, \mathcal{F} f(F,G) \uparrow J, \text{ then } \forall F \in S, \exists b \in G, \text{ } F \uparrow f^{-1}_b(J). \quad (2.15)$$
Suppose that the left-hand side of (2.15) holds and take \( F \in S \). Then \( \mathcal{F}f(F, G) \sqsubseteq J \), so as we have seen above, there must exist \( a \in F \) and \( b \in G \) such that \( a \in f_b^{-1}(J) \). It follows that \( a \in F \cap f_b^{-1}(J) \), so that \( F \sqsubseteq f_b^{-1}(J) \). Since \( F \in S \) was arbitrary, it follows that (2.15) holds. Recall that we assumed that \( J \in \mathcal{I}M \) such that \( \forall F \in S, \mathcal{F}f(F, G) \sqsubseteq J \); we now see that

\[
\forall F \in S, \mathcal{F}f(F, G) \sqsubseteq J \quad \text{by assumption,}
\]
\[
\Rightarrow \forall F \in S, \exists b \in G, F \sqsubseteq f_b^{-1}(J) \quad \text{by (2.15),}
\]
\[
\Rightarrow \forall F \in S, F \sqsubseteq I_1 \quad \text{since } I_1 = \bigcup_G f_b^{-1}(J),
\]
\[
\Rightarrow F' \sqsubseteq I_1 \quad \text{by (2.13) and (2.10),}
\]
\[
\Rightarrow \exists b \in G, F' \sqsubseteq f_b^{-1}(J) \quad \text{by def. of } I_1,
\]
\[
\Rightarrow \exists b \in G, \mathcal{F}f_b(F') \sqsubseteq J \quad \text{by Lemma 2.2.7(2),}
\]
\[
\Rightarrow \mathcal{F}f(F, G) \sqsubseteq J \quad \text{by (2.12) and L. 2.1.2(1).}
\]

Since \( J \in \mathcal{I}M \) was arbitrary, it follows that (2.11) holds. \( \blacksquare \)

The following well-known lemma is perhaps the most powerful technical result in the theory of canonical extensions of maps between lattices.

2.2.10. Lemma ([34]). Let \( e_1 : \mathbb{L}_1 \rightarrow \mathbb{L}_1^\delta, e_2 : \mathbb{L}_2 \rightarrow \mathbb{L}_2^\delta \) and \( e_M : \mathbb{M} \rightarrow \mathbb{M}^\delta \) be canonical extensions of lattices, and let \( f : \mathbb{L}_1 \times \mathbb{L}_2 \rightarrow \mathbb{M} \) be a order-preserving map which preserves binary joins in the first coordinate. Then \( f^\vee : \mathbb{L}_1^\delta \times \mathbb{L}_2^\delta \rightarrow \mathbb{M}^\delta \) preserves all non-empty joins in the first coordinate.

**Proof** Let \( T \subseteq \mathbb{L}_1^\delta \) be a non-empty set and let \( y \in \mathbb{L}_2^\delta \). To show that \( f^\vee(\vee T, y) = \bigvee_{x \in T} f^\vee(x, y) \), it suffices to show that

\[
f^\vee(\vee T, y) \leq \bigvee_{x \in T} f^\vee(x, y),
\]

(2.16)

since \( f^\vee \) is order-preserving. Using the definition of \( f^\vee \), one can show that

\[
f^\vee(\vee T, y) = \bigvee \{ f^\vee(\vee T, e_2^F(G)) \mid e_2^F(G) \leq y \},
\]

and that

\[
\bigvee_{x \in T} f^\vee(x, y) = \bigvee \{ \bigvee_{x \in T} f^\vee(x, e_2^F(G)) \mid e_2^F(G) \leq y \}.
\]

Thus we see that it suffices to show that for arbitrary \( G \in \mathcal{F}\mathbb{L}_2 \),

\[
f^\vee(\vee T, e_2^F(G)) \leq \bigvee_{x \in T} f^\vee(x, e_2^F(G)).
\]

(2.17)

Fix \( G \in \mathcal{F}\mathbb{L}_2 \) and define \( S := \{ F \in \mathcal{F}\mathbb{L}_1 \mid \exists x \in T, e_1^F(x) \leq x \} \); we see that \( \bigvee_{F \in S} e_1^F(F) = \vee T \). Now if we look at the left-hand side of (2.17) then we see that

\[
f^\vee(\vee T, e_2^F(G))
\]
\[
= f^\vee(\bigvee_{F \in S} e_1^F(F), e_2^F(G))
\]
\[
= \bigvee \{ e_M \circ \mathcal{F}f(F', G) \mid e_1^F(F') \leq \bigvee_{F \in S} e_1^F(F) \}
\]

by def. of \( f^\vee \).
The right-hand side of (2.17) reduces as follows:
\[
\bigvee_{x \in T} f^\uparrow(x, e_2^F(G)) \\
= \bigvee_{x \in T} \big\{ e_2^F \circ F f(F, G) \mid e_1^F(F) \leq x \big\} \quad \text{by def. of } f^\uparrow, \\
= \bigvee\big\{ e_2^F \circ F f(F, G) \mid F \in S \big\} \quad \text{by def. of } S, \\
= \bigwedge\big\{ e_1^F(J) \mid \forall F \in S, F f(F, G) \nleq J \big\} \quad \text{by L. 2.1.29.}
\]

Thus we see that to show that (2.17) holds, it suffices to show that for all \( F' \in \mathcal{F} \mathbb{L}_1 \) such that \( e_1^F(F') \leq \bigvee_{F \in S} e_1^F(F) \) and for all \( J \in \mathcal{T} \mathbb{M} \) such that \( \forall F \in S, F f(F, G) \nleq J \), we have that
\[
e_2^F \circ F f(F', G) \leq e_1^F(J).
\]
(2.18)

Now given the fact that \( e_1^F(F') \leq \bigvee_{F \in S} e_1^F(F) \), we know from Lemma 2.1.29 that
\[
e_1^F(F') \leq \bigwedge\big\{ e_1^F(I) \mid \forall F \in S, F \nleq I \big\},
\]
i.e. for all \( I \in \mathcal{T} \mathbb{L}_1 \), if \( \forall F \in S, F \nleq I \), then \( F' \nleq I \). But now it follows from Lemma 2.2.9 and our assumption about \( J \) that \( F f(F', G) \nleq J \), so that (2.18) holds. Since \( F' \) and \( J \) were arbitrary, it follows that (2.17) holds, which concludes our proof.

### Operators and dual operators

Obviously, we can use Lemma 2.2.10 to make claims about join-preserving maps, but that is not all. Operators form another example of maps which satisfy the conditions of Lemma 2.2.10.

**2.2.11. Definition.** Let \( f : \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \rightarrow \mathbb{M} \) be an \( n \)-ary order-preserving map between lattices. We call \( f \) an **operator** if \( f \) preserves binary joins in each coordinate, i.e. if for all \( i \leq n \), for all \( a_1, \ldots, a_n \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \) and all \( b \in \mathbb{L}_i \), we have

\[
f(a_1, \ldots, a_i \vee b, \ldots, a_n) = f(a_1, \ldots, a_i, \ldots, a_n) \vee f(a_1, \ldots, b, \ldots, a_n).
\]

If all lattices involved are complete and if \( f \) preserves all non-empty joins in each coordinate, then we call \( f \) a **complete operator**.

We call \( f \) a **normal operator** if \( f \) is an operator and for all \( i \leq n \), for all \( a_1, \ldots, a_n \in \mathbb{L}_1 \times \cdots \times \mathbb{L}_n \),

\[
a_i = 0 \Rightarrow f(a_1, \ldots, a_i, \ldots, a_n) = 0.
\]

In other words, \( f \) is a normal operator if it also preserves the empty join in each coordinate.

A **dual operator** (complete dual operator, etc.) is an \( n \)-ary map which preserves binary meets (all non-empty meets, etc.) in each coordinate.
Operators arise, for instance, in the algebraic semantics for modal logics [19, Ch. 5], where they correspond to existential modalities (usually denoted by ‘◊’).

### 2.2.12. Example

The property of being a (normal) operator is weaker than that of being a join-homomorphism. Consider the map \( f : \{0,1\} \times \{0,1\} \rightarrow \{0,1\} \), with the usual order on \( \{0,1\} \), defined as

\[
 f : (a,b) \mapsto a \land b.
\]

Then \( f \) is an operator, in fact a normal operator, because \( \{0,1\} \) is a distributive lattice. However, \( f \) is not a join-homomorphism:

\[
 f((0,1) \lor (1,0)) = f(1,1) = 1 \land 1 = 1,
\]

but

\[
 f(0,1) \lor f(1,0) = (0 \land 1) \lor (1 \land 0) = 0 \lor 0 = 0.
\]

This is quite different from the situation for directed joins; if \( g : \mathbb{D}_1 \times \cdots \times \mathbb{D}_n \rightarrow \mathbb{E} \) is a map between dcpos which preserves directed joins in each coordinate, then by Fact A.3.4, \( g \) preserves directed joins in \( \mathbb{D}_1 \times \cdots \times \mathbb{D}_n \).

In Lemma 2.2.10, we only considered binary joins, i.e. non-empty finite joins. Canonical extensions also behave well with respect to maps which preserve the empty join.

### 2.2.13. Lemma

Let \( e_1 : \mathbb{L}_1 \rightarrow \mathbb{L}_1^\delta, e_2 : \mathbb{L}_2 \rightarrow \mathbb{L}_2^\delta \) and \( e_M : \mathbb{M} \rightarrow \mathbb{M}^\delta \) be canonical extensions of lattices, and let \( f : \mathbb{L}_1 \times \mathbb{L}_2 \rightarrow \mathbb{M} \) be an order-preserving map.

1. If \( \forall b \in \mathbb{L}_2, f(0,b) = 0 \), then also \( \forall y \in \mathbb{L}_2^\delta, f^\lor(0,y) = 0 \);

2. If \( \forall b \in \mathbb{L}_2, f(1,b) = 1 \), then also \( \forall y \in \mathbb{L}_2^\delta, f^\lor(1,y) = 1 \);

**Proof** We will only prove (1), since (2) is just the order dual of (1). First, observe that

\[
 \forall F \in \mathcal{F}\mathbb{L}_1, \forall G \in \mathcal{F}\mathbb{L}_2, 0 \in F \Rightarrow 0 \in \mathcal{F}f(F,G). \tag{2.19}
\]

Take \( F \in \mathcal{F}\mathbb{L}_1 \) and \( G \in \mathcal{F}\mathbb{L}_2 \) such that \( 0 \in F \), then since \( G \) must be non-empty, there is some \( b \in G \). Now

\[
 f(0,b) \in f[F,G] \subseteq \uparrow f[F,G] = \mathcal{F}f(F,G),
\]

so since \( f(0,b) = 0 \) by assumption, we see that \( 0 \in \mathcal{F}f(F,G) \). Next, observe that it is a basic fact about canonical extensions that for any lattice \( \mathbb{L} \),

\[
 \forall a \in \mathbb{L}, \forall F \in \mathcal{F}\mathbb{L}, a \in F \text{ iff } e^F_L(F) \leq e_L(a). \tag{2.20}
\]
After all, \( a \in F \) iff \( F \uparrow a \) iff \( e_1^F(F) \leq e_1^\downarrow(F) = e_L(a) \). Now we can see that for any \( y \in L_2^\delta \),

\[
\begin{align*}
  f^\vee(0, y) \\
  = f^\vee(e(0), y) \\
  = \bigvee \{ e_M \circ \mathcal{F}(F, G) \mid e_1^F(F) \leq e_1^G(G) \leq y \} & \quad \text{since } e(0) = 0, \\
  = \bigvee \{ e_M \circ \mathcal{F}(F, G) \mid 0 \in F, e_2^F(G) \leq y \} & \quad \text{by def. of } f^\vee, \\
  \leq \bigvee \{ e_M(F') \mid 0 \in F' \} & \quad \text{by (2.20),} \\
  = \bigvee \{ e_M(F') \mid e_M(F') \leq e(0) \} & \quad \text{by (2.20),} \\
  = 0,
\end{align*}
\]

which is what we wanted to show. \[\blacksquare\]

We can now state an immediate corollary of Lemmas 2.2.10 and 2.2.13:

2.2.14. **Corollary** ([34]). *Let \( f : L_1 \times \cdots \times L_n \to M \) be an \( n \)-ary order-preserving map between lattices.*

1. If \( f \) is a (normal) operator, then \( f^\vee \) is a complete (normal) operator.

2. If \( f \) is a dual (normal) operator, then \( f^\wedge \) is a complete dual (normal) operator.

**Topological properties of operators and join-preserving maps**

It is one of the characteristic features of canonical extension that maps between lattices \( f : L \to M \) have both a lower \( (f^\vee : L^\delta \to M^\delta) \) and an upper extension \( (f^\wedge : L^\delta \to M^\delta) \). These two extensions need not necessarily be different.

2.2.15. **Definition.** We say that an order-preserving map \( f : L \to M \) is smooth if \( f^\vee = f^\wedge \). If we want to emphasize that \( f \) is smooth, we will refer to the canonical extension of \( f \) as \( f^\delta \) rather than \( f^\vee \) or \( f^\wedge \).

In light of the recurring topological themes in this chapter, it may not come as a surprise that smoothness of a map \( f : L \to M \) is a topological property.

2.2.16. **Lemma.** *An order-preserving map \( f : L \to M \) between lattices is smooth iff \( f^\vee : L^\delta \to M^\delta \) is \((\delta, \sigma)\)-continuous.*

**Proof** If \( f^\vee = f^\wedge \), then \( f^\vee \) is both \((\delta^1, \sigma^1)\)-continuous and \((\delta^1, \sigma^1)\)-continuous; hence \( f^\vee \) is also \((\delta, \sigma)\)-continuous. Conversely, if \( f^\vee \) is \((\delta, \sigma)\)-continuous, then it follows from Lemma 2.1.17 and the fact that \( f^\vee \) is order-preserving that \( f^\wedge \) is \((\delta^1, \sigma^1)\)-continuous. It follows by the order dual of Theorem 2.2.4 that \( f^\wedge \leq f^\vee \). Since \( f^\vee \leq f^\wedge \) by Lemma 2.2.3, we find that \( f^\vee = f^\wedge \). \[\blacksquare\]
Before we proceed to the main result about topological properties of operators and join-preserving maps, we prove another technical lemma which says, intuitively, that if \( f : \mathbb{L} \to \mathbb{M} \) preserves binary joins, then \( f^\vee : \mathbb{L}^\delta \to \mathbb{M}^\delta \) has a kind of weak, partial right adjoint.

**2.2.17. Lemma.** Let \( f : \mathbb{L} \to \mathbb{M} \) be an order-preserving map preserving binary joins.

\[ \forall x \in \mathbb{L}^\delta, \forall J \in \mathcal{I} \mathbb{M}, f^\vee(x) \leq e^\vee_M(J) \text{ iff } f^{-1}(J) \in \mathcal{I} \mathbb{L} \text{ and } x \leq e^\vee_L \circ f^{-1}(J). \]

**Proof** Let \( x \in \mathbb{L}^\delta \) and \( J \in \mathcal{I} \mathbb{M} \). We define \( S := \{ F \in \mathcal{F} \mathbb{L} \mid e^\vee_L(F) \leq x \} \); observe that \( S \) is always non-empty since at least \( 1 \in S \). Now if \( f^\vee(x) \leq e^\vee_M(J) \), then since \( f^\vee(x) = \bigvee_{F \in S} e^\vee_L \circ F(f) \), we see that \( \forall F \in S, e^\vee_L \circ F(f) \leq e^\vee_M(J) \). By basic properties of canonical extension it follows that \( \forall F \in S, \mathcal{F} f(F) \not\leq J \). Since \( S \neq \emptyset \), there is at least one \( F \in \mathcal{F} \mathbb{L} \) such that \( \mathcal{F} f(F) \not\leq J \), so by Lemma 2.2.7(2), \( f^{-1}(J) \in \mathcal{I} \mathbb{L} \). It also follows by Lemma 2.2.7(2) that \( \forall F \in S, F \not\leq f^{-1}(J) \), so that \( \forall F \in S, e^\vee_L(F) \leq e^\vee_L \circ f^{-1}(J) \). Since \( x = \bigvee_{F \in S} e^\vee_L(F) \), it follows that \( x \leq e^\vee_L \circ f^{-1}(J) \). The proof of the converse implication is analogous.

We now arrive at the main theorem about topological properties of operators and join-preserving maps. Parts (1) and (3) were already known from [34].

**2.2.18. Theorem.** Let \( f : \mathbb{L} \to \mathbb{M} \) be an order-preserving map between lattices.

1. (a) If \( f \) is an operator, then \( f^\vee \) is \((\sigma^1, \sigma^1)\)-continuous.
   (b) If \( f \) is a dual operator, then \( f^\vee \) is \((\sigma^1, \sigma^1)\)-continuous.

2. (a) If \( f \) preserves binary joins, then \( f^\vee : \mathbb{L}^\delta \to \mathbb{M}^\delta \) is \((\delta^1, \delta^1)\)-continuous.
   (b) If \( f \) preserves binary meets, then \( f^\wedge : \mathbb{L}^\delta \to \mathbb{M}^\delta \) is \((\delta^1, \delta^1)\)-continuous.

3. If \( f \) preserves binary joins or binary meets, then \( f \) is smooth.

4. If \( f \) preserves binary joins and binary meets, then \( f \) is smooth and \( f^\delta : \mathbb{L}^\delta \to \mathbb{M}^\delta \) is \((\delta, \delta)\)-continuous.

**Proof** We will only show the proofs for the statements about \( f^\vee \), since the proofs for the statements about \( f^\wedge \) are order dual.

(1). By Lemma 2.2.10, \( f^\vee \) preserves all non-empty joins in each coordinate, so a fortiori \( f^\vee \) preserves directed joins in each coordinate. It follows from Fact A.3.4 that \( f^\vee : \mathbb{L}^\delta_1 \times \cdots \times \mathbb{L}^\delta_n \to \mathbb{M}^\delta \) preserves directed joins.

(2). We will show that \((f^\vee)^{-1}\) maps basic \( \delta^1 \)-open sets to \( \delta^1 \)-open sets. Let \( J \in \mathcal{I} \mathbb{M} \); we need to show that \((f^\vee)^{-1}(\downarrow e^\vee_M(J))\) is \( \delta^1 \)-open. If \((f^\vee)^{-1}(\downarrow e^\vee_M(J))\) is empty then we are done. If not, then it follows from Lemma 2.2.17 that

\[ (f^\vee)^{-1}(\downarrow e^\vee_M(J)) = \downarrow e^\vee_L(f^{-1}(J)), \] (2.21)
so we see that \((f^\triangledown)^{-1}(\downarrow e^\triangledown_{M}(J))\) is in fact a basic \(\delta^1\)-open set.

(3). Suppose that \(f\) preserves binary joins; then by (2), \(f^\triangledown: L^\delta \to M^\delta\) is \((\delta^1, \delta^1)\)-continuous. Since \(\sigma^1 \subseteq \delta^1\) (Lemma 2.1.28(3)), it follows that \(f^\triangledown\) is \((\delta^1, \sigma^1)\)-continuous. On the other hand, by Theorem 2.2.4(1) we know that \(f^\triangledown\) is \((\delta^1, \sigma^1)\)-continuous. It now follows from Lemma 2.2.16 that \(f^\triangledown\) is smooth.

(4). It follows from (3) that \(f\) is smooth. Now by (2), \(f^\triangledown\) is \((\delta^1, \delta^1)\)-continuous and \(f^\delta\) is \((\delta^1, \delta^1)\)-continuous. Since \(f\) is smooth, i.e. since \(f^\triangledown = f^\delta\), it now follows from general topology that \(f^\delta := f^\triangledown\) is \((\delta, \delta)\)-continuous.

We now turn to an interesting question which we have neglected so far. We took it as part of our definition that \(e_L: L \to L^\delta\) is a lattice embedding, which means that the meet and join of \(L^\delta\) a priori ‘play nice’ with those of \(L\). If we look at meet and join as maps \(\vee_L: L \times L \to L\) and \(\wedge_L: L \times L \to L\) however, we can also ask ourselves what are the canonical extensions of \(\vee_L\) and \(\wedge_L\). Are these indeed the join and meet of \(L^\delta\)? Fortunately, the answer is yes.

**2.2.19. Lemma ([34]).** Let \(e: L \to L^\delta\) be a canonical extension. Then \((\vee^L)^\triangledown = (\vee^L)^\Delta = \vee^{L^\delta}\) and \((\wedge^L)^\triangledown = (\wedge^L)^\Delta = \wedge^{L^\delta}\).

**Proof** We will only consider \(\vee\), since the other case follows by order duality. Since \(\vee^L: L \times L \to L\) is associative, it is a join-preserving map, so by Theorem 2.2.18 \((\vee^L)^\triangledown = (\vee^L)^\Delta\). It follows from order theory that for all \(x, y \in L^\delta\),

\[
x \vee^\delta y = \bigwedge\{z \in L^\delta \mid x, y \leq z\}.
\]

By meet-density of ideal elements, this reduces to

\[
x \vee^\delta y = \bigwedge_{I \in S} e^\triangledown(I),
\]

where \(S := \{I \in \mathcal{I}_L \mid x, y \leq e^\triangledown(I)\}\). On the other hand,

\[
x(\wedge^L)^\Delta y = \bigwedge_{J \in S'} e^\triangledown(J),
\]

where \(S' := \{\mathcal{I} \vee_L(J_1, J_2) \mid x \leq e^\triangledown(J_1), y \leq e^\triangledown(J_2)\}\). Since \(\mathcal{I} \vee^L = \mathcal{I} \vee^L\), we see that if \(I \in S\) then \(\mathcal{I} \vee^L(I, I) = I \in S'\), so \(S' \subseteq S\). Conversely, if \(\mathcal{I} \vee^L(J_1, J_2) \in S'\) and \(x \leq e^\triangledown(J_1), y \leq e^\triangledown(J_2)\), then also \(x, y \leq e^\triangledown(\mathcal{I} \vee^L(J_1, J_2))\), so \(\mathcal{I} \vee^L(J_1, J_2) \in S\) and hence \(S' \subseteq S\). It follows that \((\wedge^L)^\Delta = \wedge^{L^\delta}\). □

Now if \(L\) is a distributive lattice, then we know that \(\wedge_L: L \times L \to L\) is an operator. Consequently, by Corollary 2.2.14, \((\wedge_L)^\triangledown = \wedge^\delta_L\) is a complete operator, so as a bonus we get the following well-known corollary:

**2.2.20. Corollary.** If \(L\) is a distributive lattice then so is \(L^\delta\).

**2.2.21. Remark.** Canonical extensions of distributive lattices have much stronger properties than just being distributive. We will return to this subject in §4.1.
Chapter 2. Canonical extensions: a domain theoretic approach

2.2.3 Canonical extension as a functor I: lattices only

So far in this chapter we have seen that canonical extension is a construction on lattices and maps between lattices. This raises the very natural question whether canonical extension is a functor, and if so between which categories. In this subsection we will see that canonical extension is a functor from the category of bounded lattices and lattice homomorphisms to the category of complete lattices and complete homomorphisms.

In order to establish this result, we will first prove several facts about the interaction between canonical extensions and compositions of order-preserving maps. The most basic such result is the well-known fact [34] that if we have two order-preserving maps

\[ \mathbb{L}_1 \xrightarrow{f} \mathbb{L}_2 \xrightarrow{g} \mathbb{L}_3, \]

then it is always the case that \((gf)^\triangledown \leq g^\triangledown f^\triangledown\) and dually, \(g^\triangle f^\triangle \leq (gf)^\triangle\). This result is supplemented by the observation that if we make certain continuity assumptions about \(f^\triangledown\) or \(g^\triangledown\), we can prove the reverse inequality. This fact was already known in the case of order-preserving maps between distributive lattices [39], however the result is new for the non-distributive case. Armed with these results we can then prove Theorem 2.2.24, which says that canonical extension is a functor from the category of lattices to the category of complete lattices. Theorem 2.2.24 extends a known result from [34] with a new observation about the continuity properties of canonical extensions of lattice homomorphisms. We will revisit the subject of compositions of canonical extensions of maps in §3.2.3, and the subject of functorial behaviour of canonical extension in §3.3.

We will now first state several results about canonical extensions of compositions of order-preserving maps. We begin with a well-known result.

2.2.22. Lemma ([34]). Let \(e_i: \mathbb{L}_i \to \mathbb{L}^i\) be canonical extensions of lattices \(\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3\) and let \(f: \mathbb{L}_1 \to \mathbb{L}_2\) and \(g: \mathbb{L}_2 \to \mathbb{L}_3\) be order-preserving maps. Then the following inequalities hold:

\[
(gf)^\triangledown \leq g^\triangledown f^\triangledown \leq \left\{ \begin{array}{c}
\frac{g^\triangledown f^\triangle}{g^\triangle f^\triangledown} \\
\frac{g^\triangle f^\triangledown}{g^\triangledown f^\triangle}
\end{array} \right\} \leq g^\triangle f^\triangle \leq (gf)^\triangle.
\]

Proof The inequalities \(g^\triangledown f^\triangledown \leq g^\triangledown f^\triangle\) and \(g^\triangledown f^\triangledown \leq g^\triangle f^\triangledown\) follow from Lemma 2.2.3. For the first inequality, observe that

\[
(gf)^\triangledown(x) = \bigvee \{e_3^\triangledown \circ \mathcal{F} g f (F) \mid e_1^\triangledown(F) \leq x\} = \bigvee \{e_3^\triangledown \circ \mathcal{F} g \circ \mathcal{F} f (F) \mid e_1^\triangledown(F) \leq x\} \leq \bigvee \{e_3^\triangledown \circ \mathcal{F} g (F') \mid e_2^\triangledown(F') \leq f^\triangledown(x)\},
\]

where the inequality directly above follows from the fact that \(e_1^\triangledown(F) \leq x\), then also

\[ e_2^\triangledown \circ \mathcal{F} f (F) = f^\triangledown(e_1^\triangledown(F)) \leq f^\triangledown(x). \]

The other inequalities in the statement of the lemma follow by order duality. ■
Next, we present a handful of corollaries of the above lemma.

2.2.23. COROLLARY. Let \( f: \mathbb{L}_1 \to \mathbb{L}_2 \) and \( g: \mathbb{L}_2 \to \mathbb{L}_3 \) be order-preserving maps between lattices.

1. If \( g^\vee f^\vee \) is \((\delta^1, \sigma^1)\)-continuous then \( g^\vee f^\vee = (gf)^\vee \);
2. If \( g^\vee \) is \((\sigma^1, \sigma^1)\)-continuous then \( g^\vee f^\vee = (gf)^\vee \);
3. If \( f^\vee \) is \((\delta^1, \delta^1)\)-continuous then \( g^\vee f^\vee = (gf)^\vee \).

Proof (1). If \( g^\vee f^\vee \) is \((\delta^1, \sigma^1)\)-continuous then by Theorem 2.2.4, \( g^\vee f^\vee \leq (gf)^\vee \).
By Lemma 2.2.22, \((gf)^\vee \leq g^\vee f^\vee \). Statements (2) and (3) are instances of (1).

Recall from Definition 2.2.15 that we call a map \( f: \mathbb{L} \to \mathbb{M} \) smooth if \( f^\vee = f^\delta \), and that we refer to the canonical extension of \( f \) as \( f^\delta: \mathbb{L}^\delta \to \mathbb{M}^\delta \) in that case. Additionally, recall from §A.4 that we denote the category of bounded lattices and lattice homomorphisms by \( \text{Lat} \), and the category of complete lattices and complete lattice homomorphisms by \( \text{CLat} \). We can now state a fundamental theorem about canonical extensions of lattices. Most of this theorem was already known, see e.g. [34]; part (1) is a new observation however.

2.2.24. THEOREM. Let \( \mathbb{L}, \mathbb{M} \) be lattices and let \( f: \mathbb{L} \to \mathbb{M} \) be a lattice homomorphism. Then \( f \) is smooth and

1. \( f^\delta: \mathbb{L}^\delta \to \mathbb{M}^\delta \) is a complete lattice homomorphism which is both \((\delta, \delta)\)-continuous and \((\sigma, \sigma)\)-continuous;
2. If \( f \) is injective, then so is \( f^\delta \);
3. If \( f \) is surjective, then so is \( f^\delta \).

In fact, canonical extension defines a functor from \( \text{Lat} \) to \( \text{CLat} \) and \( e_\mathbb{L}: \mathbb{L} \to \mathbb{L}^\delta \) is a natural transformation.

Proof (1). Since \( f \) preserves binary joins and binary meets, it follows by Lemma 2.2.10 that \( f^\delta \) preserves all non-empty joins and meets. It follows from Lemma 2.2.13 that \( f^\delta \) preserves 0 and 1. Now since complete homomorphisms preserve directed joins and co-directed meets a fortiori, it follows that \( f^\delta \) is \((\sigma, \sigma)\)-continuous. Finally it follows from Theorem 2.2.18(2) that \( f^\delta \) is \((\delta, \delta)\)-continuous.

(2). This follows from Lemma 2.2.6, since \( f \) preserves binary joins.
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(3). Assume \( f \) is surjective; then by (1), \( f^\delta \) is a complete homomorphism, and we already know that \( \mathcal{F} f \) is surjective (see §A.5.1). Let \( x \in \mathcal{M}^\delta \); we now make a straightforward computation:

\[
\begin{align*}
  f^\delta \left( \bigvee \{ e_L^\mathcal{F}(F) \mid e_M^\mathcal{F} \circ \mathcal{F} f(F) \leq x \} \right) \\
  = \bigvee \{ f^\delta \circ e_L^\mathcal{F}(F) \mid e_M^\mathcal{F} \circ \mathcal{F} f(F) \leq x \} & \quad \text{because } f \text{ is a complete hom.,} \\
  = \bigvee \{ e_M^\mathcal{F} \circ \mathcal{F} f(F) \mid e_M^\mathcal{F} \circ \mathcal{F} f(F) \leq x \} & \quad \text{by Lemma 2.2.3,} \\
  = \bigvee \{ e_M^\mathcal{F}(F') \mid e_M^\mathcal{F}(F') \leq x \} & \quad \text{because } \mathcal{F} f \text{ is surjective,} \\
  = x & \quad \text{by join-density of filter elements.}
\end{align*}
\]

Since \( x \in \mathcal{M}^\delta \) was arbitrary it follows that \( f^\delta \) is surjective.

To see that canonical extension is a functor, we will first look at compositions of homomorphisms: consider lattice homomorphisms \( f : L_1 \to L_2 \) and \( g : L_2 \to L_3 \). Since \( g^\delta \) is Scott-continuous by (1), it follows by Corollary 2.2.23 that \( g^\delta f^\delta = (gf)^\delta \).

To see that canonical extension preserves the identity function, we use that fact that \( F \) is a functor. Let \( x \in L^\delta \) be arbitrary, then

\[
\begin{align*}
  (id_L^\delta)(x) & = \bigvee \{ e_L^\mathcal{F} \circ \mathcal{F} id_L(F) \mid e_L^\mathcal{F}(F) \leq x \} & \text{by definition,} \\
  & = \bigvee \{ id_L^\mathcal{F} \circ e_L^\mathcal{F}(F) \mid e_L^\mathcal{F}(F) \leq x \} & \quad \text{because } \mathcal{F} \text{ is a functor,} \\
  & = \bigvee \{ e_L^\mathcal{F}(F) \mid e_L^\mathcal{F}(F) \leq x \} & \quad \text{by definition of } id_L^\mathcal{F}, \\
  & = x & \quad \text{by join-density of filter elements,} \\
  & = id_L^\delta(x)
\end{align*}
\]

This proves that canonical extension is a functor on lattices and lattice homomorphisms. The fact that \( e_L : L \to L^\delta \) is a natural transformation follows from the fact that \( f^\delta \) is an extension of \( f \).

We conclude this section with a minor observation about canonical extensions of sublattices which will be useful later.

2.2.25. Corollary. If \( \mathbb{L} \) is a sublattice of \( \mathbb{M} \), then \( \mathbb{L}^\delta \) is isomorphic to a complete sublattice of \( \mathbb{M}^\delta \).

2.2.4 Conclusions and further work

The main contribution of this section lies in the technical results concerning filters and ideals, and the results about topological properties of \( f^\triangledown \) and \( f^\triangleleft \). Most of the topological results in this section can also be found in our paper with M. Gehrke [43]. Many of these where inspired by what was known about distributive lattices from the work of Gehrke and Jónsson [39]. It was not known however that distributivity of the lattices involved, which is a central assumption in [39], is in
no way essential when one wants to discuss topological properties of canonical extensions of order-preserving maps. In the work of Ghilardi and Meloni [44], the action of join-preserving maps on filters and ideals receives the attention it deserves, although the authors nowhere refer explicitly to canonical extensions. The idea to define canonical extensions of order-preserving maps, rather than join-preserving maps or operators, via the filter and ideal completion seems to have been introduced by Gehrke and Priestley in [41], but in that paper most attention was directed at lattice homomorphisms.

The definition of $f^\uparrow$ and $f^\downarrow$ for an order-preserving map in §2.2.1 is well known from the work of Gehrke and Harding [34]. Theorem 2.2.4, which describes $f^\uparrow$ and $f^\downarrow$ as the largest and smallest continuous extensions of an order-preserving map $f$, respectively, was previously only known to hold for distributive lattices. It raises questions which we will return to in Remark 3.2.22. The result concerning preservation of order embeddings (Lemma 2.2.6) is also an improvement over what was previously known (viz. that the canonical extension of an injective lattice homomorphism is again injective).

The technical results about order-preserving maps applied to filters and ideals in §2.2.2 are new, although similar results can be found in [44]. It was already known from [34] that canonical extensions of join-preserving maps and operators are very well-behaved (Theorem 2.2.18); the topological results we presented are new however.

Further work

- In this section, we have made much use of the filter completion and the ideal completion, which we borrowed from domain theory. It would be interesting to see if there are more domain-theoretic tools available that we can use to better understand and develop the theory of canonical extensions. We will see an example of this in the next section, where we use dcpo presentations to describe canonical extensions.

- Another interesting question is to see whether the domain theory tools and topological methods of this section can also be applied in the setting of monotone partially ordered algebras, as studied by Dunn et al. in [32].

- A more specific question which needs to be resolved is whether the canonical extension of any order embedding is again an order embedding, cf. Lemma 2.2.6.

2.3 Canonical extensions via dcpo presentations

Thus far, we have explored canonical extensions of lattices and order-preserving maps via topological methods. We will conclude this chapter with a section, based
on results from [42], in which we show that canonical extensions can also be understood via dcpo presentations, a technique from domain theory [60]. Apart from being interesting in its own right, the perspective on canonical extensions using dcpo presentations sheds a different light on the issue of extending maps between lattices to maps between their canonical extensions, and ultimately on the question whether inequations valid on a distributive lattice with operators $A$ are also valid on its canonical extension $A^\delta$, i.e. the question when inequations are canonical.

The idea of dcpo presentations is to give a unique description of a dcpo $D$, that is, of a directed complete partial order, by specifying a partially ordered or pre-ordered set of generators $P$ such that each element of $D$ is the directed join of the generators below it. If one imposes no relations on the generators other than the order, that is, if one freely adds all directed joins to $P$, then this is equivalent to taking the ideal completion of $P$. Thus we can give presentations of algebraic dcpos. If, on the other hand, one imposes relations of the shape $a \leq \bigvee U$, for $\{a\} \cup U \subseteq P$, then we may get a presentation for any dcpo, see [60].

To see how we may use this to get a dcpo presentation of the canonical extension of a lattice $L$, recall from Theorem 2.1.25 that there exists an order embedding $g: L^\delta \rightarrow I \mathcal{F} L$ such that $g \circ e^F_L = \downarrow_{\mathcal{F} L}$.

\[
\begin{array}{ccc}
L & \xrightarrow{e_L} & L^\delta \\
\downarrow_{I_L} & & \downarrow_{I \mathcal{F} L} \\
F L & \xrightarrow{e^F_L} & \mathcal{F} L \\
\end{array}
\]

where $g = (e^F_L)^{-1} \circ \downarrow_{L^\delta}$ and $e^F_L$ are embeddings. What this diagram tells us is that the canonical extension of $L$ ‘sits between’ $F L$ and $I \mathcal{F} L$, the structure one obtains by freely adding all directed joins to $F L$. The idea of the dcpo presentation of canonical extension is to be selective and only add some directed joins to $F L$, so that we obtain $L^\delta$. Because of the way that filters, ideals and canonical extensions dualize with respect to order (see Lemma 2.1.26), we can just as well regard $L^\delta$ as an object sitting in between $I L$ and $I \mathcal{F} L$; this would lead to a description of $L^\delta$ via a co-dcpo presentation over $I L$. We choose not to engage in this exercise of dualization.

Since dcpo presentations are characterized externally, that is, by conditions on their behaviour with respect to maps, it is quite natural to expect that we can use them to describe canonical extensions of maps between lattices. In this section, we will restrict our attention to maps of the type $f: L^n \rightarrow L$, with $n$ a natural number. This choice is dictated by economy rather than necessity: we will develop just enough of the technique of extending maps between lattices to canonical extensions via dcpo presentations to allow us to prove a canonicity result in §3.3.3. The key observation is that under the right assumptions, the canonical extension of a map can be seen as an instance of an extension via dcpo presentations, so that one can apply results about dcpo presentations to canonical extensions.
2.3. Canonical extensions via dcpo presentations

2.3.1 Dcpo presentations

In this subsection, we introduce dcpo presentations, which are a technical tool for uniquely specifying a dcpo without spelling out its entire structure. This is an instance of a very general algebra technique, namely that of specifications by generators and relations.

2.3.1. Definition. A dcpo presentation [60] is a triple \( \langle P, \sqsubseteq, \triangleleft \rangle \) where

- \( \langle P, \sqsubseteq \rangle \) is a pre-order;
- \( \triangleleft \subseteq P \times \mathcal{P}(P) \) is a binary relation such that \( a \triangleleft U \) only if \( U \subseteq P \) is non-empty and directed.

An order-preserving map \( f: P \to \mathcal{D} \) to a dcpo \( \mathcal{D} \) is cover-stable if for all \( a \triangleleft U \), \( f(a) \leq \bigvee f[U] \).

In other words, a dcpo presentation consists of a pre-ordered set of generators \( \langle P, \sqsubseteq \rangle \) together with set of relations of the form \( a \leq \bigvee U \). But what does it mean for a dcpo presentation to uniquely describe, i.e. to present a dcpo?

2.3.2. Definition. A dcpo presentation \( \langle P, \sqsubseteq, \triangleleft \rangle \) presents a dcpo \( \mathcal{D} \) if there exists a cover-stable order-preserving map \( \eta: P \to \mathcal{D} \) such that for all dcpos \( \mathcal{E} \), if \( f: P \to \mathcal{E} \) is a cover-stable order-preserving map then there exists a unique Scott-continuous \( f': \mathcal{D} \to \mathcal{E} \) such that \( f' \circ \eta = f \). If this is the case, we say that \( \langle P, \sqsubseteq, \triangleleft \rangle \) presents \( \mathcal{D} \) via \( \eta \).

\[
\begin{array}{c}
\mathcal{D} \\
\eta \downarrow \\
\mathcal{E} \\
\end{array}
\]

\[
P \xrightarrow{f} \mathcal{E}
\]

We may ask ourselves if every dcpo presentation uniquely describes, i.e. presents a dcpo. This is indeed the case; one can show this by constructing a dcpo from a given presentation using so-called \( C \)-ideals. For more information we refer the reader to [60].

2.3.3. Fact. Every dcpo presentation presents a dcpo.

We conclude this subsection with two trivial examples of dcpo presentations.

2.3.4. Example. If \( P = \langle P, \leq \rangle \) is a poset, then \( \langle P, \leq, \emptyset \rangle \) presents \( \mathcal{I}P \) via \( \downarrow: P \to \mathcal{I}P \). This follows from the universal property of the ideal completion. What makes this example trivial is the fact that there are no relations imposed on the generators.

If \( \mathcal{D} \) is a dcpo, then \( \langle D, \leq, \triangleleft \mathcal{D} \rangle \), where \( a \triangleleft \mathcal{D} U \) iff \( a \leq \bigvee U \), presents \( \mathcal{D} \) itself via \( \text{id}_\mathcal{D}: \mathcal{D} \to \mathcal{D} \). This is a trivial example because there are as many generators
2.3.2 A dcpo presentation of the canonical extension

We now define a dcpo presentation given a lattice \( L \), with the aim of showing that this dcpo presentation presents \( L^\delta \). This is a two-stage process: first we take \( L \) and we define a presentation \( \Delta(L) \), using \( F_L \) as the set of generators. Then, we show that \( \Delta(L) \) presents \( L^\delta \).

2.3.5. Definition. Given a lattice \( L \), we define a dcpo presentation \( \Delta(L) := \langle F_L, \supseteq, \triangleleft_L \rangle \), where for all \( F \in F_L \) and \( S \subseteq F_L \) directed,

\[
F \triangleleft_L S \text{ iff } \forall I \in I_L, [\forall F' \in S, F' \not\subseteq I] \Rightarrow F \not\subseteq I.
\]

We want to emphasize that this definition is not being pulled out of a hat. Firstly, since the filter elements of \( L^\delta \) are join-dense in \( L^\delta \), it is not a strange idea to take the filters of \( L \) as generators. Secondly, we know by Lemma 2.1.29 that if \( e : L \rightarrow L^\delta \) is the canonical extension of \( L \), then

\[
\bigvee \{e^F(F) \mid F \in S\} = \bigwedge \{e^I(I) \mid \forall F' \in S, F' \not\subseteq I\},
\]

and we will see in the proof of the theorem below that this equation is essentially equivalent to the relations of the shape \( F \triangleleft_L S \) we are imposing on \( \Delta(L) \).

2.3.6. Theorem. Let \( L \) be a lattice and let \( e : L \rightarrow L^\delta \) be its canonical extension. Then \( \Delta(L) \) presents \( L^\delta \) via \( e^F : F_L \rightarrow L^\delta \).

Proof Observe that \( e^F : F_L \rightarrow L^\delta \) is order-preserving by Lemma 2.1.28(1); we now first need to show that \( e^F \) is cover-stable. We will show something stronger: for all \( F \in F_L \) and \( S \subseteq F_L \) directed,

\[
F \triangleleft_L S \iff \forall F' \in S, F' \not\subseteq I. \tag{2.22}
\]

The key to this observation is Lemma 2.1.29, which states that

\[
\bigvee_{F' \in S} e^F(F') = \bigwedge \{e^I(F) \mid \forall F' \in S, F' \not\subseteq I\}. \tag{†}
\]

Now

- \( e^F(F) \leq \bigvee_{F' \in S} e^F(F') \)
- \( e^F(F) \leq \bigwedge \{e^I(F) \mid \forall F' \in S, F' \not\subseteq I\} \) by Lemma 2.1.29,
- \( e^F(F) \leq e^I(F) \) by order theory,
- \( \forall I \in I_L, [\forall F' \in S, F' \not\subseteq I] \Rightarrow e^F(F) \leq e^I(F) \) (†),
- \( F \triangleleft_L S \) by def. of \( \triangleleft_L \),

as there are elements in the dcpo being presented and the order is already fully specified: the main reason for considering presentations of objects via generators and relations, rather than the objects themselves, is that the presentations can be simpler to describe than the objects they are presenting. In the current example, this is not the case.
where (1) follows from the basic fact about canonical extensions that $e^\mathcal{F}(F) \leq e^\mathcal{F}(I)$ iff $F \downarrow I$. It follows that (2.22) holds.

Next, suppose that $f: \mathcal{F} \downarrow \mathcal{D}$ is an order-preserving cover-stable map to a dcpo $\mathcal{D}$. We define $f': \mathcal{L}^\delta \to \mathcal{D}$ by setting 

$$f': x \mapsto \bigvee \{f(F) \mid e^\mathcal{F}(F) \leq x\}.$$ 

We need to show (1) that $f'$ is well-defined and Scott-continuous, (2) that $f' \circ e^\mathcal{F} = f$ and (3) that $f'$ is unique with respect to properties (1) and (2).

(1). For any $x \in \mathcal{L}^\delta$, the set $\downarrow x$ is an ideal. Since $e^\mathcal{F}$ is a $\vee$-homomorphism (Lemma 2.1.28(1)), it follows that $(e^\mathcal{F})^{-1}(\downarrow x) = \{F \mid e^\mathcal{F}(F) \leq x\}$ is also an ideal and hence, directed. Since $f$ is order-preserving it follows that $f'(x)$ is a directed join, so that indeed $f'$ is well-defined. To see that $f'$ is Scott-continuous, take $S \subseteq \mathcal{L}^\delta$ directed. Observe that

$$\bigvee S = \bigvee_{x \in S} \bigvee \{e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\}$$ 

by join-density of filter elements,

$$= \bigvee \bigcup_{x \in S} \{e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\}$$ 

by associativity of $\bigvee$,

$$= \bigvee e^\mathcal{F} \big[\bigcup_{x \in S} \{F \mid e^\mathcal{F}(F) \leq x\}\big]$$ 

by elementary set theory.

It is not hard to see that $\bigcup_{x \in S} \{F \mid e^\mathcal{F}(F) \leq x\}$ is a directed union of directed sets; consequently, we will simply assume that $S$ is a directed set of filter elements; say $S = \{e^\mathcal{F}(F) \mid F \in S'\}$ where $S' \subseteq \mathcal{F} \downarrow \mathcal{L}$ is directed. Now observe that

$$f'(\bigvee_{F \in S'} e^\mathcal{F}(F)) = \bigvee \{f(F') \mid e^\mathcal{F}(F') \leq \bigvee_{F \in S'} e^\mathcal{F}(F)\} = \bigvee \{f(F') \mid F' \downarrow \mathcal{L} S'\},$$

where the last equality follows by (2.22). Since $f$ is cover-stable, $F' \downarrow \mathcal{L} S'$ implies $f(F') \leq \bigvee f[S']$. We see that

$$f'(\bigvee_{F \in S'} e^\mathcal{F}(F)) = \bigvee \{f(F') \mid F' \downarrow \mathcal{L} S'\} \leq \bigvee_{F \in S'} f(F) = \bigvee_{F \in S'} f'(e^\mathcal{F}(F)),$$

so that it follows that $f'$ is Scott-continuous.

(2). To see that $f' \circ e^\mathcal{F} = f$, observe that

$$f' \circ e^\mathcal{F} = \bigvee \{f(F') \mid e^\mathcal{F}(F') \leq e^\mathcal{F}(F)\}.$$ 

Because $e^\mathcal{F}$ is an order embedding, the join in the RHS above has a maximal element, viz. $f(F)$. It follows that $f' \circ e^\mathcal{F} = f$.

(3). Suppose that $f'': \mathcal{L}^\delta \to \mathcal{D}$ is Scott-continuous and that $f'' \circ e^\mathcal{F} = f$. Take $x \in \mathcal{L}^\delta$, then since $x = \bigvee \{e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\}$ is a directed join, we see that

$$f''(x) = f''(\bigvee \{e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\}) = \bigvee \{f'' \circ e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x\}$$

$$= \bigvee \{f(F) \mid e^\mathcal{F}(F) \leq x\} = f'(x).$$

It follows that $f' = f''$ so that $f'$ is unique. $\square$
What have we learned now? Firstly, we have discovered a new characterization of the canonical extension of a lattice $L$, namely as a certain dcpo generated by the filters of $L$. Secondly, we have paved the way for applying general results about dcpo algebras to canonical extensions of lattice-based algebras, as we will see later.

### 2.3.3 Extending maps via dcpo presentations

We will now briefly look at extensions of maps via dcpo presentations. The goal is to be able to use results about dcpo presentations to prove facts about canonical extensions. Specifically, we would like to be able to lift a map $f: L^n \to L$ through our two-stage construction, first to a map defined on $\mathcal{F}L$ and then via the dcpo presentation to a map on $L^\delta$. We first state the facts we need for the second stage of extending $f$.

Let $\langle P, \sqsubseteq, \ll \rangle$ be a dcpo presentation and let $f: P^n \to P$ be an order-preserving map. We say $f$ is cover-stable if $f$ preserves covers in each coordinate, i.e. for all $1 \leq i \leq n$, for all $a_1, \ldots, a_n \in P$, for all $U \subseteq P$,

$$a_i \ll U \Rightarrow f(a_1, \ldots, a_n) \ll \{ f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) \mid b \in U \}.$$

#### 2.3.7. Fact ([60]).

Let $\langle P, \sqsubseteq, \ll \rangle$ be a dcpo presentation which presents a dcpo $D$ via $\eta: P \to D$. If $f: P^n \to P$ is a cover-stable order-preserving map, then there exists a unique Scott-continuous map $\overline{f}: D^n \to D$ which extends $f$, i.e. such that $\overline{f} \circ \eta^n = \eta \circ f$.

Suppose we are given a map $f: L^n \to L$, and recall that $\mathcal{F}f(F_1, \ldots, F_n) := \uparrow f[F_1, \ldots, F_n]$. How do we know if $f$ is an operator, i.e. if $f$ preserves binary joins in each coordinate?

#### 2.3.8. Lemma.

Let $L$ be a lattice. If $f: L^n \to L$ is an operator, then $\mathcal{F}f: (\mathcal{F}L)^n \to \mathcal{F}L$ is cover-stable with respect to $\Delta(L)$. Consequently, $\mathcal{F}f$ extends to a map $\overline{\mathcal{F}\mathcal{f}}: (L^\delta)^n \to L^\delta$; moreover, $\overline{\mathcal{F}\mathcal{f}} = f^\oplus$.

**Proof** Let $e: L \to L^\delta$ be the canonical extension of $L$ and let $f: L^n \to L$ be an operator; to lighten the notation, we assume that $n = 2$.

First, we need to show that $\mathcal{F}f$ is cover-stable with respect to $\Delta(L)$. It suffices to show that $\mathcal{F}f$ is cover-stable in its first coordinate. So suppose that $\{ F, G \} \cup S \subseteq \mathcal{F}L$ such that $F \trianglelefteq_{\mathcal{L}} S$; we want to show that $\mathcal{F}f(F, G) \trianglelefteq_{\mathcal{L}} \{ \mathcal{F}f(F', G) \mid F' \in S \}$.

By definition of $\trianglelefteq_{\mathcal{L}}$, this amounts to showing that if

$$\forall I \in \mathcal{I}L_1, \forall F' \in S, F \triangleright I \Rightarrow F' \triangleright I,$$

then

$$\forall I \in \mathcal{I}L_1, F \triangleright I \Rightarrow \exists I' \in \mathcal{I}L_1, \forall F' \in S, F' \triangleright I' \Rightarrow F' \triangleright I'.$$
then also
\[
\forall J \in \mathcal{I} \forall F' \in S, \mathcal{F} f(F', G) \not\ll J \Rightarrow \mathcal{F} f(F, G) \not\ll J.
\]
But that is exactly the statement of Lemma 2.2.9, so it follows immediately that \( \mathcal{F} f \) is cover-stable.

Now, observe that by Fact 2.3.7, we know that \( \mathcal{F} f \) has a Scott-continuous extension \( \overline{\mathcal{F}} f \) such that \( \overline{\mathcal{F}} f \circ (e^\mathcal{F} \times e^\mathcal{F}) = e^\mathcal{F} \circ \mathcal{F} f \). We will show that \( \overline{\mathcal{F}} f = f^\forall \). Take \( x, y \in L^\delta \), then
\[
\overline{\mathcal{F}} f(x, y) = \overline{\mathcal{F}} f \left( \bigvee \{ e^\mathcal{F}(F) \mid e^\mathcal{F}(F) \leq x \}, \bigvee \{ e^\mathcal{F}(G) \mid e^\mathcal{F}(G) \leq y \} \right),
\]
by join-density of filter elements. Now since each of the joins above is directed (Lemma 2.1.28) and \( \overline{\mathcal{F}} f \) is Scott-continuous, we see that
\[
\overline{\mathcal{F}} f(x, y) = \bigvee \{ \overline{\mathcal{F}} f(e^\mathcal{F}(F), e^\mathcal{F}(G)) \mid e^\mathcal{F}(F) \leq x, e^\mathcal{F}(G) \leq y \}
= \bigvee \{ e^\mathcal{F} \circ \mathcal{F} f(F, G) \mid e^\mathcal{F}(F) \leq x, e^\mathcal{F}(G) \leq y \}
= f^\forall(x, y),
\]
by F. 2.3.7, where the last equality follows by definition of \( f^\forall \). Since \( x, y \in L^\delta \) were arbitrary, it follows that \( \overline{\mathcal{F}} f = f^\forall \).  

Thus, we see that canonical extensions of operators can be described using dcpo presentation techniques. This concludes our discussion of dcpo presentations for now. We will use what we have learned here later on, in §3.3.3.

### 2.3.4 Conclusions and further work

This section is based on a paper of M. Gehrke and the author [42], which was written to demonstrate how a canonicity result for distributive lattices with operators from [38] can be seen as a special case of a result concerning dcpo algebras from [60], see §3.3.3.

What we have seen here (and what we will see in §3.3.3) is part of the intersection of the structures and results which can be described both using dcpo algebras and canonical extensions. It would be interesting to further explore the overlapping area of the two fields.