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The Role of the Endpoint in Weighted Polynomial Approximation with Varying Weights

A. B. J. Kuijlaars

Abstract. For a weight function $w: [a, b] \rightarrow (0, \infty)$, we consider weighted polynomials of the form $w^n P_n$ where the degree of P_n is at most n . The class of functions that can be approximated with such polynomials depends on the behavior of the density $v(t)$ of the extremal measure associated with w . We show that every approximable function must vanish at the endpoint a if $v(t)$ behaves like $(t - a)^\beta$ as $t \rightarrow a$ with $\beta > -\frac{1}{2}$. We also present an analogous result for internal points. Our results solve some open problems posed by V. Totik and disprove a conjecture of G. G. Lorentz on incomplete polynomials.

1. Introduction and Statement of Results

In recent years there has been much interest in characterizing the class of functions f that are uniform limits on some closed set $E \subset \mathbf{R}$ of weighted polynomials of the form $w^n P_n$, where $P_n \in \Pi_n$, the space of polynomials of degree $\leq n$, and $w: E \rightarrow [0, \infty)$ is a weight function, see [10] and the references given there. If f is the uniform limit of a sequence of weighted polynomials $w^n P_n$, then we say that f is *w-approximable*. We always assume here that w is admissible in the sense of [8] and [10]. That is:

- (i) w is continuous.
- (ii) The set $\{x \in E | w(x) > 0\}$ has positive logarithmic capacity.
- (iii) If E is unbounded, then $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in E$.

Associated with w is the extremal measure μ_w and its support S_w which is a compact subset of E . It is known that every w -approximable function vanishes on $E \setminus S_w$. The most general result in the opposite direction is due to Totik. To state that result he introduced the notion of the restricted support S^w of the measure μ_w . A point x from the interior of S_w belongs to S^w if μ_w is absolutely continuous in a neighborhood of x and has positive and continuous density there. Then the following result holds.

Theorem 1.1 [10, Theorem 4.2]. *Every continuous function on E that vanishes outside the restricted support S^w is w -approximable.*

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In this paper we assume that $E = [a, b]$ is a bounded interval and that S_w is equal to E . In that case we have

$$(1.1) \quad w(x) = \exp(U^{\mu_w}(x) - F_w), \quad x \in [a, b].$$

Here U^{μ_w} is the logarithmic potential of the measure μ_w ,

$$U^{\mu_w}(x) = - \int \log|x - t| d\mu_w(t),$$

and F_w is a constant. So our assumptions imply in particular that w is strictly positive on $[a, b]$.

If the extremal measure μ_w has the form $d\mu_w(t) = v(t) dt$, where v is a positive and continuous function on (a, b) then Theorem 1.1 shows that every continuous function on $[a, b]$ vanishing at a and b is w -approximable. Note that, by definition, the endpoints a and b are not in the restricted support of μ_w .

Here we investigate the role of the endpoints in these type of approximation problems and also the role of internal zeros of the density function.

To state the results we use the following nonstandard notation. Let f and g be positive functions in a neighborhood of the point $a \in [-\infty, \infty]$. Then we write

$$f(t) \sim g(t) \quad (t \rightarrow a)$$

if the limit $\lim_{t \rightarrow a} f(t)/g(t)$ exists and is nonzero. The following are our main results.

Theorem 1.2. *Let $w: [a, b] \rightarrow [0, \infty)$ have the form (1.1) with a probability measure μ_w satisfying*

$$(1.2) \quad d\mu_w(t) = v(t) dt, \quad t \in (a, b),$$

where v is a function on (a, b) with a finite number of zeros and discontinuities. Suppose a $\beta > -\frac{1}{2}$ exists such that

$$(1.3) \quad v(t) \sim (t - a)^\beta \quad (t \rightarrow a).$$

Then every w -approximable function vanishes at a .

Theorem 1.3. *Under the same conditions as in Theorem 1.2, let $c \in (a, b)$ and assume $v(c) = 0$. Suppose a $\beta > 0$ exists such that*

$$(1.4) \quad v(t) \sim |t - c|^\beta \quad (t \rightarrow c).$$

Then every w -approximable function vanishes at c .

Remark 1. The proofs that we give also apply to the case where a function is the limit of a subsequence of weighted polynomials. So, in fact, the following slightly stronger statements hold.

Suppose for some subsequence $\{n_k\}_{k \in \mathbb{N}}$, $n_k \rightarrow \infty$, there are polynomials $P_{n_k} \in \Pi_{n_k}$ such that $w^{n_k} P_{n_k} \rightarrow f$ uniformly on $[a, b]$. If (1.3) holds for $\beta > -\frac{1}{2}$, then $f(a) = 0$. If (1.4) holds for $\beta > 0$, then $f(c) = 0$.

Remark 2. The conditions $\beta > -\frac{1}{2}$ and $\beta > 0$ in Theorems 1.2 and 1.3, respectively, are sharp. In fact, if $\beta = 0$ in Theorem 1.3, then c belongs to the restricted support of μ_w , and so by Theorem 1.1 w -approximable functions that do not vanish at c exist. Similarly, if $\beta = -\frac{1}{2}$ in Theorem 1.2, w -approximable functions that do not vanish at a exist. This may be proved using the inverse of the transformation that is introduced in Section 5.1.

However, for $\beta < -\frac{1}{2}$ in Theorem 1.2 and $\beta < 0$ in Theorem 1.3 the problem is open.

Remark 3. Theorem 1.3 solves to a large extent the problem of clearing the role of internal zeros. This problem was posed recently by Totik [10, p. 109]. Of course, Theorem 1.3 is not the final answer to this problem. It is very likely that the theorem remains valid under the following weaker condition on v :

$$v(t) \leq K|t - c|^\beta, \quad K > 0, \quad \beta > 0,$$

for t in a neighborhood of c .

On the other hand, if v tends to 0 very slowly, it is possible that w -approximable functions that do not vanish at c exist, as an example of Totik shows, see Example 4.6 of [10].

The paper is organized as follows. In Section 2 we present three examples to illustrate our results. In particular in Section 2.2 we apply Theorem 1.2 to disprove a conjecture of Lorentz on incomplete polynomials. The rest of the paper contains the proofs of Theorems 1.2 and 1.3. The proof of Theorem 1.2 is the most substantial part of our work. In Section 3 we reduce it to proving a result on weighted polynomials for a weight on the unbounded interval $[1, \infty)$, see Proposition 3.1 below. The proof of this proposition is given in Section 4. It involves a discretization of logarithmic potentials, which requires various careful estimates. Finally, in Section 5 we use Theorem 1.2 to prove Theorem 1.3.

2. Examples

2.1. Freud Weights

For the so-called Freud weights

$$w(x) = \exp(-\gamma_\lambda |x|^\lambda), \quad \gamma_\lambda = \frac{\Gamma(\lambda/2)\Gamma(1/2)}{2\Gamma(\lambda/2 + 1/2)}, \quad \lambda > 0,$$

on the interval $[-1, 1]$, it is known that the extremal measure has the density

$$v(t) = \frac{\lambda}{\pi} \int_{|t|}^1 \frac{y^{\lambda-1}}{\sqrt{y^2 - t^2}} dy, \quad t \in [-1, 1],$$

see [8]. At the endpoints ± 1 this density satisfies (1.3) with $\beta = \frac{1}{2}$. Therefore by Theorem 1.2 every w -approximable function vanishes at ± 1 .

This result was obtained earlier by Lubinsky and Totik [7].

2.2. Incomplete Polynomials

For the weight $w(x) = x^\theta$, $x \in [0, 1]$, it is known that the support of the extremal measure is $[\alpha^2, 1]$ with $\alpha := \theta/(1 + \theta)$ and the density is

$$v(t) = \frac{1 + \theta}{\pi t} \sqrt{\frac{t - \alpha^2}{1 - t}}, \quad t \in [\alpha^2, 1],$$

see [8]. At the left endpoint the density satisfies (1.3) with $\beta = \frac{1}{2}$. Thus it follows from Theorem 1.2 that if weighted polynomials $x^{n\theta} P_n(x)$ tend to a function f uniformly on $[\alpha^2, 1]$, then necessarily $f(\alpha^2) = 0$.

This result can be used to settle a conjecture of Lorentz [5, Conjecture 2] on incomplete polynomials, see also [9]. A polynomial P is α -incomplete if it has a zero of order $\geq \alpha \deg P$ at 0. It was proved by von Golitschek [3] and Saff and Varga [9] that a continuous function f on $[0, 1]$ is the uniform limit of a sequence of α -incomplete polynomials (with strictly increasing degrees) if and only if f vanishes on $[0, \alpha^2]$. In [5] Lorentz made the conjecture (which he himself called doubtful) that every continuous function on $[\alpha^2, 1]$ is the limit of α -incomplete polynomials uniformly on $[\alpha^2, 1]$. It follows from our results that this conjecture is not true.

Corollary 2.1. *Let $0 < \alpha < 1$. If a sequence of α -incomplete polynomials (with strictly increasing degrees) converges uniformly on $[\alpha^2, 1]$ to a function f , then $f(\alpha^2) = 0$.*

Proof. Any α -incomplete polynomial may be written as $x^k P_n(x)$, $P_n \in \Pi_n$, with $\alpha(n + k) \leq k \leq \alpha(n + k) + 1$. If $\theta := \alpha/(1 - \alpha)$, this means that any α -incomplete polynomial can be written as

$$x^k P_n(x), \quad n\theta \leq k \leq n\theta + \frac{1}{1 - \alpha}, \quad P_n \in \Pi_n.$$

Now suppose f is the uniform limit on $[\alpha^2, 1]$ of α -incomplete polynomials $x^{k_n} P_n(x)$ as $n \rightarrow \infty$ through a subsequence Λ of the natural numbers. Here we have $P_n \in \Pi_n$ and $n\theta \leq k_n \leq n\theta + (1 - \alpha)^{-1}$ for every $n \in \Lambda$. Passing to a subsequence if necessary, we may assume that $k_n - n\theta$ converges with limit L , say. Then $\{x^{n\theta} P_n\}_{n \in \Lambda}$ converges to $f(x)x^{-L}$ uniformly on $[\alpha^2, 1]$. So in view of the above result and Remark 1, we have $f(\alpha^2) = 0$. ■

2.3. Fast Decreasing Polynomials

For the weight function $w(x) = \exp(x^2)$ on $[-1, 1]$, the extremal measure has the density

$$v(t) = \frac{2t^2}{\pi \sqrt{1 - t^2}}, \quad t \in [-1, 1],$$

see [10]. The origin is an internal zero of the density and the density satisfies (1.4) with $\beta = 2$. Therefore by Theorem 1.3 every w -approximable function vanishes at 0. This answers a question of Totik [10, p. 110].

In fact, our proof shows that there is a constant $C_1 > 1$ and a positive integer n_0 such that

$$\|\exp(nx^2) P_n(x)\|_{[-1, 1]} \geq C_1, \quad n \geq n_0, \quad P_n \in \Pi_n, \quad P_n(0) = 1.$$

On the other hand, Lubinsky and Totik [6] showed that it is possible to find a constant C_2 and polynomials $P_n \in \Pi_n$ with $P_n(0) = 1$ such that

$$|P_n(x)| \leq C_2 \exp(-nx^2), \quad x \in [-1, 1].$$

So it follows that the constant C_2 cannot be too close to 1. Also not too close to 1 if we consider only large integers n .

3. Transformation of Theorem 1.2

Let w , v , and β be as in Theorem 1.2. Without loss of generality we may assume $a = 0$ and $b = 1$.

3.1. An Extremal Problem

Define, for every n ,

$$\lambda_n := \inf\{\|w^n P_n\| \mid (w^n P_n)(0) = 1, P_n \in \Pi_n\},$$

where $\|\cdot\|$ denotes the supnorm on $[0, 1]$. Theorem 1.2 follows from the following result on the numbers λ_n :

$$(3.1) \quad \liminf_{n \rightarrow \infty} \lambda_n > 1.$$

Indeed, if the theorem were not true we could find a continuous function f and polynomials $P_n \in \Pi_n$ such that

$$f(0) = 1, \quad w^n P_n \rightarrow f \quad \text{uniformly on } [0, 1].$$

Obviously, we may assume that

$$(3.2) \quad w^n(0)P_n(0) = 1.$$

First we show that we may assume that f vanishes at the zeros and discontinuities of v and at the endpoint 1. Let $t_1, \dots, t_k$ be the zeros and discontinuities of v . Then

$$w^{n+k+1}(x)P_n(x)(1-x) \prod_{j=1}^k (x-t_j) \rightarrow f(x)w^k(x)(1-x) \prod_{j=1}^k (x-t_j)$$

uniformly on $[0, 1]$. Therefore $f(x)w^k(x)(1-x) \prod_{j=1}^k (x-t_j)$ is w -approximable and this function vanishes at 1 and at the points t_j (and not at 0). So we may assume that f itself has this property.

Next we claim that we may also assume that

$$0 \leq f(x) \leq 1, \quad x \in [0, 1].$$

To prove this claim note that the restricted support of μ_w is equal to $(0, 1) \setminus \{t_1, \dots, t_k\}$. Because f vanishes at the points t_j and at 1 we can find a continuous function g vanishing outside the restricted support such that $0 \leq f + g \leq 1$ on $[0, 1]$. By Theorem 1.1 the function g is w -approximable. Then clearly $f + g$ is w -approximable too. This proves the claim.

Finally, it follows from (3.1) and (3.2) that there is a constant $C > 1$ such that $\|w^n P_n\| \geq C$ for every large n . Then also $\|f\| \geq C$, which contradicts the fact that $0 \leq f \leq 1$. Thus Theorem 1.2 follows from assertion (3.1).

3.2. Transformation to the Interval $[1, \infty)$

Instead of working with polynomials P_n satisfying (3.2) it is more convenient to work with monic polynomials. This can be achieved by a transformation to the unbounded interval $[1, \infty)$ as follows.

We define the weight

$$(3.3) \quad W(x) := \frac{w(1/x)}{xw(0)}, \quad x \in [1, \infty)$$

and the extremal error

$$(3.4) \quad E_n := \inf\{\|W^n Q_n\|_{[1, \infty)} \mid Q_n \in \Pi_n \text{ is monic}\}.$$

Then it is not hard to see that $E_n = \lambda_n$. Indeed, to a polynomial $P_n \in \Pi_n$ satisfying $w^n(0)P_n(0) = 1$ corresponds a monic polynomial $Q_n \in \Pi_n$ given by

$$Q_n(x) := w^n(0)x^n P_n\left(\frac{1}{x}\right)$$

and vice versa. Since $W^n(x)Q_n(x) = w^n(1/x)P_n(1/x)$ for $x \in [1, \infty)$, the relation $E_n = \lambda_n$ follows. Note incidentally, that the weight W is not admissible, since condition (iii) in the Introduction is violated. However, study of these types of weights might be useful for other purposes as well.

Thus we have proved Theorem 1.2 if we can show that

$$(3.5) \quad \liminf_{n \rightarrow \infty} E_n > 1.$$

To obtain (3.5) we use a de La Vallée Poussin type argument, see p. 77 of [1]. We look for monic polynomials $Q_n(x) = \prod_{j=1}^n (x - \zeta_j)$, $n \geq n_0$, whose zeros satisfy

$$(3.6) \quad 1 < \zeta_1 < \zeta_2 < \cdots < \zeta_n < \infty,$$

such that there are numbers x_j , $j = 1, \dots, n+1$, satisfying

$$(3.7) \quad 1 \leq x_1 < \zeta_1 < x_2 < \cdots < x_n < \zeta_n < x_{n+1} < \infty$$

and

$$(3.8) \quad |W^n(x_j)Q_n(x_j)| \geq C > 1, \quad j = 1, \dots, n+1.$$

Here C is a constant that does not depend on n and j . If we can find such numbers ζ_j and x_j then this implies $E_n \geq C$ for $n \geq n_0$ and then (3.5) follows.

3.3. A Discretization Problem for Potentials

Estimates of the form (3.8) are more tractable if we take logarithms. From (1.1) and (3.3) we obtain, for $x \in [1, \infty)$,

$$\begin{aligned} \log W(x) &= \log w\left(\frac{1}{x}\right) - \log w(0) - \log x \\ &= U^{\mu_w}\left(\frac{1}{x}\right) - U^{\mu_w}(0) - \log x \\ &= -\int_0^1 \log \left| \frac{1}{x} - t \right| d\mu_w(t) + \int_0^1 \log |t| d\mu_w(t) - \log x \\ &= -\int_0^1 \log \left| x - \frac{1}{t} \right| d\mu_w(t). \end{aligned}$$

Then after the transformation $t = 1/s$ we find

$$(3.9) \quad \log W(x) = - \int_1^\infty \log|x - s| d\nu(s) = U^\nu(x), \quad x \in [1, \infty),$$

where the measure $d\nu(s)$ is the image of $d\mu_w(t)$ under the mapping $t = 1/s$. Since μ_w has the form (1.2) and ν satisfies (1.3) the measure ν has the form

$$(3.10) \quad d\nu(s) = u(s) ds, \quad s \in [1, \infty),$$

with a density u satisfying

$$(3.11) \quad u(s) \sim s^{-1-\alpha} \quad (s \rightarrow \infty).$$

Here $\alpha := \beta + 1 > \frac{1}{2}$.

We associate with the polynomial $Q_n(x) = \prod_{j=1}^n (x - \zeta_j)$ the normalized counting measure on its zeros:

$$(3.12) \quad \mu_n := \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j}.$$

Then we have $\log|Q_n| = -nU^{\mu_n}$. Using this and (3.9) we find that inequalities (3.8) are equivalent to

$$(3.13) \quad U^\nu(x_j) - U^{\mu_n}(x_j) \geq \frac{C}{n}, \quad j = 1, \dots, n+1,$$

where C is a positive constant independent of j and n .

In summary, we have shown that Theorem 1.2 follows from the following proposition.

Proposition 3.1. *Let ν be a probability measure on $[1, \infty)$ of the form (3.10) with a density u satisfying (3.11) with $\alpha > \frac{1}{2}$. Then a constant $C > 0$ and a positive integer n_0 exist such that, for every $n \geq n_0$, we can find numbers $x_1, \dots, x_{n+1}$ and $\zeta_1, \dots, \zeta_n$ such that (3.7) and (3.13) hold.*

Proposition 3.1 is an example of a collection of results centered around the question “How to discretize a logarithmic potential?” Results of this type were obtained in, e.g., [4], [6], and [10].

4. Proof of Proposition 3.1

Let ν , u , and α be as in Proposition 3.1. For the proof of Proposition 3.1 we need to introduce the numbers $x_1, \dots, x_{n+1}$ and $\zeta_1, \dots, \zeta_n$. This is done in three steps.

4.1. The Points $x_1, \dots, x_n$ and $\zeta_1, \dots, \zeta_{n-1}$

Let $n \in \mathbb{N}$. For the choice of the points $x_1, x_2, \dots, x_n$ and $\zeta_1, \dots, \zeta_{n-1}$, we follow a method due to Totik, see [6]. We take $x_1 = 1$ and $x_2, \dots, x_n$ such that

$$(4.1) \quad \nu([x_j, x_{j+1}]) = \frac{1}{n}, \quad j = 1, \dots, n-1, \quad \nu([x_n, \infty)) = \frac{1}{n}.$$

Then we take $\zeta_1, \dots, \zeta_{n-1}$ such that

$$(4.2) \quad \zeta_j = n \int_{x_j}^{x_{j+1}} s \, d\nu(s), \quad j = 1, \dots, n-1.$$

Of course, these points also depend on n . In places where confusion might arise, we write $x_{1,n}, \dots, x_{n,n}$ and $\zeta_{1,n}, \dots, \zeta_{n-1,n}$.

We have not yet defined ζ_n and x_{n+1} . These two numbers are chosen in Sections 4.2 and 4.3 below. For the moment we let ζ_n be an arbitrary number $> x_n$ and we take μ_n to be the measure (3.12). Then we write

$$(4.3) \quad U^v(x) - U^{\mu_n}(x) = \sum_{k=1}^n \Delta_k(x), \quad x \in [1, \infty)$$

with

$$(4.4) \quad \Delta_k(x) := - \int_{x_k}^{x_{k+1}} \log \left| \frac{x-s}{x-\zeta_k} \right| d\nu(s), \quad k = 1, \dots, n-1,$$

and

$$(4.5) \quad \Delta_n(x) := - \int_{x_n}^{\infty} \log \left| \frac{x-s}{x-\zeta_n} \right| d\nu(s).$$

The proof of the following lemma is straightforward, see [6].

Lemma 4.1. *Let $k \leq n-1$ and $x \notin (x_k, x_{k+1})$. Then $\Delta_k(x) > 0$.*

For future estimates we need results on the asymptotic behavior of $x_n = x_{n,n}$ as $n \rightarrow \infty$. Define

$$(4.6) \quad L := \frac{1}{\alpha} \lim_{s \rightarrow \infty} u(s) s^{1+\alpha}.$$

This limit exists because of (3.11).

Lemma 4.2. *The following limit relations hold:*

$$(4.7) \quad \lim_{n \rightarrow \infty} \frac{x_n^\alpha}{n} = L$$

and

$$(4.8) \quad \lim_{n \rightarrow \infty} \sup_{t \in [1, \infty)} \left| \frac{nx_n u(x_n t)}{\alpha t^{-1-\alpha}} - 1 \right| = 0.$$

Proof. It is clear that $\lim_n x_n = \infty$. Let $\varepsilon > 0$. From (4.6) it follows that, for n large enough, we have

$$(4.9) \quad |u(s) - L\alpha s^{-1-\alpha}| < \varepsilon \alpha s^{-1-\alpha}, \quad s \geq x_n.$$

Integrating this inequality from x_n to ∞ , we find

$$\left| \int_{x_n}^{\infty} u(s) \, ds - \int_{x_n}^{\infty} L\alpha s^{-1-\alpha} \, ds \right| \leq \varepsilon \int_{x_n}^{\infty} \alpha s^{-1-\alpha} \, ds,$$

which is

$$(4.10) \quad \left| \frac{1}{n} - Lx_n^{-\alpha} \right| \leq \varepsilon x_n^{-\alpha}.$$

This proves (4.7). We also find from (4.9) that

$$\left| \frac{u(x_nt)}{\alpha(x_nt)^{-1-\alpha}} - L \right| < \varepsilon, \quad t \in [1, \infty).$$

Combining this with (4.10) and letting $\varepsilon \rightarrow 0$ we obtain (4.8). ■

4.2. The Point ζ_n

We take $\zeta_n = Ax_n$ where the constant $A > 1$ has to be determined in such a way that, for every $x \in [1, x_n]$, we have $n\Delta_n(x) \geq C_1$, where C_1 is a positive constant.

For $x \in [1, x_n]$ we have

$$\Delta_n(x) = - \int_{x_n}^{\infty} \log \left(\frac{s-x}{\zeta_n-x} \right) u(s) ds.$$

Substitute $s = x_nt$ and take $x = x_ny$ where $0 \leq y \leq 1$ to obtain

$$n\Delta_n(x_ny) = \int_1^{\infty} \log \left(\frac{A-y}{t-y} \right) nx_n u(x_nt) dt.$$

From (4.8) it follows that

$$(4.11) \quad \lim_{n \rightarrow \infty} n\Delta_n(x_ny) = \int_1^{\infty} \log \left(\frac{A-y}{t-y} \right) \alpha t^{-1-\alpha} dt$$

and it is not hard to show that the convergence is uniform for $y \in [0, 1]$.

We proceed to calculate the integral in (4.11). A change of variables $t \mapsto 1/t$ and an integration by parts leads to

$$\log \left(\frac{A-y}{1-y} \right) - \int_0^1 t^{\alpha-1} (1-ty)^{-1} dt = \log(A-y) - \log(1-y) - \alpha^{-1} F(1, \alpha; 1+\alpha; y),$$

where $F(1, \alpha; 1+\alpha; y)$ denotes a hypergeometric function. Expanding this into a power series with respect to y , we get

$$\log A - \frac{1}{\alpha} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{\alpha+k} - \frac{1}{kA^k} \right) y^k.$$

Now it is easy to show that, for $A > \exp(1/\alpha)$, all terms in the series are positive. So, for such A , we see from (4.11) that $\lim_{n \rightarrow \infty} n\Delta_n(x) \geq C_1 > 0$ uniformly for $x \in [1, x_n]$.

In Section 4.3 it is important that $A < \alpha/(\alpha-1)$ in case $\alpha > 1$. It is easily seen that $\exp(1/\alpha) < \alpha/(\alpha-1)$ for $\alpha > 1$. So if $\alpha > 1$ we choose A from the interval $(\exp(1/\alpha), \alpha/(\alpha-1))$. Now we have proved the following lemma.

Lemma 4.3. *There are constants $A > 1$, $C_1 > 0$ and a positive integer n_1 such that with $\zeta_n = Ax_n$ we have*

$$\Delta_n(x) \geq \frac{C_1}{n} \quad \text{for all } n \geq n_1 \quad \text{and } x \in [1, x_n].$$

Moreover, if $\alpha > 1$, we can take $A < \alpha/(\alpha-1)$.

4.3. The Point x_{n+1}

We take $A > 1$ and $\zeta_n = Ax_n$ as in Lemma 4.3. We also assume $A < \alpha/(\alpha - 1)$ if $\alpha > 1$. We put $x_{n+1} = x_{n+1,n} = Bx_n$ with $B > A$ where B has to be determined. We need to find a lower bound for $\Delta_n(x_{n+1,n})$. From

$$\Delta_n(x_{n+1,n}) = - \int_{x_n}^{\infty} \log \left| \frac{s - Bx_n}{Ax_n - Bx_n} \right| u(s) ds$$

we obtain, after a substitution $s = x_n t$,

$$n\Delta_n(x_{n+1,n}) = - \int_1^{\infty} \log \left| \frac{t - B}{B - A} \right| nx_n u(tx_n) dt.$$

Then using (4.8) we find

$$(4.12) \quad \lim_{n \rightarrow \infty} n\Delta_n(x_{n+1,n}) = - \int_1^{\infty} \log \left| \frac{t - B}{B - A} \right| \alpha t^{-\alpha-1} dt = I_1 + I_2.$$

Here I_1 is the integral from 1 to B and I_2 is the integral from B to ∞ . For I_1 we have

$$I_1 = \int_1^B \log \left(\frac{B - t}{B - A} \right) d(t^{-\alpha} - B^{-\alpha}).$$

After integration by parts and a change of variables $t \mapsto Bt$ we obtain

$$(4.13) \quad I_1 = (B^{-\alpha} - 1) \log \left(\frac{B - 1}{B - A} \right) + B^{-\alpha} \int_{1/B}^1 \frac{t^{-\alpha} - 1}{1 - t} dt.$$

For I_2 we find after the change of variables $t \mapsto B/t$,

$$I_2 = -B^{-\alpha} \int_0^1 \log \left(\frac{1/t - 1}{1 - A/B} \right) \alpha t^{\alpha-1} dt,$$

which can be evaluated explicitly, using integration by parts and an integral representation for the digamma function, see (13) on p. 16 of [2]. The result is

$$(4.14) \quad I_2 = B^{-\alpha} \left[\psi(\alpha) - \psi(1) + \log \left(1 - \frac{A}{B} \right) \right],$$

where ψ is the digamma function.

Now we distinguish three cases: $\alpha = 1$, $\frac{1}{2} < \alpha < 1$ and $\alpha > 1$. In the following relations all o and $\mathcal{O}$ symbols are for $B \rightarrow \infty$.

Case I: $\alpha = 1$. In this case (4.14) becomes

$$I_2 = B^{-1} \log \left(1 - \frac{A}{B} \right) = \mathcal{O}(B^{-2}),$$

while (4.13) is

$$I_1 = (B^{-1} - 1) \log \left(\frac{B - 1}{B - A} \right) + B^{-1} \log B = B^{-1} \log B + \mathcal{O}(B^{-1}).$$

From this it is clear that, for large B , the sum of I_1 and I_2 is positive.

Case II: $\frac{1}{2} < \alpha < 1$. In this case we note that [2, (13), p. 16],

$$\int_0^1 \frac{v^{-\alpha} - 1}{1 - v} dv = \psi(1) - \psi(1 - \alpha)$$

so that we can write (4.13) as

$$I_1 = B^{-\alpha}(\psi(1) - \psi(1 - \alpha)) + \mathcal{O}(B^{-1}).$$

Since (4.14) is

$$I_2 = B^{-\alpha}(\psi(\alpha) - \psi(1) + \mathcal{O}(B^{-1}))$$

we get

$$I_1 + I_2 = B^{-\alpha}(\psi(\alpha) - \psi(1 - \alpha)) + \mathcal{O}(B^{-1}).$$

Since ψ is increasing on the interval $(0, 1)$ and $\frac{1}{2} < \alpha < 1$, we have $\psi(\alpha) > \psi(1 - \alpha)$. So for B sufficiently large, we have $I_1 + I_2 > 0$. (Note that at this point the proof would fail for $\alpha \leq \frac{1}{2}$.)

Case III: $\alpha > 1$. Noting that

$$\int_{1/B}^1 \frac{v^{-\alpha} - 1}{1 - v} dv = \int_{1/B}^1 v^{-\alpha} dv + \int_{1/B}^1 \frac{v^{-\alpha+1} - 1}{1 - v} dv = B^{\alpha-1} \frac{1}{\alpha - 1} + o(1)$$

we see that (4.13) gives

$$I_1 = -\log\left(\frac{B-1}{B-A}\right) + B^{-1}\left(\frac{1}{\alpha-1} + o(1)\right) = B^{-1}\left(1 - A + \frac{1}{\alpha-1} + o(1)\right).$$

Obviously (4.14) implies $I_2 = \mathcal{O}(B^{-\alpha})$. So

$$I_1 + I_2 = B^{-1}\left(1 - A + \frac{1}{\alpha-1} + o(1)\right).$$

Since $A < \alpha/(\alpha - 1)$, see Lemma 4.3, we have $1 - A + 1/(\alpha - 1) > 0$ and again we find that, for B large enough, $I_1 + I_2 > 0$.

So, in all three cases, $I_1 + I_2 > 0$ for B large enough. Because of (4.12), we have proved the following lemma.

Lemma 4.4. *There are constants $C_2 > 0$, $B > A$, and a positive integer n_2 such that with $x_{n+1} = Bn^{1/\alpha}$, we have*

$$\Delta_n(x_{n+1}) > \frac{C_2}{n} \quad \text{for all } n \geq n_2.$$

4.4. Conclusion of the Proof of Proposition 3.1

Let n_0 be the maximum of n_1 and n_2 and let $n \geq n_0$. From Lemmas 4.1 and 4.3 we see that

$$\sum_{k=1}^n \Delta_k(x_j) \geq \frac{C_1}{n} \quad j = 1, \dots, n,$$

and from Lemmas 4.1 and 4.4 it follows that

$$\sum_{k=1}^n \Delta_k(x_{n+1}) \geq \frac{C_2}{n}.$$

So because of (4.3) the estimates (3.13) holds. This proves Proposition 3.1. ■

5. Proof of Theorem 1.3

For the proof of Theorem 1.3 we first consider the special case of a symmetric density v vanishing at the origin. In Section 5.2 the general case is deduced from this special case.

5.1. The Symmetric Case

Suppose that $a = -1$, $b = 1$, $c = 0$, and w (and hence v) is an even function. We make a transformation which turns the origin into an endpoint. Consider the following weight $\tilde{w}$ on $[0, 1]$:

$$(5.1) \quad \tilde{w}(y) := w(\sqrt{y})^2, \quad y \in [0, 1].$$

To compute the extremal measure associated with $\tilde{w}$ we use (1.1) and (5.1) to obtain, for $y \in [0, 1]$,

$$\log \tilde{w}(y) = 2U^{\mu_w}(\sqrt{y}) - 2F_w = -2 \int_{-1}^1 \log|\sqrt{y} - t|v(t) dt - 2F_w.$$

From the symmetry of v it follows that

$$\begin{aligned} \log \tilde{w}(y) &= -2 \left(\int_0^1 \log|\sqrt{y} - t|v(t) dt + \int_0^1 \log|\sqrt{y} + t|v(t) dt \right) - 2F_w \\ &= -2 \int_0^1 \log|y - t^2|v(t) dt - 2F_w. \end{aligned}$$

Then the change of variables $t^2 = s$ gives

$$\log \tilde{w}(y) = - \int_0^1 \log|y - s| \frac{v(\sqrt{s})}{\sqrt{s}} ds - 2F_w.$$

Therefore the extremal measure for $\tilde{w}$ has a density on $[0, 1]$ which is equal to

$$(5.2) \quad \tilde{v}(s) = \frac{v(\sqrt{s})}{\sqrt{s}}, \quad s \in [0, 1].$$

Now let f be a w -approximable function, say $w^n P_n \rightarrow f$ uniformly on $[-1, 1]$ with $P_n \in \Pi_n$. Then also $w^n(x)P_n(-x) \rightarrow f(-x)$ uniformly on $[-1, 1]$, so that

$$w^{2n}(x)P_n(x)P_n(-x) \rightarrow f(x)f(-x) \quad \text{uniformly on } [-1, 1].$$

There is a polynomial $Q_n \in \Pi_n$ such that $Q_n(x^2) = P_n(x)P_n(-x)$. Then if we write $g(x^2) = f(x)f(-x)$, make the substitution $x^2 = y$, and use (5.1), we get

$$(5.3) \quad \tilde{w}^n(y)Q_n(y) \rightarrow g(y) \quad \text{uniformly on } [0, 1].$$

From (1.4) and (5.2) it follows that

$$\tilde{v}(s) \sim s^{\beta/2-1/2} \quad (s \rightarrow 0).$$

Since $\beta/2 - \frac{1}{2} > -\frac{1}{2}$, we get from Theorem 1.2 and (5.3) that $g(0) = 0$ and therefore $f(0) = 0$.

This proves Theorem 1.3 in the symmetric case.

5.2. The General Case

First we need a lemma.

Lemma 5.1. *Let w be an admissible weight on the interval $[-1, 1]$ such that the support of the extremal measure is $[-1, b]$ for some $b \in (0, 1)$. Assume the restricted support of w is equal to $(-1, b) \setminus E$ where E is a finite set. Let f be a continuous function f which vanishes on $[b, 1]$ and for which a sequence $P_n \in \Pi_n$ exists such that*

$$w^n P_n \rightarrow f \quad \text{uniformly on } [-1, b].$$

Then

$$w^n P_n \rightarrow f \quad \text{uniformly on } [-1, 1].$$

Proof. We may restrict ourselves to functions f that vanish in a neighborhood of the endpoint b . Indeed, if this does not hold, then we can subtract from f a continuous function g which vanishes outside the restricted support and agrees with f in a neighborhood of b . Then g is w -approximable by Theorem 1.1, and it suffices to prove the lemma for $f - g$. So assume f vanishes on $[b - \eta, b]$ with $\eta > 0$.

Let μ_w be the extremal measure associated with w . Then we have, with a constant F_w [10, p. 4],

$$\begin{aligned} \log w(x) &= U^{\mu_w}(x) - F_w, & x \in [-1, b], \\ \log w(x) &\leq U^{\mu_w}(x) - F_w, & x \in [b, 1]. \end{aligned}$$

Let ω_x be the harmonic measure on $[-1, b]$ of the point $x \in (b, 1]$. Then it is easy to see that there is a constant $C > 0$ such that $\omega_x([b - \eta, b]) \geq C$ for all $x \in (b, 1]$. Because $n \log w + \log |P_n|$ is dominated by $n(U^{\mu_w} - F_w) + \log |P_n|$ on $(b, 1]$ and the latter function is subharmonic on $\overline{\mathbb{C}} \setminus [-1, b]$, we have that, for $x \in [b, 1]$,

$$n \log w(x) + \log |P_n(x)| \leq \int_{-1}^b (n(U^{\mu_w} - F_w) + \log |P_n|) d\omega_x = \int_{-1}^b \log |w^n P_n| d\omega_x.$$

From the convergence of $w^n P_n$ on $[-1, b]$, it easily follows that the integrals

$$\int_{-1}^{b-\eta} \log |w^n P_n| d\omega_x$$

are uniformly bounded from above. Since $w^n P_n$ converges uniformly to 0 on $[b - \eta, b]$ and $\omega_x([b - \eta, b]) \geq C$ for all $x \in [b, 1]$, the integrals

$$\int_{b-\eta}^b \log |w^n P_n| d\omega_x$$

tend uniformly to $-\infty$ as $n \rightarrow \infty$. Thus $n \log w(x) + \log |P_n(x)|$ tends to $-\infty$ uniformly for $x \in [b, 1]$.

This proves the lemma. ■

Now we can finish the proof of Theorem 1.3.

We may assume that $c = 0$, $a = -1$, and $b \in (0, 1)$. As noted in (1.1) the equation $w(x) = \exp(U^{\mu_w}(x) - F_w)$ holds for $x \in [-1, b]$, and we define $w(x)$ by this expression on the full interval $[-1, 1]$. We also define $v(t)$ to be equal to 0 on $(b, 1]$.

We introduce a symmetric weight $\tilde{w}$ as follows:

$$\tilde{w}(x) := \sqrt{w(x)w(-x)}, \quad x \in [-1, 1].$$

Then we have for $x \in [-1, 1]$,

$$\begin{aligned} \log \tilde{w}(x) &= -\frac{1}{2} \left(\int_{-1}^1 \log |x-t| v(t) dt + \int_{-1}^1 \log |-x-t| v(t) dt \right) - F_w \\ &= -\int_{-1}^1 \log |x-t| \frac{v(t) + v(-t)}{2} dt - F_w. \end{aligned}$$

So the support of the extremal measure of $\tilde{w}$ is equal to $[-1, 1]$ and it has the density

$$(5.4) \quad \tilde{v}(t) = \frac{v(t) + v(-t)}{2}, \quad t \in [-1, 1].$$

Now suppose f is w -approximable on the interval $[-1, b]$, say $w^n P_n \rightarrow f$ uniformly on $[-1, b]$. Without loss of generality we may assume that f vanishes at b , see the argument given after (3.2). Extend f to the interval $[-1, 1]$ by defining it to be identically zero on $[b, 1]$. Then by Lemma 5.1 we have

$$w^n P_n \rightarrow f \quad \text{uniformly on } [-1, 1].$$

Then also $w^n(-x) P_n(-x) \rightarrow f(-x)$ uniformly for $x \in [-1, 1]$ and therefore

$$\tilde{w}^{2n}(x) P_n(x) P_n(-x) \rightarrow f(x) f(-x) \quad \text{uniformly on } [-1, 1].$$

From (1.4) and (5.4) it is clear that $\tilde{v}(t) \sim |t|^\beta$ ($t \rightarrow 0$). Therefore, by the result of Section 5.1, it follows that $f(x) f(-x)$ vanishes for $x = 0$ and so $f(0) = 0$.

This completes the proof of Theorem 1.3. ■

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Note Added in Proof. The cases $\beta < -\frac{1}{2}$ in Theorem 1.2 and $\beta < 0$ in Theorem 1.3, referred to in Remark 2 of the introduction, were recently settled by the author: A. B. J. Kuijlaars, *Weighted approximation with varying weights: the case of a power-type singularity*, J. Math. Anal. Appl. (to appear). In these cases Theorems 1.2 and 1.3 continue to hold, but the proof is completely different.

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