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MODAL OPERATORS ON COMPACT REGULAR FRAMES AND DE VRIES ALGEBRAS

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ABSTRACT. In [7] we introduced the category MKHaus of modal compact Hausdorff spaces, and showed these were concrete realizations of coalgebras for the Vietoris functor on compact Hausdorff spaces, much as modal spaces are coalgebras for the Vietoris functor on Stone spaces. Also in [7] we introduced the categories MKRFrm and MDV of modal compact regular frames, and modal de Vries algebras as algebraic counterparts to modal compact Hausdorff spaces, much as modal algebras are algebraic counterparts to modal spaces. In [7], MKRFrm and MDV were shown to be dually equivalent to MKHaus, hence equivalent to one another.

Here we provide a direct, choice-free proof of the equivalence of MKRFrm and MDV. We also detail connections between modal compact regular frames and the Vietoris construction for frames [19, 20], discuss a Vietoris construction for de Vries algebras, and how it is linked to modal de Vries algebras. Also described is an alternative approach to the duality of MKRFrm and MKHaus obtained by using modal de Vries algebras as an intermediary.

1. INTRODUCTION

In [7] we began a program of lifting structures and techniques of modal logic, based fundamentally on Stone spaces and Boolean algebras, to the setting of compact Hausdorff spaces, de Vries algebras, and compact regular frames. Here, we consider aspects of this work more closely linked to the study of point-free topology than to modal logic. While we briefly recall some important facts from [7], the reader would benefit from having access to this paper when reading this note.

A modal space, or descriptive frame, (X, R) is a Stone space X with binary relation R satisfying certain properties equivalent to requiring the associated map from X into its Vietoris space $\mathcal{V}(X)$ be continuous. With the so-called p-morphisms between them, the category MS of modal spaces is isomorphic to the category of coalgebras for the Vietoris functor on Stone spaces. This lies at the heart of the coalgebraic treatment of modal logic. A modal algebra $(B, \Diamond)$ is a Boolean algebra with unary operation $\Diamond$ that preserves finite joins. The category MA of modal algebras and the homomorphisms between them is dually equivalent to MS via a lifting of Stone duality. These equivalences and dual equivalences tie the coalgebraic, algebraic, and relational treatments of modal logic.

In [7] the situation was lifted from the setting of Stone spaces to compact Hausdorff spaces. We defined a modal compact Hausdorff space (X, R) to be a compact Hausdorff space with binary relation R satisfying conditions equivalent to having the associated map from X to its Vietoris space $\mathcal{V}(X)$ be continuous. Then with morphisms again being p-morphisms, we showed the category MKHaus of modal compact Hausdorff spaces is isomorphic to the category of coalgebras for the Vietoris functor on the category KHaus of compact Hausdorff spaces. For algebraic counterparts to modal compact Hausdorff spaces, we lifted Isbell duality between KHaus and compact regular frames, and de Vries duality between KHaus and de Vries algebras, obtaining categories MKRFrm of modal compact regular frames, and MDV of modal de Vries algebras, each dually equivalent to MKHaus. For various reasons, the category MDV was a bit poorly behaved. We defined two full subcategories of MDV, the categories LMDV and UMDV of lower and upper continuous modal de Vries algebras, that were better behaved, and showed both were equivalent to MDV. The situation is summarized in Figure 1 below.

The functors in Figure 1 are described in [7]. Those between MKRFrm and MKHaus lift the usual point and open set functors between compact regular frames and compact Hausdorff spaces, and those between MDV and MKHaus lift the usual end and regular open set functors between de Vries algebras and compact Hausdorff spaces. As such, they require the axiom of choice. The composite of these functors then gives

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$$
\begin{tikzcd}
 & \mathsf{MDV} \arrow[dl, "\mathrm{L}"'] \arrow[dr, "\mathrm{U}"] \arrow[d, shift right, "(-)_*"'] & \\
\mathsf{LMDV} & \mathsf{MKHaus} \arrow[l, "(-)^L"'] \arrow[r, "(-)^U"] \arrow[u, shift right, "(-)^*"'] \arrow[d, shift right, "\Omega"'] & \mathsf{UMDV} \\
 & \mathsf{MKRFrm} \arrow[u, shift right, "\mathfrak{p}"'] &
\end{tikzcd}
$$

FIGURE 1

an equivalence between MKRFrm and MDV, but again, this requires the axiom of choice. The equivalences between MDV and its subcategories LMDV and UMDV are choice-free.

A primary purpose here is to give a direct, choice-free proof of the equivalence of MKRFrm and each of MDV, LMDV and UMDV. To do so, we construct functors $\mathfrak{L} : \mathsf{MKRFrm} \to \mathsf{LMDV}$ and $\mathfrak{U} : \mathsf{MKRFrm} \to \mathsf{UMDV}$ that lift the Booleanization functor in two ways, and a functor $\mathfrak{R} : \mathsf{MDV} \to \mathsf{MKRFrm}$ that lifts the round ideal functor. After the preliminaries in Section 2, this equivalence is established in Section 3.

The definition of modal compact regular frames involves identities for the modal operators that appear in Johnstone's construction of Vietoris frames [19, 20]. This is not surprising as modal compact regular frames arise as algebraic counterparts of coalgebras for the Vietoris functor on compact Hausdorff spaces. The details of this connection are given in Section 4. In this section we also discuss a counterpart of the Vietoris construction for de Vries algebras.

The equivalence of MKRFrm and MDV of Section 3 composed with the dual equivalence of MDV and MKHaus of [7] provides an alternative approach to the duality of MKRFrm and MKHaus. Restricted to KRFrm and KHaus, this composite is a particular case of Hofmann-Lawson duality [17], and closely resembles Stone duality. In the modal setting, it resembles the familiar duality between modal algebras and modal spaces. Details of this alternative approach are given in Section 5.

2. PRELIMINARIES

We briefly recall the primary definitions. The reader should consult [7] for complete details.

Definition 2.1. *A* frame *is a complete lattice L where finite meets distribute over infinite joins, and a* frame homomorphism *is a map between frames preserving finite meets and infinite joins. A frame is* compact *if $\bigvee S = 1$ implies there is a finite subset $T \subseteq S$ with $\bigvee T = 1$. Using $\neg a$ for the pseudocomplement of an element a, we say a is* well inside *b, and write $a \prec b$, if $\neg a \vee b = 1$. A frame L is a* regular frame *if for each $b \in L$ we have $b = \bigvee\{a : a \prec b\}$. The category of compact regular frames and the frame homomorphisms between them is denoted* KRFrm.

For more about compact regular frames see, e.g., [18, 4, 19, 22].

Definition 2.2. *A* modal compact regular frame *(abbreviated:* MKR*-frame) is a triple $\mathcal{L} = (L, \Box, \Diamond)$ where L is a compact regular frame, and $\Box, \Diamond$ are unary operations on L satisfying the following conditions.*

(1) *$\Box$ preserves finite meets, so $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.*
(2) *$\Diamond$ preserves finite joins, so $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.*
(3) *$\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.*
(4) *$\Box, \Diamond$ preserve directed joins, so $\Box \bigvee S = \bigvee\{\Box s : s \in S\}$, $\Diamond \bigvee S = \bigvee\{\Diamond s : s \in S\}$ for any up-directed S.*

An MKR*-morphism is a frame homomorphism h that satisfies $h(\Box a) = \Box h(a)$ and $h(\Diamond a) = \Diamond h(a)$. The category of modal compact regular frames and their morphisms, composed by ordinary function composition, is denoted* MKRFrm.

We next describe de Vries algebras. Here, as is common, we use the symbol $\prec$ for a certain type of relation on a Boolean frame (complete Boolean algebra). This is a different usage than in Definition 2.1, although there are many connections. For further details see [11, 5, 7, 8], as well as Section 3 below.

Definition 2.3. *A* de Vries algebra *is a pair* $(A, \prec)$ *where* A *is a Boolean frame and* $\prec$ *is a binary relation on* A*, called a* proximity*, satisfying*

(1) $1 \prec 1$.
(2) $a \prec b$ *implies* $a \le b$.
(3) $a \le b \prec c \le d$ *implies* $a \prec d$.
(4) $a \prec b, c$ *implies* $a \prec b \wedge c$.
(5) $a \prec b$ *implies* $\neg b \prec \neg a$.
(6) $a \prec b$ *implies there exists* c *with* $a \prec c \prec b$.
(7) $a \ne 0$ *implies there exists* $b \ne 0$ *with* $b \prec a$.

A morphism between de Vries algebras is a function α *that satisfies (i)* $\alpha(0) = 0$*, (ii)* $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$*, (iii)* $a \prec b$ *implies* $\neg\alpha(\neg a) \prec \alpha(b)$*, and (iv)* $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$.

The motivating example of a de Vries algebra is the complete Boolean algebra $\mathcal{RO}X$ of regular open sets of a compact Hausdorff space X with relation $\prec$ on $\mathcal{RO}X$ defined by $S \prec T$ if $\mathbf{C}S \subseteq T$ where $\mathbf{C}$ is usual topological closure. A continuous map $f : X \to Y$ between compact Hausdorff spaces gives a de Vries morphism $\mathbf{IC}f^{-1}[-]$ from $\mathcal{RO}Y$ to $\mathcal{RO}X$ where $\mathbf{I}$ is usual topological interior. In this setting, one can see that the ordinary function composite of de Vries morphisms need not be a de Vries morphism.

Definition 2.4. *For de Vries morphisms* α *and* β*, define their composite to be* $\beta \star \alpha$ *where*

$$(\beta \star \alpha)(a) = \bigvee\{\beta\alpha(b) : b \prec a\}.$$

Let DeV *be the category of de Vries algebras and their morphisms under this* $*$ *composition.*

Remark 2.5. The idea of a proximity has a long history, see [21] for details. A number of authors have considered structures closely related to de Vries algebras; see, e.g., [23, 12, 3, 13, 24]. The crucial notion of a de Vries morphism essential for obtaining categorical duality appears to originate in [11]. Further discussion can be found in [5, 8].

Definition 2.6. *A* modal de Vries algebra *(abbreviated:* MDV*-algebra) is a triple* $\mathfrak{A} = (A, \prec, \Diamond)$ *where* $(A, \prec)$ *is a de Vries algebra and* $\Diamond$ *is a unary operation on* A *that satisfies the following conditions.*

(1) $\Diamond 0 = 0$.
(2) $a_1 \prec b_1$ *and* $a_2 \prec b_2$ *imply* $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$.

A morphism between modal de Vries algebras is a de Vries morphism α *for which* $a \prec b$ *implies both* $\alpha(\Diamond a) \prec \Diamond\alpha(b)$ *and* $\Diamond\alpha(a) \prec \alpha(\Diamond b)$*. Let* MDV *be the category of modal de Vries algebras and morphisms with composition being the* $*$ *composition of Definition 2.4.*

Two full subcategories of MDV play an important role in [7], and also in our considerations here.

Definition 2.7. *An* MDV*-algebra* $(A, \prec, \Diamond)$ *is called* lower continuous *if* $\Diamond a = \bigvee\{\Diamond b : b \prec a\}$ *and* upper continuous *if* $\Diamond a = \bigwedge\{\Diamond b : a \prec b\}$*. Let* LMDV *and* UMDV *be the full subcategories of* MDV *consisting of all lower, respectively upper, continuous* MDV*-algebras.*

We recall that in [7, Sec. 4.3] it was shown that each member of MDV is isomorphic to a member of LMDV and to a member of UMDV, this despite the fact that a modal de Vries algebra need be neither lower nor upper continuous. This somewhat counterintuitive situation is due to the fact that composition in MDV is not function composition, and isomorphisms are not structure preserving bijections.

3. Equivalence of MKRFrm, MDV, LMDV, and UMDV

In this section we provide direct equivalences between MKRFrm and each of MDV, LMDV, and UMDV. These proofs do not rely on the axiom of choice, as did ones in [7].

Definition 3.1. *For a de Vries algebra* $(A, \prec)$ *and* $S \subseteq A$*, define* ${\downarrow}S = \{a : a \le s \text{ for some } s \in S\}$*, and* $\mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}S = \{a : a \prec s \text{ for some } s \in S\}$*. An ideal* I *of* A *is called* round *if* $I = \mathrel{\rotatebox[origin=c]{-90}{$\twoheadrightarrow$}}I$.

It is known (see, e.g., [3, Lem. 2] or [8, Prop. 4.6]) that the collection $\mathfrak{R}A$ of all round ideals of A is a subframe of the frame of all ideals of A.

Definition 3.2. *For $\mathfrak{A} = (A, \prec, \Diamond)$ an* MDV*-algebra, define $\Box$ on A by setting $\Box a = \neg \Diamond \neg a$ for all $a \in A$.*

Lemma 3.3. *Let $\mathfrak{A} = (A, \prec, \Diamond)$ be an* MDV*-algebra and $a \prec b$, $a_1 \prec b_1$, $a_2 \prec b_2$. Then*

(1) *$\Diamond a \prec \Diamond b$ and $\Box a \prec \Box b$.*
(2) *$\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$ and $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$.*
(3) *$\Box(a_1 \vee a_2) \prec \Box b_1 \vee \Diamond b_2$ and $\Box a_1 \wedge \Diamond a_2 \prec \Diamond(b_1 \wedge b_2)$.*

Proof. The definition of an MDV-algebra gives $\Diamond a \prec \Diamond b$ and $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$. In any de Vries algebra we have $a \prec b$ iff $\neg b \prec \neg a$. This gives $\Box a \prec \Box b$ and $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$. So (1) and (2) are established. For (3) use interpolation to find $a_1 \prec c_1 \prec d_1 \prec b_1$ and $a_2 \prec c_2 \prec d_2 \prec b_2$. Then $a_1 \vee a_2 \prec c_1 \vee c_2$ and $\neg d_2 \prec \neg c_2$, so by (2), $\Box(a_1 \vee a_2) \wedge \Box\neg d_2 \prec \Box((c_1 \vee c_2) \wedge \neg c_2)$. As $(c_1 \vee c_2) \wedge \neg c_2 \leq c_1 \prec d_1$, applying (1) gives $\Box(a_1 \vee a_2) \wedge \Box\neg d_2 \prec \Box d_1$, hence $\Box(a_1 \vee a_2) \leq \Box d_1 \vee \neg \Box \neg d_2 = \Box d_1 \vee \Diamond d_2$. Finally use (1) once again to obtain $\Box d_1 \vee \Diamond d_2 \prec \Box b_1 \vee \Diamond b_2$. This gives the first statement in (3). Using that $x \prec y$ iff $\neg y \prec \neg x$, the second statement in (3) is equivalent to $\Box(\neg b_1 \vee \neg b_2) \prec \Diamond\neg a_1 \vee \Box\neg a_2$, which is equivalent to the first. $\Box$

Definition 3.4. *For $\mathfrak{A} = (A, \prec, \Diamond)$ an* MDV*-algebra, define $\mathfrak{R}\mathfrak{A} = (\mathfrak{R}A, \overline{\Box}, \overline{\Diamond})$ where $\mathfrak{R}A$ is the frame of round ideals of A and $\overline{\Box}, \overline{\Diamond}$ are given by $\overline{\Box}(I) = \twoheaddownarrow\Box[I]$ and $\overline{\Diamond}(I) = \twoheaddownarrow\Diamond[I]$.*

Proposition 3.5. *If $\mathfrak{A}$ is an* MDV*-algebra, then $\mathfrak{R}\mathfrak{A}$ is an* MKR*-frame.*

Proof. It is well-known that $\mathfrak{R}A$ is a subframe of the ideal frame of A that is compact regular (see, e.g., [3] or [6]). It is easy to see that $\overline{\Box}(I)$ and $\overline{\Diamond}(I)$ are round ideals so $\overline{\Box}, \overline{\Diamond}$ are well defined. By Lemma 3.3.1, both $\Box, \Diamond$ are proximity preserving on A, so we can alternately describe $\overline{\Box}(I) = \downarrow\Box[I]$ and $\overline{\Diamond}(I) = \downarrow\Diamond[I]$.

We must verify the conditions of Definition 2.2. As $\Diamond 0 = 0$ and $\Box 1 = 1$, we have $\overline{\Diamond}0 = 0$ and $\overline{\Box}1 = 1$. Clearly $\overline{\Box}$ and $\overline{\Diamond}$ are order-preserving, so $\overline{\Diamond}(I) \vee \overline{\Diamond}(J) \subseteq \overline{\Diamond}(I \vee J)$ and $\overline{\Box}(I \wedge J) \subseteq \overline{\Box}(I) \wedge \overline{\Box}(J)$. If $a_1 \in I$ and $a_2 \in J$, then roundness gives $b_1 \in I$ and $b_2 \in J$ with $a_1 \prec b_1$ and $a_2 \prec b_2$. Then Lemma 3.3.2 gives $\Diamond(a_1 \vee a_2) \prec \Diamond b_1 \vee \Diamond b_2$, showing $\overline{\Diamond}(I \vee J) \subseteq \overline{\Diamond}(I) \vee \overline{\Diamond}(J)$, and $\Box a_1 \wedge \Box a_2 \prec \Box(b_1 \wedge b_2)$, showing $\overline{\Box}(I \wedge J) \subseteq \overline{\Box}(I) \wedge \overline{\Box}(J)$. Thus $\overline{\Diamond}$ is finitely additive and $\overline{\Box}$ is finitely multiplicative. Also, Lemma 3.3.3 gives $\Box(a_1 \vee a_2) \prec \Box b_1 \vee \Diamond b_2$ and $\Box a_1 \wedge \Diamond a_2 \prec \Diamond(b_1 \wedge b_2)$, showing $\overline{\Box}(I \vee J) \subseteq \overline{\Box}(I) \vee \overline{\Diamond}(J)$ and $\overline{\Box}(I) \wedge \overline{\Diamond}(J) \subseteq \overline{\Diamond}(I \wedge J)$. Finally, directed joins in $\mathfrak{R}A$ are given by unions, and it follows easily that both $\overline{\Box}$ and $\overline{\Diamond}$ preserve directed joins. $\Box$

Theorem 3.6. *The assignment $\mathfrak{A} \mapsto \mathfrak{R}\mathfrak{A}$ can be extended to a functor $\mathfrak{R} : \mathsf{MDV} \to \mathsf{MKRFrm}$ by setting $\mathfrak{R}\alpha = \twoheaddownarrow\alpha[\cdot]$ for an* MDV*-morphism $\alpha : \mathfrak{A} \to \mathfrak{B}$.*

Proof. It is known [6, Rem. 3.10] that the "restriction" of $\mathfrak{R}$ gives a functor $\mathfrak{R} : \mathsf{DeV} \to \mathsf{KRFrm}$, so it remains only to show that the frame homomorphism $\mathfrak{R}\alpha$ is an MKR-morphism. This means we must show $(\mathfrak{R}\alpha)(\overline{\Diamond}I) = \overline{\Diamond}((\mathfrak{R}\alpha)I)$ and $(\mathfrak{R}\alpha)(\overline{\Box}I) = \overline{\Box}((\mathfrak{R}\alpha)I)$ for each round ideal I of $\mathfrak{A}$. This follows directly once we show $a \prec b$ implies (i) $\alpha(\Diamond a) \prec \Diamond\alpha(b)$, (ii) $\Diamond\alpha(a) \prec \alpha(\Diamond b)$, (iii) $\alpha(\Box a) \prec \Box\alpha(b)$, and (iv) $\Box\alpha(a) \prec \alpha(\Box b)$.

Items (i) and (ii) are part of the definition of an MDV-morphism. For (iii), use interpolation to find $a \prec c \prec d \prec b$ and recall that an MDV-morphism also satisfies $x \prec y$ implies $\alpha(\neg y) \prec \neg\alpha(x)$ and $\neg\alpha(y) \prec \alpha(\neg x)$. Then as $\Diamond\neg c \prec \Diamond\neg a$ we have $\alpha(\Box a) = \alpha(\neg\Diamond\neg a) \prec \neg\alpha(\Diamond\neg c)$, and as $\neg d \prec \neg c$ we have $\Diamond\alpha(\neg d) \prec \alpha(\Diamond\neg c)$, hence $\alpha(\Box a) \prec \neg\alpha(\Diamond\neg c) \prec \neg\Diamond\alpha(\neg d)$. But $d \prec b$ gives $\neg\alpha(b) \prec \alpha(\neg d)$, hence $\alpha(\Box a) \prec \neg\Diamond\alpha(\neg d) \prec \neg\Diamond\neg\alpha(b) = \Box\alpha(b)$. This gives (iii), and a similar calculation provides (iv). $\Box$

Next we construct a functor from MKRFrm to MDV. In fact, we will construct two functors, one will have image in LMDV and the other in UMDV.

Lemma 3.7 ([7, Lem. 3.6])**.** *Let $\mathcal{L} = (L, \Box, \Diamond)$ be an* MKR*-frame and $a, b \in L$. Then*

(1) *$\Diamond a \leq \neg \Box \neg a$ and $\Box a \leq \neg \Diamond \neg a$.*
(2) *If $a \prec b$, then $\Diamond a \prec \Diamond b$ and $\Box a \prec \Box b$.*
(3) *If $a \prec b$, then $\neg \Box \neg a \prec \Diamond b$ and $\neg \Diamond \neg a \prec \Box b$.*
(4) *If $a \prec b$, then $\Box a \prec \neg \Diamond \neg b$ and $\Diamond a \prec \neg \Box \neg b$.*

Recall that for a compact regular frame L, the operation $\neg\neg$ is a closure operator on L whose fixed points $\mathfrak{B}L$ are a de Vries algebra with proximity given by the restriction of the well inside relation $\prec$ on L [6, Lem. 3.1]. Meets in $\mathfrak{B}L$ agree with those in L, joins are given by applying the closure operator $\neg\neg$ to the join in L. We use $\sqcup$ for finite joins in $\mathfrak{B}L$ and $\bigsqcup$ for infinite joins.

Definition 3.8. *For $\mathcal{L} = (L, \Box, \Diamond)$ an* MKR*-frame, define $\Diamond^L$, $\Diamond^U$ on $\mathfrak{B}L$ by $\Diamond^L a = \neg\neg\Diamond a$ and $\Diamond^U a = \neg\Box\neg a$, and following our convention, define $\Box^L = \neg\Diamond^L\neg$ and $\Box^U = \neg\Diamond^U\neg$.*

Proposition 3.9. *For $\mathcal{L} = (L, \Box, \Diamond)$ an* MKR*-frame, $\mathfrak{L}\mathcal{L} = (\mathfrak{B}L, \Diamond^L)$ is a lower continuous* MDV*-algebra, and $\mathfrak{U}\mathcal{L} = (\mathfrak{B}L, \Diamond^U)$ is an upper continuous* MDV*-algebra.*

Proof. Clearly $\Diamond^L 0 = 0$ and $\Diamond^U 0 = 0$. Let $a_1, a_2, b_1, b_2 \in \mathfrak{B}L$ with $a_1 \prec b_1$ and $a_2 \prec b_2$. Then $a_1 \vee a_2 \prec b_1 \vee b_2$. As $x \prec y$ implies $\neg\neg x \prec y$ we have $a_1 \sqcup a_2 = \neg\neg(a_1 \vee a_2) \prec b_1 \vee b_2$. Lemma 3.7.2 and the additivity of $\Diamond$ give $\Diamond(a_1 \sqcup a_2) \prec \Diamond b_1 \vee \Diamond b_2 \leq \neg\neg\Diamond b_1 \sqcup \neg\neg\Diamond b_2$, so $\Diamond^L(a_1 \sqcup a_2) \prec \Diamond^L b_1 \sqcup \Diamond^L b_2$. This shows $\Diamond^L$ is de Vries additive, so $\mathfrak{L}\mathcal{L}$ is an MDV-algebra. For de Vries additivity of $\Diamond^U$, we have $a_1 \sqcup a_2 \prec b_1 \sqcup b_2$, and as $x \prec y$ iff $\neg y \prec \neg x$ in any de Vries algebra, $\neg(b_1 \sqcup b_2) \prec \neg(a_1 \sqcup a_2)$. Then Lemma 3.7.2 gives $\Box\neg(b_1 \sqcup b_2) \prec \Box\neg(a_1 \sqcup a_2)$, hence $\neg\Box\neg(a_1 \sqcup a_2) \prec \neg\Box\neg(b_1 \sqcup b_2)$. Using DeMorgan's law and the fact that $\Box$ is multiplicative, this gives $\Diamond^U(a_1 \sqcup a_2) \prec \Diamond^U b_1 \sqcup \Diamond^U b_2$, and shows $\mathfrak{U}\mathcal{L}$ is an MDV-algebra.

To see $\mathfrak{L}\mathcal{L}$ is lower continuous, let $a \in \mathfrak{B}L$. Recall we use $\bigvee$ for joins in L and $\bigsqcup$ for joins in $\mathfrak{B}L$. As L is regular, $a = \bigvee\{b : b \in L \text{ and } b \prec a\}$. Since $b \prec a$ implies $\neg\neg b \prec a$, we have $a = \bigvee\{c : c \in \mathfrak{B}L \text{ and } c \prec a\}$. As $\Diamond$ is additive, $\Diamond a = \bigvee\{\Diamond c : c \in \mathfrak{B}L \text{ and } c \prec a\}$, and it follows that $\neg\neg\Diamond a = \neg\neg\bigvee\{\neg\neg\Diamond c : c \in \mathfrak{B}L \text{ and } c \prec a\}$. Thus $\Diamond^L a = \bigsqcup\{\Diamond^L c : c \prec a\}$, showing $\mathfrak{L}\mathcal{L}$ is lower continuous.

To see $\mathfrak{U}\mathcal{L}$ is upper continuous, recall meets in $\mathfrak{B}L$ agree with those in L. For $a \in \mathfrak{B}L$, we have $\neg a \in \mathfrak{B}L$ and $\neg a = \bigvee\{c : c \in \mathfrak{B}L \text{ and } c \prec \neg a\}$. Noting that the $c \in \mathfrak{B}L$ with $c \prec \neg a$ are exactly the $\neg b$ with $b \in \mathfrak{B}L$ and $a \prec b$, we have $\neg a = \bigvee\{\neg b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. As $\Box$ preserves directed joins, $\Box\neg a = \bigvee\{\Box\neg b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. Then as $\neg\bigvee x_i = \bigwedge\neg x_i$ in any frame, and $\Diamond^U = \neg\Box\neg$, we have $\Diamond^U a = \bigwedge\{\Diamond^U b : b \in \mathfrak{B}L \text{ and } a \prec b\}$. Thus $\mathfrak{U}\mathcal{L}$ is upper continuous. □

Theorem 3.10. *The assignments $\mathcal{L} \mapsto \mathfrak{L}\mathcal{L}$ and $\mathcal{L} \mapsto \mathfrak{U}\mathcal{L}$ can be extended to functors $\mathfrak{L} : \mathsf{MKRFrm} \to \mathsf{LMDV}$ and $\mathfrak{U} : \mathsf{MKRFrm} \to \mathsf{UMDV}$ by setting $\mathfrak{L}h = \mathfrak{U}h = \neg\neg h$ for an* MKR*-morphism $h : \mathcal{L} \to \mathcal{M}$.*

Proof. The "restrictions" of $\mathfrak{L}, \mathfrak{U}$ to KRFrm are known [6, Lem. 3.4] to give a functor $\mathfrak{B} : \mathsf{KRFrm} \to \mathsf{DeV}$. It remains to show the de Vries morphisms $\mathfrak{L}h : \mathfrak{L}\mathcal{L} \to \mathfrak{L}\mathcal{M}$ and $\mathfrak{U}h : \mathfrak{U}\mathcal{L} \to \mathfrak{U}\mathcal{M}$ are modal de Vries morphisms. This means we must show that $a \prec b$ in $\mathfrak{B}L$ implies (i) $\neg\neg h(\Diamond^L a) \prec \Diamond^L\neg\neg h(b)$, (ii) $\Diamond^L\neg\neg h(a) \prec \neg\neg h(\Diamond^L b)$, (iii) $\neg\neg h(\Diamond^U a) \prec \Diamond^U\neg\neg h(b)$, and (iv) $\Diamond^U\neg\neg h(a) \prec \neg\neg h(\Diamond^U b)$.

Before proving these items, we collect some facts. As h is a frame homomorphism, it preserves proximity and order, and satisfies $h(\neg x) \leq \neg h(x)$; and as h is an MKR-morphism, $x \prec y$ implies $h(\Diamond x) \prec \Diamond h(y)$, $\Diamond h(x) \prec h(\Diamond y)$, $h(\Box x) \prec \Box h(y)$, and $\Box h(x) \prec h(\Box y)$. Lemma 3.7 shows $\Diamond, \Box$ preserve proximity. Finally, in any frame, $x \prec y$ iff $\neg\neg x \prec y$.

As $a \prec b$, we have $\Diamond^L a = \neg\neg\Diamond a \prec \Diamond b$. So $h(\Diamond^L a) \prec h(\Diamond b) = \Diamond h(b) \leq \Diamond^L\neg\neg h(b)$. From this, (i) follows. Also $a \prec b$ implies $\neg\neg h(a) \prec h(b)$, hence $\Diamond\neg\neg h(a) \prec \Diamond h(b) = h(\Diamond b) \leq \neg\neg h(\Diamond^L b)$, and from this (ii) follows. As $a \prec b$, we have $\neg a \vee b = 1$. Thus $\Box(\neg a \vee b) = 1$, and the definition of an MKR-frame gives $\Box\neg a \vee \Diamond b = 1$. Then, by Lemma 3.7.1, $1 = h(\Box\neg a \vee \Diamond b) \leq h(\neg\neg\Box\neg a) \vee \Diamond h(b) \leq \neg h(\neg\Box\neg a) \vee \neg\Box\neg h(b) = \neg h(\Diamond^U a) \vee \Diamond^U h(b)$, giving $h(\Diamond^U a) \prec \Diamond^U h(b)$, and (iii) follows. Finally, $a \prec b$ gives $h(a) \prec h(b)$, and as in (iii), $\Box\neg h(a) \vee \Diamond h(b) = 1$. So $\neg\neg\Box\neg\neg\neg h(a) \vee h(\Diamond b) = 1$, giving $\Diamond^U(\neg\neg h(a)) \prec h(\Diamond b) \leq \neg\neg h(\Diamond^U b)$. □

Theorem 3.11. *There is an equivalence between MKRFrm and LMDV given by $\mathfrak{L}$ and the restriction of $\mathfrak{R}$ to LMDV; and an equivalence between MKRFrm and UMDV given by $\mathfrak{U}$ and the restriction of $\mathfrak{R}$ to UMDV.*

Proof. Suppose $\mathcal{L} = (L, \Box, \Diamond)$ is an MKR-frame, $\mathfrak{A} = (A, \prec, \Diamond)$ is a lower continuous MDV-algebra, and $\mathfrak{C} = (C, \prec, \Diamond)$ is an upper continuous MDV-algebra. Define $h : \mathfrak{R}\mathfrak{L}\mathcal{L} \to \mathcal{L}$ and $k : \mathfrak{R}\mathfrak{U}\mathcal{L} \to \mathcal{L}$ by $h(I) = k(I) = \bigvee I$. Also, define $\alpha : \mathfrak{A} \to \mathfrak{L}\mathfrak{R}\mathfrak{A}$ and $\beta : \mathfrak{C} \to \mathfrak{U}\mathfrak{R}\mathfrak{C}$ by $\alpha(a) = \twoheaddownarrow a$ and $\beta(c) = \twoheaddownarrow c$. It is known [6, Sec. 3] that on the level of compact regular frames and de Vries algebras h, k and α, β are natural isomorphisms. It remains only to show h, k are MKR-isomorphisms and α, β are MDV-isomorphisms.

To show h is an MKR-isomorphism, we must show $h(\overline{\Diamond^L}(I)) = \Diamond h(I)$ and $h(\overline{\Box^L}(I)) = \Box h(I)$ for I a round ideal of the regular elements of L. In Proposition 3.5 we noted $\overline{\Diamond^L}(I) = \downarrow\Diamond^L[I]$ and $\overline{\Box^L}(I) = \downarrow\Box^L[I]$. Then $h(\overline{\Diamond^L}(I)) = \bigvee\Diamond^L[I]$ and $h(\overline{\Box^L}(I)) = \bigvee\Box^L[I]$. Also, as $\Diamond$ and $\Box$ preserve directed joins, $\Diamond h(I) = \bigvee\Diamond[I]$ and $\Box h(I) = \bigvee\Box[I]$. So to show h is an isomorphism, we must show $\bigvee\Diamond^L[I] = \bigvee\Diamond[I]$ and $\bigvee\Box^L[I] = \bigvee\Box[I]$. Similarly, to show k is an isomorphism, we must show $\bigvee\Diamond^U[I] = \bigvee\Diamond[I]$ and $\bigvee\Box^U[I] = \bigvee\Box[I]$. But for $a \in L$ regular, Definition 3.8 gives $\Diamond^L a = \neg\neg\Diamond a$, $\Box^L a = \neg\Diamond\neg a$, $\Diamond^U a = \neg\Box\neg a$, and $\Box^U a = \neg\neg\Box a$. So if a, b are regular with $a \prec b$, Lemma 3.7 gives $\Diamond a \leq \Diamond^L a \leq \Diamond^U a \prec \Diamond b$ and $\Box a \leq \Box^U a \leq \Box^L a \prec \Box b$. The required equalities of the above joins follow easily from these inequalities and the roundness of I.

To show α is an MDV-isomorphism, we must show $\alpha(\Diamond a) = (\overline{\Diamond})^L \alpha(a)$. Recall $(\overline{\Diamond})^L I = \neg\neg \Diamond [I]$ where pseudocomplement $\neg$ in the frame of round ideals is given by $\neg I = \twoheaddownarrow \neg \bigvee I$, hence $\neg\neg I = \twoheaddownarrow \bigvee I$ [6, Lem. 3.5]. We then have $(\overline{\Diamond})^L \alpha(a) = \neg\neg \Diamond [\twoheaddownarrow a] = \twoheaddownarrow \bigvee \Diamond [\twoheaddownarrow a]$. As $\mathfrak{A}$ is lower continuous, $\bigvee \Diamond [\twoheaddownarrow a] = \Diamond a$, and the result follows. To show β is an isomorphism we must show $\beta(\Diamond c) = (\overline{\Diamond})^U \beta(c)$. Recall $(\overline{\Diamond})^U I = \neg \overline{\Box} \neg I$ where $\overline{\Box} I = \twoheaddownarrow \Box [I]$ and $\Box = \neg \Diamond \neg$. So $(\overline{\Diamond})^U \beta(c) = \neg \overline{\Box} \neg \twoheaddownarrow c = \neg \overline{\Box} \twoheaddownarrow \neg c = \twoheaddownarrow \neg \bigvee \Box [\twoheaddownarrow \neg c] = \twoheaddownarrow \neg \bigvee \{\Box b : b \prec \neg c\}$. Using the infinite DeMorgan law in a Boolean algebra, the fact that in an MDV-algebra $\neg \Box b = \Diamond \neg b$ and $b \prec \neg c$ iff $c \prec \neg b$, we have $(\overline{\Diamond})^U \beta(c) = \twoheaddownarrow \bigwedge \{\Diamond a : c \prec a\}$. Then as $\mathfrak{C}$ is upper continuous, this is $\twoheaddownarrow \Diamond c$, giving the result. $\square$

Corollary 3.12. *Without choice, the categories* MKRFrm, LMDV, UMDV, *and* MDV *are equivalent; and with choice, they are all dually equivalent to* MKHaus.

Proof. We have just proved without choice that MKRFrm, LMDV, and UMDV are equivalent, and in [7, Sec. 4.3] we proved without choice that LMDV and UMDV are equivalent to MDV. In [7, Sec. 3], using choice, we proved MKRFrm and MKHaus are equivalent. $\square$

4. Connections to the Vietoris construction

In this section we relate MKR-frames to Johnstone's construction of the Vietoris functor on frames. We also discuss a counterpart of the Vietoris functor for de Vries algebras. We begin with a brief summary of Isbell duality between KRFrm and KHaus [18, 4, 19].

A *point* of a frame L is a frame homomorphism $p : L \to 2$ into the 2-element frame. The set of points of L is topologized by $\{\varphi(a) : a \in L\}$ where $\varphi(a) = \{p : p(a) = 1\}$. This topological space is denoted $\mathfrak{p}L$. The functor $\mathfrak{p} : \mathsf{KRFrm} \to \mathsf{KHaus}$ takes a frame L to its space of points, and a frame homomorphism $h : L \to M$ to the continuous map $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ where $\mathfrak{p}h(q) = q \circ h$. The functor $\mathfrak{p}$ with the open set functor $\Omega : \mathsf{KHaus} \to \mathsf{KRFrm}$ provide Isbell duality.

We next give a brief summary of Johnstone's results ([19, Sec. III.4], [20]).

Definition 4.1. *For a frame L, let L^* be the set of all formal symbols $L^* = \{\Box_a, \Diamond_a : a \in L\}$ and $F(L^*)$ be the free frame over L^*. Let θ be the frame congruence on $F(L^*)$ generated by the following:*

1. $\Box_{a\wedge b} = \Box_a \wedge \Box_b$ *and* $\Box_1 = 1$ *where* $a, b \in L$.
2. $\Diamond_{a\vee b} = \Diamond_a \vee \Diamond_b$ *and* $\Diamond_0 = 0$ *where* $a, b \in L$.
3. $\Box_{a\vee b} \le \Box_a \vee \Diamond_b$ *and* $\Box_a \wedge \Diamond_b \le \Diamond_{a\wedge b}$ *where* $a, b \in L$.
4. $\Box_{\bigvee S} = \bigvee\{\Box_s : s \in S\}$ *and* $\Diamond_{\bigvee S} = \bigvee\{\Diamond_s : s \in S\}$ *where* $S \subseteq L$ *is directed.*

Then set $\mathcal{W}(L) = F(L^*)/\theta$ and call this the Vietoris frame *of L.*

This construction on objects extends to give a functor $\mathcal{W}$, called the Vietoris frame functor, from the category of frames to itself. A frame homomorphism $g : L \to M$ lifts to $\mathcal{W}(g) : \mathcal{W}(L) \to \mathcal{W}(M)$ that maps the generator $\Box_a/\theta$ to $\Box_{ga}/\theta$ and the generator $\Diamond_a/\theta$ to $\Diamond_{ga}/\theta$. The following specializes Johnstone's results on this functor to our setting of compact regular frames.

Theorem 4.2 (Johnstone)**.** *The Vietoris frame functor $\mathcal{W}$ restricts to a functor on* KRFrm. *Here, if L is a compact regular frame isomorphic to the frame of open sets of the compact Hausdorff space X, then $\mathcal{W}(L)$ is isomorphic to the frame of open sets of the Vietoris space of X. Further, for $\mathcal{V}$ the Vietoris functor on* KHaus *and $\Omega, \mathfrak{p}$ the open set and point functors providing a dual equivalence between* KHaus *and* KRFrm, *we have $\mathcal{W}$ is naturally isomorphic to $\Omega \circ \mathcal{V} \circ \mathfrak{p}$.*

We now come to the key result relating MKR-frames and the Vietoris frame functor.

Proposition 4.3. *If L is a compact regular frame, then each frame homomorphism $h : \mathcal{W}(L) \to L$ gives an* MKR*-frame structure $\mathcal{L}_h = (L, \Box_h, \Diamond_h)$ on L where $\Box_h a = h(\Box_a/\theta)$ and $\Diamond_h a = h(\Diamond_a/\theta)$ for each $a \in L$. This provides a bijective correspondence between frame homomorphisms $h : \mathcal{W}(L) \to L$ and* MKR*-frames having underlying frame L.*

Proof. For a frame homomorphism $h : \mathcal{W}(L) \to L$, the operations $\Box_h$ and $\Diamond_h$ on L are obviously well-defined. We must show they satisfy the conditions of Definition 2.2. For $a, b \in L$, we have $\Box_h(a \wedge b) = h(\Box_{a\wedge b}/\theta) = h((\Box_a \wedge \Box_b)/\theta) = h(\Box_a/\theta \wedge \Box_b/\theta) = h(\Box_a/\theta) \wedge h(\Box_b/\theta) = \Box_h a \wedge \Box_h b$. Here, the second equality follows from the definition of θ. Also $\Box_h 1 = h(\Box_1/\theta) = h(1/\theta) = 1$, establishing the first condition of Definition 2.2. The second condition is similar. For the third condition, $\Box_h(a \vee b) = h(\Box_{a\vee b}/\theta) \le h((\Box_a \vee \Diamond_b)/\theta) = h(\Box_a/\theta) \vee h(\Diamond_b/\theta) =$

$\Box_h a \vee \Diamond_h b$, with the other item in the third condition similar. For the final condition, if $S \subseteq L$ is up-directed, then $\Box_h(\bigvee S) = h(\Box_{\bigvee S}/\theta) = h(\bigvee\{\Box_s : s \in S\}/\theta) = \bigvee\{h(\Box_s/\theta) : s \in S\} = \bigvee\{\Box_h s : s \in S\}$. Here we have used that frame congruences and frame homomorphisms preserve arbitrary joins. The other item in the fourth condition is obviously similar.

The above paragraph shows each frame homomorphism $h : \mathcal{W}(L) \to L$ induces an MRK-structure on L as indicated. If $h, h' : \mathcal{W}(L) \to L$ are frame homomorphisms that induce the same structure, then $h(\Box_a/\theta) = h'(\Box_a/\theta)$ and $h(\Diamond_a/\theta) = h'(\Diamond_a/\theta)$ for each $a \in L$. So h and h' agree on a generating set of $\mathcal{W}(L)$, hence are equal.

It remains to show each MKR-frame structure on L is induced by a frame homomorphism $h : \mathcal{W}(L) \to L$. Suppose $\mathcal{L} = (L, \Box, \Diamond)$ is an MKR-frame. Define $g : L^* \to L$ by $g(\Box_a) = \Box a$ and $g(\Diamond_a) = \Diamond a$ for each $a \in L$. As $F(L^*)$ is the free frame over the set L^*, the map g extends to a frame homomorphism $\overline{g} : F(L^*) \to L$. We claim the kernel of $\overline{g}$ contains θ. Indeed, if $a, b \in L$, then as the MKR-frame $\mathcal{L}$ satisfies $\Box(a \wedge b) = \Box a \wedge \Box b$, we have $\overline{g}(\Box_{a\wedge b}) = \overline{g}(\Box_a) \wedge \overline{g}(\Box_b) = \overline{g}(\Box_a \wedge \Box_b)$, showing the pair $\Box_{a\wedge b}$ and $\Box_a \wedge \Box_b$ belongs to the kernel of $\overline{g}$. Similar arguments show all pairs in the generating set of θ belong to the kernel of $\overline{g}$, showing θ is contained in the kernel of $\overline{g}$. Thus, there is a frame homomorphism $h : F(L^*)/\theta \to L$ with $h \circ \kappa = \overline{g}$ where κ is the canonical homomorphism $\kappa : F(L^*) \to F(L^*)/\theta$. Then $h(\Box_a/\theta) = \overline{g}(\Box_a) = \Box a$ and $h(\Diamond_a/\theta) = \overline{g}(\Diamond_a) = \Diamond a$ for all $a \in L$, showing h induces the structure $\mathcal{L}$ on L. □

Algebras for the Vietoris frame functor on KRFrm are morphisms $h : \mathcal{W}(L) \to L$. So the above result shows these algebras are concretely realized by MKR-frames. The algebras for $\mathcal{W}$ form a category where a morphism between algebras $h : \mathcal{W}(L) \to L$ and $h' : \mathcal{W}(M) \to M$ is a frame homomorphism $g : L \to M$ where the square formed from h, h', g and $\mathcal{W}(g)$ commutes. Then $g(\Box a) = gh(\Box_a/\theta) = h'\mathcal{W}(g)(\Box_a/\theta) = h'(\Box_{ga}/\theta) = \Box ga$, with a similar calculation showing $g(\Diamond a) = \Diamond ga$. This provides the following.

Theorem 4.4. *The category of algebras for the Vietoris frame functor $\mathcal{W}$ on* KRFrm *is isomorphic to the category* MKRFrm *of modal compact regular frames.*

From a general categorical argument, it follows that the category of algebras for the Vietoris frame functor $\mathcal{W}$ on KRFrm is dually equivalent to the category of coalgebras for the Vietoris functor $\mathcal{V}$ on KHaus. These coalgebras are morphisms $\rho : X \to \mathcal{V}(X)$ from a compact Hausdorff space X into its Vietoris space. In [7] we showed that the category MKHaus of modal compact Hausdorff spaces was isomorphic to the category of coalgebras for $\mathcal{V}$. This provides an alternative proof to the following result established directly in [7].

Theorem 4.5. *The categories* MKRFrm *and* MKHaus *are dually equivalent.*

As DeV is equivalent to KRFrm and dually equivalent to KHaus, there is a version of the Vietoris functor on DeV as well. This can be realized either by sending a de Vries algebra $\mathfrak{A}$ to the regular open sets of the Vietoris space of the compact Hausdorff space of ends of $\mathfrak{A}$, or by sending $\mathfrak{A}$ to the regular elements of the Vietoris frame of the compact regular frame of round ideals of $\mathfrak{A}$. Just as MKRFrm is isomorphic to the category of algebras for the Vietoris frame functor, so is MDV isomorphic to the category of algebras for the Vietoris de Vries functor. This yields an alternative route to the following result established directly in [7].

Theorem 4.6. *The categories* MDV *and* MKHaus *are dually equivalent.*

It would be desirable to have a direct construction of the Vietoris de Vries functor $\mathcal{W}$. While we do not have such, a few remarks may be useful.

Remark 4.7. Extend the definition of a continuous relation on a compact Hausdorff space to that of a continuous relation from a compact Hausdorff space X to a compact Hausdorff space Y. This is a subset $R \subseteq X \times Y$ where the image of a point is closed, the inverse image of an open set is open, and the inverse image of a closed set is closed. As these conditions imply the image under R of a closed set is closed (see [7, Lem. 7.10.2]) the relational composite of continuous relations is continuous. So there is a category $\mathsf{KHaus}^{\mathsf{R}}$ of compact Hausdorff spaces and continuous relations between them.

The inclusion functor $I : \mathsf{KHaus} \to \mathsf{KHaus}^R$ has a right adjoint F taking a space X to its Vietoris space $\mathcal{V}(X)$ and a continuous relation R from X to Y to the function $\rho_R : \mathcal{V}(X) \to \mathcal{V}(Y)$ where ρ_R takes a closed set A to $R[A]$. The key point is $\mathsf{KHaus}^R(IX, Y) \simeq \mathsf{KHaus}(X, FY)$ since continuous relations R from X to Y are in bijective correspondence with continuous functions from X to the Vietoris space $\mathcal{V}(Y)$ where $R \subseteq X \times Y$ corresponds to the function $\rho_R : X \to \mathcal{V}(Y)$ given by $\rho_R(x) = R[x]$. Clearly the Vietoris functor $\mathcal{V}$ on KHaus is the composite $F \circ I$ of the inclusion functor I from KHaus into KHaus^R and its right adjoint.

These remarks can be used to discuss analogs of the Vietoris functor in other settings.

Remark 4.8. The above situation restricts to the setting Stone of Stone spaces, and Stone^R of Stone spaces and the continuous relations between them. One can show (see [16]) Stone^R is dually equivalent to the category $\mathsf{BA}^\vee$ of Boolean algebras and the maps between them that preserve finite joins, and that this duality restricts to the usual one between Stone and BA. It follows from the above remark that the inclusion functor $I : \mathsf{BA} \to \mathsf{BA}^\vee$ has a left adjoint $F : \mathsf{BA}^\vee \to \mathsf{BA}$ and the composite $F \circ I$ is the Vietoris functor on BA. Here the left adjoint has a simple description, for a Boolean algebra B, we have $F(B)$ is the free Boolean algebra over the join semilattice reduct of B (see, e.g., [1, 14, 9]).

This situation is applicable to the de Vries setting.

Remark 4.9. Let $\mathsf{DeV}^\vee$ be the category of de Vries algebras, with the maps between them that are de Vries additive and are lower continuous in the sense that $\alpha(a) = \bigvee\{\alpha(b) : b \prec a\}$. Here composition is the same composition $*$ as in DeV. One can show $\mathsf{DeV}^\vee$ is dually equivalent to KHaus^R, and that this duality restricts to the usual one between DeV and KHaus. The proofs essentially follow from pieces of [7, Sec. 5]. As is the Boolean case, the inclusion functor $I : \mathsf{DeV} \to \mathsf{DeV}^\vee$ has a left adjoint $F : \mathsf{DeV}^\vee \to \mathsf{DeV}$ and the composite $F \circ I$ is the Vietoris functor on DeV. Unlike the Boolean case, we have no direct construction of this left adjoint F, hence no direct construction of the Vietors functor on DeV.

Similarly, an extension $\mathsf{KRFrm}^\vee$ of KRFrm could be formed that is dual to KHaus^R and equivalent to $\mathsf{DeV}^\vee$. Here objects would be as in KRFrm, but unlike $\mathsf{DeV}^\vee$, morphisms would be pairs of maps $\Diamond, \Box : L \to M$ that satisfy conditions similar to those of Definition 2.2.

Finally, we remark that the category KHaus^R of compact Hausdorff spaces and continuous relations seems of natural interest, and is perhaps worth of additional study.

5. Further remarks

In [7] Isbell and de Vries dualities were lifted to the modal setting to establish the dualities between MKRFrm and MKHaus, and between MDV and MKHaus. In Section 3 we showed a direct choice-free equivalence of MKRFrm and MDV. Composing this with the dual equivalence of MDV and MKHaus gives an alternative path to the dual equivalence of MKRFrm and MKHaus. Restricted to KRFrm and KHaus, this composite is a particular case of Hofmann-Lawson duality [17] between locally compact frames and locally compact sober spaces, and closely resembles Stone duality. In the modal setting, this composite resembles the familiar duality between modal algebras and modal spaces (descriptive frames).

For KRFrm and KHaus, Hofmann-Lawson duality works as follows (see, e.g., [2, 19, 22]). Let L be a compact regular frame and let $\prec$ be the well inside relation on L. For $S \subseteq L$, let $\twoheaduparrow S = \{a : s \prec a \text{ for some } s \in S\}$. We say a filter F of L is *round* if $F = \twoheaduparrow F$. We note that round filters are also called regular or completely regular filters. By a *prime round filter* we mean a prime filter that is round. The essential point is the following.

Theorem 5.1. *For a compact regular frame L, its space of points $\mathfrak{p}L$ is homeomorphic to the space X of its prime round filters topologized by the sets $\varphi(a) = \{x \in X : a \in x\}$.*

It follows that the point functor $\mathfrak{p}$ used in Isbell duality may be replaced by a functor taking a compact regular frame L to its space of prime round filters, and a frame homomorphism $h : L \to M$ to the continuous map h^{-1} between the associated spaces of prime round filters.

Remark 5.2. A prime round filter F has been defined as a prime filter that is additionally round. One can show that this is equivalent to each of the following conditions: (i) F is a meet prime element in the lattice of round filters ordered by set inclusion, (ii) $F = \twoheaduparrow G$ for some prime filter G, and (iii) F is a completely prime filter. The definition of round ideal is dual to the definition of round filter. Being a meet-prime element in the lattice of round ideals is equivalent to being equal to $\twoheaddownarrow I$ for some prime ideal I, and is also equivalent to being equal to $\twoheaddownarrow m$ for some meet prime element m. However, it is *not* equivalent to being a prime ideal that is round. We use the term a *prime round ideal* for a meet prime element in the lattice of round ideals.

Stone duality is often realized via prime ideals rather than prime filters. In the setting of compact regular frames, there is a similar path using prime round ideals. The key point is that the space of points of a compact regular frame L is homeomorphic to the space of prime round ideals of L topologized by the sets $\varphi(a) = \{I : a \nleq \bigvee I\}$. This provides a functor, where a frame homomorphism h is taken to the continuous map $\twoheaddownarrow h^{-1}$ between the spaces of prime round ideals.

Remark 5.3. Hofmann-Lawson duality has many similarities with Stone duality. For example, the Prime Ideal Theorem, stated for compact regular frames, takes the following form: If F and I are disjoint round filter and round ideal, then there are disjoint prime round filter containing F and prime round ideal containing I.

Another similarity is how round ideals and filters are connected to the topology of the dual space of a compact regular frame. Recall the basic fact that in Stone duality, open sets of the dual space X of a Boolean algebra B correspond to ideals of B, and closed sets of X correspond to filters of B. For a compact regular frame L, the open sets of its dual space are the $\varphi(a)$ where $a \in L$. For any round ideal I, we have $I = \twoheaddownarrow \bigvee I$, so there is a bijection between round ideals of L and elements of L, so open sets of the dual space correspond to round ideals of L. Similarly, closed sets of the dual space correspond to round filters of L. Here the underlying point is that each closed set in a compact Hausdorff space is the intersection of the open sets that contain it. As round filters of a compact regular frame are exactly Scott open filters and closed subsets of a compact Hausdorff space are exactly compact saturated subsets, this correspondence between round filters and closed sets of the dual space amounts to the Hofmann-Mislove theorem [15, Thm. II-1.20] for compact regular frames.

We next consider how the restriction of Hofmann-Lawson duality to KRFrm extends to the modal setting. As we will see, it closely resembles the extension of Stone duality to modal algebras. By [7, Def. 3.11], for an MKR-frame $\mathcal{L} = (L, \Box, \Diamond)$ a relation R is defined on its space of points by $p\,R\,q$ iff $q(a) = 1$ implies $p(\Diamond a) = 1$ for each $a \in L$. Viewing the space of points of L via its prime round filters, this amounts to defining a relation R on the prime round filters by xRy iff $y \subseteq \Diamond^{-1}(x)$. This is the approach most commonly taken in defining a relation on the dual space of a modal algebra.

There is more to say about the definition of the relation R on the dual space of an MKR-frame. In modal logic, the $\Box$ and $\Diamond$ operators are definable from each other, and the relation R on the dual space of a modal algebra may be defined either by setting xRy iff $y \subseteq \Diamond^{-1}(x)$ or by setting xRy iff $\Box^{-1}(x) \subseteq y$. For an MKR-frame, the operators $\Box$ and $\Diamond$ are also definable from each other [7, Rem. 3.7]. The following proposition shows that the relation R on its dual space of prime round filters of an MKR-frame may also be equivalently defined by either approach.

Proposition 5.4. *Let $\mathcal{L} = (L, \Box, \Diamond)$ be an* MKR*-frame and let x and y be prime round filters of L. The following are equivalent.*

(1) $y \subseteq \Diamond^{-1}(x)$.
(2) $\Box^{-1}(x) \subseteq y$.

Proof. (1) $\Rightarrow$ (2) Let $\Box a \in x$. By [7, Rem. 3.7] we have $\Box a = \bigvee\{\neg \Diamond \neg c : c \prec a\}$, and as x is a round filter, it is completely prime, so there is $c \prec a$ with $\neg \Diamond \neg c \in x$. Then $\Diamond \neg c$ is not in x, and as $y \subseteq \Diamond^{-1}(x)$, we have $\neg c$ is not in y. But $c \prec a$ gives $\neg c \vee a = 1$, so $\neg c$ not being in y implies $a \in y$. Thus $\Box a \in x$ implies $a \in y$, so $\Box^{-1}(x) \subseteq y$. (2) $\Rightarrow$ (1) Let $a \in y$. As $a = \bigvee\{c : c \prec a\}$ we have $c \in y$ for some $c \prec a$, and by interpolation there is b with $c \prec b \prec a$. Then $\neg c$ is not in y, and $\Box^{-1}(x) \subseteq y$ gives $\Box\neg c$ is not in x. Note $c \prec b$ gives $\Box\neg b \prec \Box\neg c$, hence $\neg \Box \neg b \vee \Box\neg c = 1$. Therefore, as $\Box\neg c$ is not in x, we have $\neg \Box \neg b \in x$. By [7, Rem. 3.7] we have $\Diamond a = \bigvee\{\neg \Box \neg b : b \prec a\}$, so $\Diamond a \in x$. Thus, $a \in y$ implies $\Diamond a \in x$, so $y \subseteq \Diamond^{-1}(x)$. $\square$

This shows that for an MKR-frame, the relation on its dual space may be defined either through $\Diamond$ by xRy iff $y \subseteq \Diamond^{-1}(x)$ or via $\Box$ by setting xRy iff $\Box^{-1}(x) \subseteq y$. This is linked to the fact that the operations $\Box$ and $\Diamond$ are definable from one another. This is perhaps a bit unexpected. These MKR-frames are examples of positive modal algebras [10], algebras consisting of bounded distributive lattices with operators $\Box$ and $\Diamond$ satisfying the first three conditions of Definition 2.2. Dual spaces of positive modal algebras are constructed through their prime filters, and relations defined by the above conditions are considered, but in general are not equal. Also in this setting of positive modal algebras, the operations $\Box$ and $\Diamond$ are not in general definable from one another.

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References

1. S. Abramsky, *A Cook's tour of the finitary non-well-founded sets*, We Will Show Them: Essays in honour of Dov Gabbay (S. Artemov et al., ed.), College Publications, 2005, pp. 1–18.
2. B. Banaschewski, *The duality of distributive continuous lattices*, Canad. J. Math. **32** (1980), no. 2, 385–394.
3. ———, *Compactification of frames*, Math. Nachr. **149** (1990), 105–115.
4. B. Banaschewski and C. J. Mulvey, *Stone-Čech compactification of locales. I*, Houston J. Math. **6** (1980), no. 3, 301–312.
5. G. Bezhanishvili, *Stone duality and Gleason covers through de Vries duality*, Topology Appl. **157** (2010), no. 6, 1064–1080.
6. ———, *De Vries algebras and compact regular frames*, Applied Categorical Structures **20** (2012), 569–582.
7. G. Bezhanishvili, N. Bezhanishvili, and J. Harding, *Modal compact Hausdorff spaces*, Journal of Logic and Computation (2012), DOI 10.1093/logcom/exs030.
8. G. Bezhanishvili and J. Harding, *Proximity frames and regularization*, Applied Categorical Structures (2013), DOI 10.1007/s10485-013-9308-9.
9. N. Bezhanishvili and A. Kurz, *Free modal algebras: a coalgebraic perspective*, Algebra and coalgebra in computer science, Lecture Notes in Comput. Sci., vol. 4624, Springer, Berlin, 2007, pp. 143–157.
10. S. Celani and R. Jansana, *Priestley duality, a Sahlqvist theorem and a Goldblatt-Thomason theorem for positive modal logic*, Log. J. IGPL **7** (1999), no. 6, 683–715.
11. H. de Vries, *Compact spaces and compactifications. An algebraic approach*, Ph.D. thesis, University of Amsterdam, 1962.
12. C. H. Dowker, *Mappings of proximity structures*, General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961), Academic Press, New York, 1962, pp. 139–141.
13. J. L. Frith, *The category of uniform frames*, Cahiers Topologie Géom. Différentielle Catég. **31** (1990), no. 4, 305–313.
14. S. Ghilardi, *An algebraic theory of normal forms*, Ann. Pure Appl. Logic **71** (1995), no. 3, 189–245.
15. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *Continuous lattices and domains*, Cambridge University Press, Cambridge, 2003.
16. P. R. Halmos, *Algebraic logic*, Chelsea Publishing Co., New York, 1962.
17. K. H. Hofmann and J. D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. **246** (1978), 285–310.
18. J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5–32.
19. P. T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982.
20. ———, *Vietoris locales and localic semilattices*, Continuous lattices and their applications (Bremen, 1982), Lecture Notes in Pure and Appl. Math., vol. 101, Dekker, New York, 1985, pp. 155–180.
21. S. A. Naimpally and B. D. Warrack, *Proximity spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 59, Cambridge University Press, London, 1970.
22. J. Picado and A. Pultr, *Frames and locales*, Frontiers in Mathematics, Birkhäuser/Springer Basel AG, Basel, 2012.
23. Y. M. Smirnov, *On proximity spaces*, Mat. Sbornik N.S. **31(73)** (1952), 543–574, (Russian).
24. M. B. Smyth, *Stable compactification. I*, J. London Math. Soc. **45** (1992), no. 2, 321–340.

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