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Cosmological two-fluid bulk viscosity

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ABSTRACT
A simple two-fluid model of cosmological bulk viscosity, in which small deviations from thermal equilibrium account for the viscous bulk pressure, is substantiated by kinetic theory. Some peculiar issues regarding its relation to the radiative fluid model are discussed. The microphysical picture underlying the viscous dissipation is made precise. We also consider a reactive ‘cross’ viscosity associated with deviations from detailed balance, which includes the so-called creation pressure of the cosmological fluid. For collisional interactions between the fluid components, the reactive viscous pressure is not an independent mechanism for entropy production. Entropy from cross effects may be generated through an effective isentropic particle source. In both instances new results are obtained for the reactive viscosity, and applied to a representative case of non-equilibrium decay.

Key words: hydrodynamics – radiative transfer – cosmology: theory.

1 INTRODUCTION
It is hardly known that bulk (or volume) viscosity first appeared in HA Lorentz’s work on the theory of sound (Kox 1990). In classical fluids it is related to the internal degrees of freedom of the constituent particles. The bulk viscosity of a simple gas of point particles, originally obtained by Israel (1963), is a relativistic effect that vanishes in the non-relativistic and extreme-relativistic regimes.

In contrast to the separate components, a mixture of non-relativistic and purely relativistic particles generally does possess a non-vanishing bulk viscosity, as argued by Weinberg (1971) for a cosmological mixture of radiation quanta (e.g. photons or neutrinos) and material particles. Although it was indicated in Weinberg (1972) that this ‘radiative’ bulk viscosity found its origin in the breaking of thermal equilibrium between matter and radiation, the detailed dissipative mechanism, according to Udey & Israel (1982), remained somewhat obscure. In a semiquantitative model that reproduced the radiative bulk viscosity coefficient up to a numerical factor, these latter authors attributed the viscous heating to a microscopic heat current that flows to re-equalize the temperatures.

The physical origin of cosmological bulk viscosity was re-examined by Zimdahl (1996b), who argued that different individual cooling rates of two interacting perfect fluids are sufficient for a non-vanishing bulk viscosity of the fluid as a whole. In case one of the components is fully relativistic, the fluid qualitatively resembles the radiative fluid considered by Weinberg (1971) and Udey & Israel (1982). The bulk viscosity coefficient calculated by Zimdahl (1996b) then agrees with the radiative result upon the assumption that the system is matter-dominated. We will follow up on this point which is intriguing as the radiative bulk viscosity was originally obtained in the setting of the radiation-dominated early universe.

Radiative bulk viscosity is strongly suppressed by a radiation-dominated equation of state, and as such is far insufficient to account for the entropy content of the universe (Weinberg 1971). Understanding the huge radiation entropy was a major motivation to study viscosity and other modes of dissipation in the cosmological fluid. Following original ideas of Zel’dovich (1970), Hu (1982), and Prigogine et al. (1989) on matter creation in the early universe, various authors (e.g. Calvão, Lima & Waga 1992; Lima & Germano 1992; Zimdahl & Pavon 1993, 1994; Gariel & le Denmat 1995; Zimdahl 1996a, 1997; Silva et al. 2002) have elaborated on the viscous bulk pressure effects of cosmological particle production. Meanwhile the interest of viscous fluid cosmology has ranged from entropy generation via cosmic inflation to dark energy and dark matter fluids (e.g. Zimdahl, Velten & Hipolito 2011; Velten et al. 2013, and references therein).

In this paper, we focus on the nature of bulk viscosity in relativistic and radiative fluid mixtures. We first consider the non-reacting fluid mixture. Kinetic theory provides a foundation for the heuristic two-fluid model of Zimdahl (1996b). Through the kinetic approach it also transpires that the main result is equivalent to an earlier expression given in the monograph by De Groot, van Leeuwen & van Weert (1980). In the expanding reactive fluid, an additional (cross) viscosity is associated with deviations from detailed balance. According to Zimdahl (1997), this ‘reactive’ bulk viscosity would be a far larger dissipative effect than the conventional bulk viscosity driven by expansion. For this reactive pressure and its relation to particle and entropy production, we arrive at significant differences with previous results.

Section 2 is a brief synopsis of the non-reactive fluid mixture at the semiphenomenological level, while Section 3 outlines a kinetic
theory approach based on the relativistic Boltzmann equation. In Section 4, we quantify the underlying microscopic picture of bulk viscosity, and comment on the relation with radiative bulk viscosity.

Section 5 extends the original heuristic model to reactive fluids, and Section 6 is devoted to viscous and reactive entropy production. In principle the effective reactive bulk viscosity may be identified from the reactive entropy source strength. However, in these considerations an issue of sign emerges. We conclude that the viscous cross term in the reactive entropy production is the opposite of the reactive cross term in the viscous entropy production. As a consequence the two cancel and do not, as individual sources, affect the total non-equilibrium entropy production.

In Section 7, we discuss our result for the reactive bulk pressure, and identify the ‘creation pressure’ coefficient of the cosmological fluid. The effective viscous cross pressure is to be distinguished from the cross viscosity in the linear transport relations coupling bulk viscosity and reaction rate. We furthermore consider the net entropy cross term due to an effective isentropic particle source. The results are applied in two illustrative cases of non-equilibrium decay in Section 8. Concluding final remarks are given in Section 9.

There are two appendices to the paper. In Appendix A, it is shown that the binary fluid bulk viscosity coefficient is in agreement with an expression in De Groot et al. (1980). Appendix B addresses some issues regarding the conditions of thermal equilibrium and matter dominance in the case of the radiative fluid mixture.

Although the context of the subject is cosmological, it is sufficient for our purpose to deal with our fluid models in special-relativistic terms. We adopt the metric of signature $-2$, and choose units such that the velocity of light and the Boltzmann constant take the value one.

2 NON-EQUILIBRIUM PRESSURE

We first consider a semiphenomenological two-fluid model in which small deviations from thermal equilibrium account for the viscous bulk pressure. The temperature differences originate from the different adiabatic cooling rates of the fluid components and the overall cooling rate of the fluid (Zimdahl 1996b). The basic argument is that the total thermal pressure of the components differs from the equilibrium pressure of the fluid as a whole,

$$\sum_k p_k(T_k) = p(T) + \Pi.$$  \hfill (1)

It is intuitively obvious that the non-equilibrium pressure $\Pi$ is the viscous bulk pressure, and this is substantiated by kinetic theory in the next section. The equilibrium temperature $T$ is defined implicitly through the condition

$$\sum_k p_k(T_k) = p(T)$$  \hfill (2)

on the total energy density. The temperature differences $\delta T_k \equiv T_k - T$ are supposed to be small enough to allow for a linear treatment. As the particle numbers of the (non-reactive) system are conserved, the viscous pressure is then given by

$$\Pi = \sum_k n_k \delta T_k,$$  \hfill (3)

with $n_k$ the number density of species $k$. Here and in the following, we adopt the classical ideal gas equations of state, $p_k = n_k T_k$ and $p = nT$, with $n$ the total number density. In (1) and (2) the dependence on the number densities has been suppressed. Condition (2) imposes the constraint

$$\sum_k n_k c_{V_k} \delta T_k = 0$$  \hfill (4)

on the temperature differences, with $c_{V_k}$ the heat capacity per particle of species $k$, at constant volume.

The temperature deviations $\delta T_k$ arise over small time intervals of the order of the mean collision time $\tau$, during which the components evolve effectively as freely expanding perfect fluids with

$$\dot{T} = -(\gamma - 1)T \Theta.$$  \hfill (5)

Here, $\Theta$ is the overall expansion rate, and $\gamma_k = c_p/c_{V_k}$ is the appropriate adiabatic exponent. In units of the Boltzmann constant, we have the connection $c_{V_k} = 1/(\gamma_k - 1)$. The overdot indicates the proper (comoving) time derivative. Similarly, the adiabatic evolution of the equilibrium temperature is given by

$$\dot{T} = -(\gamma - 1)T \Theta,$$  \hfill (6)

with $\gamma = c_p/c_{V}$ the specific heat ratio of the composite system, and $c_V = 1/(\gamma - 1)$. During the small free time interval $\tau$, the (initially equal) temperatures change by amounts $\tau T_k$ and $\tau T$, respectively, resulting in temperature discrepancies

$$\delta T_k = -\tau (\gamma_k - \gamma) T \Theta.$$  \hfill (7)

These satisfy condition (4) by virtue of the thermodynamic identity

$$\sum_k n_k c_{V_k} (\gamma_k - \gamma) = 0.$$  \hfill (8)

We shall refer to (4), or more generally (2), as the condition of fit (COF) on the temperature, or energy density, respectively. The implicit COF on the number densities is $\delta n_k = 0$. The conditions of fit are of paramount importance in obtaining the proper form of the bulk viscosity, and in this paper we intend to highlight this aspect.

As a first instance, let us consider the bulk viscosity as given by (3). It is seen explicitly that it is due to thermal non-equilibrium effects (induced by the expansion). We can incorporate condition (4) by rewriting the expression as

$$\Pi = \sum_k n_k c_{V_k} (\gamma_k - \gamma) \delta T_k + (\gamma - 1) \sum_k n_k c_{V_k} \delta T_k.$$  \hfill (9)

In the next section, the first term on the right-hand side will be shown to correspond to the dissipative viscous pressure as it occurs in the irreversible entropy production. The second term vanishes, leaving

$$\Pi = -\sum_k n_k c_{V_k} T (\gamma_k - \gamma)^2 \tau \Theta \equiv -\zeta \Theta$$  \hfill (10)

upon inserting the deviations (7). The last equality defines the (manifestly positive) bulk viscosity coefficient $\zeta$. From (10) one readily recovers the radiative bulk viscosity, and various earlier forms of the binary fluid bulk viscosity coefficient, as discussed in Section 4.

3 KINETIC APPROACH

For a description at the level of kinetic theory, we adopt the Maxwell–Jüttner distribution,

$$f_k(x, p_k) = (2\pi \hbar \gamma)^{-3} \exp[(\mu_k - p_k^0 U_\nu)/T_k].$$  \hfill (11)

characterizing the fluid components at temperature $T_k$. It is a function of the (on-shell) particle four-momenta $p_k = p_k^0 e^\mu$ and, through the local state variables, of the space–time co-ordinates $x = x^\mu$. The
overall fluid four-velocity is $U^\mu$ (normalized to $c = 1$), so $p_k^\mu U_\mu$ is the covariant particle energy. The quantity $\mu_\kappa$ is the chemical potential of component $\kappa$.

The time-space evolution of the distribution function is governed by the relativistic Boltzmann equation,

$$p_k^\mu \partial_{\mu} f_k = C_k[f_k'] = -\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T. \quad (12)$$

for binary collisions between the species ($k, l = 1, 2; k \neq l$). For elastic collisions $p_i, p_i' \rightarrow p_i, p_i'$ one has the collision terms

$$C_k[f_k', f_k] = -\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T. \quad (13)$$

The Lorentz-invariant integration elements are $d\omega \partial \omega = d^3 p_i / \sqrt{p_0}$, and similarly for the primed final momenta. The quantities $W_{\mu\nu}$ are microscopic transition rates obeying the detailed balance symmetry

$$W_{\mu\nu} = W(p_i, p_i' \rightarrow p_i, p_i') = W(p_i', p_i' \rightarrow p_i, p_i) \equiv W_{\nu\mu}. \quad (14)$$

We cast the collision term (13) with (11) into the linearized form

$$C_k[f_k] = -\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T. \quad (15)$$

with the normalized energy variable $u_k = p_k^\mu U_\mu / T$, and similarly for its primed counterpart. The zero-order distribution

$$f_k^{0} = (2\pi)^{-3} \exp[\mu_k - p_k^\mu U_\mu / T] \quad (16)$$

is the distribution at mutual thermal equilibrium of the fluid components, which can be used here as the deviations $\delta T_k$ are of first order in the mean free time. The latter, in turn, is supposed to be small compared to the hydrodynamic (expansion) time, to allow for a linear treatment.

The distribution (16) is also used to evaluate the hydrodynamic streaming, or driving term, i.e. the left-hand side of the Boltzmann equation. The driving term involves the time-space derivatives of the chemical potentials, temperature, and hydrodynamic velocity. For bulk viscosity, spatial gradients are irrelevant except as contained in the expansion rate $\Theta = \partial \mu / \partial U^\mu$, which is the relative rate of change of the number densities according to the continuity equation

$$\dot{n}_k = -n_k \partial \mu / \partial U^\mu. \quad (17)$$

The adiabatic temperature evolution is

$$\delta T = -(\gamma - 1) \Theta \partial \mu / \partial U^\mu. \quad (18)$$

To evaluate the time derivative of the chemical potential we invert the relation

$$n_k = (2\pi)^{-3} [4\pi^2 K_2(z_k) \Theta^2] \exp[\mu_k / T] \quad (19)$$

to write $\mu_k$ as a function of $n_k$ and $T$, so that

$$\mu_k = T n_k / n_k + (\mu_k - e_k) \Theta / T \quad (20)$$

with the partial internal energy per particle

$$e_k = [z_2 K_2(z_k) / K_2(z_k) - 1] \Theta. \quad (21)$$

In these expressions, $K_2$ and $K_3$ are modified Bessel functions of the second kind of argument $z_k \equiv m_k / T$. Subsequently, (17) and (18) can be used to obtain the expansion part of the driving term

$$p_k^\mu \partial \mu f_k^{0} = f_k^{0} [(4/3 - \gamma) u_k^2 + ((\gamma - 1)e_k / T - 1) u_k - z_k^2 / 3] \Theta \partial \mu / \partial U^\mu. \quad (22)$$

The collision term is essentially a rate integral for particles with energy variable $u_k$. To obtain an appropriate mean free time this energy dependence is integrated out. A first integration of (15) over the momenta of species $k$ yields the particle production rate per unit volume

$$\int d\omega \delta T_f C_{\mu\nu} = -\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T. \quad (23)$$

The integral vanishes in accordance with the conservation of particle numbers in the collisions. Consistent with this, the integral of the expansion term also vanishes,

$$\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T = 0 \quad (24)$$

as may be verified with (22). Thus the number balance equations are trivially satisfied. The next higher integral moment,

$$\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T \quad (25)$$

is the energy transfer rate. Interchanging the labels reverses the sign, so consistency here requires the corresponding moment of the driving term,

$$\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T \quad (26)$$

to also change sign upon an interchange of labels, i.e. we require the solubility condition

$$\sum_k n_k cV_k (\gamma_\kappa - \gamma) = 0. \quad (27)$$

The condition is the thermodynamic identity already encountered in (8). While consistent, the energy equations form a dependent set and contain the same information. To determine the individual differences $\delta T_k$ an independent equation is needed. This independent equation is provided by the COF (4) which allows us to eliminate one of the temperature deviations and write (25) as

$$\int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu}\delta T / T \quad (28)$$

We may parametrize the energy source term in terms of a mean collision time $\tau$ by setting it equal to $-\delta T_k / \tau$. This amounts to defining

$$1 / \tau = (ncV / n_k cV_k n_k cV_k) \quad \times \int d\omega \partial \omega f_k' d\omega' f_k d\omega' f_k' W_{\mu\nu} f_k^0 W_{\nu} u_k (u_k - u_k'). \quad (29)$$

The pre-factor is a harmonic mean,

$$\frac{n_k cV_k}{n_k cV_k} \frac{n_k cV_k}{n_k cV_k} = 1 / n_k cV_k + 1 / n_k cV_k \quad (30)$$

equivalent to a harmonic mean of the equilibrium energy densities. The energy equations then read

$$n_k cV_k (\gamma_\kappa - \gamma) T \partial \mu / \partial U^\mu = -\tau^{-1} n_k cV_k \delta T_k \quad (31)$$

Summing over $k$ gives zero on both sides of the equality sign, as follows from (27) and (4), so the conditions of fit are consistent with the solubility requirement. The solutions,

$$\delta T_k = -\tau (\gamma_\kappa - \gamma) T \partial \mu / \partial U^\mu \quad (32)$$

are indeed identical to those obtained in (7). What we have gained over the heuristic approach of the previous section is that we have a
For the bulk-viscous pressure. The temperature deviations (32) subsequently yield the result (10), with the bulk viscosity coefficient $\zeta$. From the foregoing it may be inferred that the expression holds as well for reactive collisions between the species, provided the particle numbers are conserved, i.e. in reactive equilibrium ($\delta n_k = 0$).

4 TWO-FLUID AND RADIATIVE BULK VISCOSITY

The form (44) of the viscous pressure not only shows that thermal disequilibrium effects lie at its origin. Unlike its initial counterpart (3) it is also indicative of the microphysical picture of the bulk-viscous dissipation. The temperature differences constitute a thermodynamic ‘force’ whose associated ‘flow’ of thermal energy here becomes manifest. This can be made precise by writing the usual bilinear expression for the entropy production as

$$T \sigma = -\Pi \Theta = -\sum_k \xi_k \delta T_k / T,$$

with partial flows $\xi_k$ and forces $\delta T_k / T$. According to (44) and (32) we then have, respectively,

$$\xi_k = n_{kC} \nu_k (y_k - \gamma) T \Theta = -n_{kC} \nu_k \delta T_k / \tau,$$

i.e. the mean rate of thermal energy transfer, itself proportional to $\delta T_k$. The entropy production rate thus assumes the canonical form

$$\sigma = \sum_k n_{kC} \nu_k \tau (\delta T_k / T)^2,$$

quadratic in the microscopic thermodynamical forces. The picture is a direct scalar analogue of conductive energy flow, with $\xi_k$ corresponding to the flux of thermal energy, and $\delta T_k$ to its driving temperature gradient.

Specifying the temperature deviations gives the bulk viscosity and viscous entropy production in terms of the macroscopic driving force $\Theta \equiv \partial_\mu U^n$. The bulk viscosity is then quantified by the coefficient

$$\zeta = \sum_k n_{kC} \nu_k (y_k - \gamma)^2 \tau \equiv \sum_k \xi_k.$$

For the binary mixture, $\xi_2$ can be expressed in terms of $\xi_1$ through the identity (27),

$$n_{2C} \nu_2 (y_2 - \gamma) = -n_{1C} \nu_1 (y_1 - \gamma),$$

leading to the total coefficient

$$\zeta = n_{1C} \nu_1 T (y_1 - \gamma) (y_1 - \gamma) \tau.$$

The equality also holds with the labels interchanged. The forms are equivalent to the main result of Zimdahl (1996b),

$$\zeta = -n C V T (y_1 - \gamma) (y_2 - \gamma) \tau.$$

In Appendix A, we show that they are also in agreement with an earlier, physically less transparent expression for the non-reactive binary mixture given in De Groot et al. (1980).

The two-fluid bulk viscosity coefficient has not previously appeared in the convenient form (48). In the partial bulk viscosity $\zeta_k$ one readily recognizes the radiative bulk viscosity coefficient. For the photon component of a radiative mixture, $\zeta_\gamma$ corresponds exactly to

$$\zeta_\gamma = 4a T^2 (4/3 - \gamma)^2 \tau.$$

In this case, one has $n_{kC} \nu_k = \rho_k / \partial T$, with $\rho_k = a T^4$ the energy density of radiation, and $y_\gamma = 4/3$. 

From (50), one infers that the viscosity becomes ‘radiative’ if the matter component ($k = 2$) dominates the energy density, so that $\gamma_2 \approx \gamma$. The latter condition also implies $\delta T_2 \approx 0$, i.e. the matter is in thermal equilibrium at temperature $T$. This would appear at odds with the clear distinction between the matter temperature (Thomas 1930) and the equilibrium temperature (Eckart 1940) of the radiative fluid emphasized in (Weinberg 1971), where the radiative bulk viscosity was actually derived from this non-vanishing difference.

Zimdahl (1996b), however, points out a difference between the matter deviation $\delta T_m$ and the deviation of the matter temperature $T_m$ in (Weinberg 1971). This would imply a same difference between the matter temperatures $T_2$ and $T_m$. The difference becomes negligible if the system is matter-dominated, meaning that $\delta T_m$ and $\delta T_2$ both vanish under that condition.

Actually, the matter temperatures are a priori the same. The temperature deviations are defined through the difference in the respective energy densities. To first order this is

$$\rho(T_m) - \rho(T) \approx n c_V \delta T_m,$$

and similarly for the matter temperature $T_2$. In Weinberg (1971), this difference is identified with a radiative dissipative term,

$$n c_V \delta T_m = 4 a T^{(4/3 - \gamma)} \tau \Theta,$$

leading directly to the radiative bulk viscosity coefficient (52). While (54) is consistent with $\delta T_m \approx 0$ for a matter-dominated fluid ($\alpha k \ll n c_V T$), the very condition is out of context in the radiative-dominated universe. In Appendix B, we consider (54) in the light of the COF, and relate the bulk viscosity explicitly to the non-vanishing temperature difference $\delta T_2 = \delta T_m$.

### 5 REACTIVE FLUID

In relativistic fluids non-conservation of particle numbers is a common feature. It has long been recognized that the phenomenological effect of particle production is that of a viscous bulk pressure. Accordingly, matter creation in the early universe may be conveniently incorporated in a phenomenological imperfect fluid description. The effective reactive bulk viscosity is known as ‘creation pressure’, and is assumed to take the form

$$\Pi_k = -\alpha \psi / \Theta,$$

with an ad hoc kinetic coefficient $\alpha > 0$ (Calvão et al. 1992). The quantity $\psi$ is the particle production rate per unit proper volume, entering as a source term in the number balance equation

$$\dot{n} = -n \Theta + \psi.$$

As before, $\Theta$ is the cosmic expansion rate. Like the conventional bulk viscosity, the effective viscous pressure due to particle creation is negative upon expansion.

A corresponding extended relation between the equivalent reactive bulk viscosity and the production rates of a two-fluid model may be inferred from the non-equilibrium reactive entropy production,

$$\sigma_k = - \sum_k \psi_k \mu_k / T_k,$$

with $\psi_k$ the net proper volume production rate of species $k$. In the two-fluid formalism the chemical potential $\mu_k$ depends on $T_k$, and to first order in the temperature deviations the entropy production (57) is the sum of the ‘chemical’ entropy production,

$$\sigma^{(0)} = - \sum_k \psi_k \mu_k (T) / T,$$

and a cross term,

$$\sigma^{(1)} = \sum_k \psi_k c_V \delta T_k / T,$$

involving reactive as well as thermal non-equilibrium effects. This same cross term was previously obtained for the particular case of isentropic fluid components. By setting it equal to the effective dissipation $-\pi_{\text{react}} / T$, a ‘reactive’ bulk viscosity $\pi_{\text{react}}$ was identified (Zimdahl 1997). In evaluating this quantity, however, we will arrive at rather different results than in the paper cited. The differences will be discussed in Section 7.

The temperature deviations may be obtained from the time evolution of the temperatures. The reactive generalization of (5) is

$$\delta T_k = - (\gamma_k - 1) T_k \Theta + (E_k - \psi_k) / n_k c_V / \tau,$$

as follows in straightforward manner from the energy balance equation for the non-equilibrium components, together with the number balance,

$$\dot{n}_k = -n_k \Theta + \psi_k.$$

The quantity $E_k$ in (60) is the energy source density. Overall energy conservation is expressed by $\sum_k E_k = 0$. In line with the collision-time approach we adopt

$$E_k = e_k \psi_k - n_k c_V \delta T_k / \tau,$$

which leads to

$$\delta T_k = - \tau (\gamma_k - 1) \Theta - \tau \dot{T}.$$  

The (zero-order) equation for the equilibrium temperature,

$$\dot{T} = -(\gamma - 1) T \Theta - (n c_V)^{-1} \sum_k e_k \psi_k,$$

subsequently gives

$$\delta T_k = - \tau (\gamma_k - \gamma) \Theta + \tau (n c_V)^{-1} \sum_j e_j \psi_j.$$  

The temperature deviations are equally affected by the reactive energy exchange, so that the temperature difference between the fluid components depends only on the expansion rate. We will see presently that this leaves the conventional bulk viscosity insensitive to the reactive part of (65).

With (65) the entropy production (59) becomes

$$\sigma^{(1)} = - \tau (\gamma - 1) \left( h \psi_k - \sum_k h_k \psi_k \right) \Theta / T + \tau (n c_V)^{-1} \left[ \sum_k (e_k / T_k) \psi_k \right]^2,$$

with $h$ and $h_k$ the total and partial enthalpies per particle, respectively, and $\psi$ the net total particle production. Equating this source strength to the effective viscous dissipation, $\sigma^{(1)} = -\pi_{\text{react}} / T$, yields a reactive viscosity

$$\pi_{\text{react}} = \tau (\gamma - 1) (h \psi_k - \sum_k h_k \psi_k) + \mathcal{O} (\psi_k^2).$$

The explicit linear part, due to the expansion term of (65), is the evident cross term. It contains a creation pressure associated with total particle production as well as a contribution from particle exchange. The creation pressure, however, appears to have the wrong (positive) sign.
For reference, let us determine the non-equilibrium pressure \( P \), from which a linear relationship may be expected. The expansion term of the temperature differences (65) contributes only to the part
\[
\sum_k n_k c_{V_k}(\gamma_k - \gamma) \delta T_k = -\tau \sum_k n_k c_{V_k}(\gamma_k - \gamma)^2 \Theta, \tag{68}
\]
yielding the bulk viscosity proper, \( \Pi = -\xi \Theta \). The reactive exchange term, on the other hand, is just picked up by the sum
\[
\sum_k n_k c_{V_k} \delta T_k = \tau \sum_k c_k \psi_k, \tag{69}
\]
contributing a ‘chemical’ relaxation pressure which, in terms of the enthalpies per particle, can be expressed as
\[
\tau(\gamma - 1) \sum_k (h_k - T) \psi_k = -\tau(\gamma - 1) \left( h \psi - \sum_k h_k \psi_k \right) + \tau T \psi. \tag{70}
\]
With \( \psi = -\delta n/\tau \), we thus have from (68) and (70) a total non-equilibrium pressure
\[
\sum_k \left( n_k \delta T_k + T \delta n_k \right) = -\xi \Theta + \Pi. \tag{71}
\]
The left-hand side is the obvious reactive extension of the original viscous pressure model (3), with \( \delta n_k \) representing the deviations from detailed balance. The right-hand side features the effective reactive bulk viscosity
\[
\Pi = -\tau(\gamma - 1) \left( h \psi - \sum_k h_k \psi_k \right), \tag{72}
\]
which takes the same form as the linear part of (67), but now with the opposite (negative) sign. This sign issue is further investigated in the next section. With no other contributions to (72), the initial decomposition has provided an unambiguous separation of the non-equilibrium pressure into a ‘thermal’ and a ‘chemical’ part. The result (72) is discussed in Section 7.

6 ENTROPY PRODUCTION

To resolve the sign issue raised in the previous section, we consider the full (thermal and reactive) entropy production,
\[
\sigma = \sum_k (\dot{E}_k - \mu_k \psi_k)/T_k. \tag{73}
\]
For the source term (62), the thermal part is
\[
\sum_k \dot{E}_k/T_k = -\sum_k c_{V_k} \delta T_k/T \tag{74}
\]
\[+ \sum_k n_k c_{V_k} \tau^{-1} (\delta T_k/T)^2. \]
The first term on the right is the opposite of the cross term (59) in the reactive entropy production. Therefore, these cross terms cancel and do not emerge as explicit sources in the total entropy production,
\[
\sigma = \sum_k n_k c_{V_k} \tau^{-1} (\delta T_k/T)^2 - \sum_k \mu_k \psi_k/T. \tag{75}
\]
The last term here is the ‘chemical’ entropy production \( \sigma^{(0)} \), as given by (58). The first term on the right represents the entropy production in canonical form associated with the thermodynamical forces \( \delta T_k/T \). The temperature deviations (65) give this part of the entropy production as
\[
\sigma^{(1)} = \tau \sum_k n_k c_{V_k}(\gamma_k - \gamma)^2 \Theta^2 + \tau (n c_{V_k})^{-1} \left[ \sum_k (\epsilon_k/T) \right] \psi^2. \tag{76}
\]
In the first term on the right, we recognize the standard bulk-viscous entropy production, \( \xi \Theta^2/T = -\Pi \Theta/T \), while the second term is the quadratic rate term previously encountered in (66) as part of \( \sigma^{(1)} \). Therefore, we can eliminate it and write
\[
\sigma^{(1)} = -[\Pi - \tau(\gamma - 1)(h \psi - \sum_k h_k \psi_k)] \Theta + \sigma^{(1)}_r. \tag{77}
\]
The total entropy production (75) thus takes the form
\[
\sigma = -(\Pi + \Pi \Theta) + \sigma^{(1)}_r + \sigma^{(0)}_r, \tag{78}
\]
with \( \Pi \) the effective reactive viscosity as given by (72). This establishes the correct sign in (72).

With (78) the total entropy production is expressed as the sum of all effective viscous and reactive sources. The total viscous entropy production, \(-[\Pi + \Pi \Theta] \Theta/T \), evidently corresponds to the energy transfer accounted for by \( \dot{E}_k \). The reactive term in (76) occurs as the expression for the entropy production does not take account of the fitting condition,
\[
\sum_k n_k c_{V_k} \delta T_k = 0, \tag{79}
\]
imposed on the temperature deviations in order to split off the proper bulk-viscous pressure \( \Pi \). The condition discards the reactive part of the deviations (65), thereby removing the quadratic rate term from \( \sigma^{(1)} \). As this applies equally to (66), relation (78) stands unaltered. The COF (79) ensures a priori that only the expansion part of (65) is employed to determine the irreversible bulk-viscous pressure. The full expression could none the less be used without ambiguity in (68) in which, unlike (66) and (76), the COFs have been implemented.

We conclude that the effective reactive viscosity \( \Pi \), turns out inconsequential for the (total) entropy production. The viscous cross term in the reactive entropy production is just the opposite of the reactive cross term in the viscous entropy production. As a corollary we remark that the second law of thermodynamics does not pose any sign requirements on these cross terms.

7 REACTIVE BULK PRESSURE

7.1 Creation and exchange pressure

With due regard to the sign issue addressed in the previous section, the result (72) for the reactive bulk pressure is formally consistent with the linear cross term in the entropy production. We note that it is intimately connected to a non-vanishing composition change, quantified by
\[
n_k \dot{x}_k = \psi_k - x_k \psi, \tag{80}
\]
where \( x_k = n_k/n \) stands for the number concentration of component \( k \). From this relation we immediately have
\[
n \sum_k h_k \dot{x}_k = - \left( h \psi - \sum_k h_k \psi_k \right), \tag{81}
\]
giving the essential connection. The first term on the right-hand side gives rise to a negative reactive pressure,
\[
\Pi = -\tau(\gamma - 1) h \psi, \tag{82}
\]
associated with net particle production. This corresponds to the so-called creation pressure,
\[ \Pi_\epsilon = -\alpha \psi / \Theta, \]  
(83)
of the cosmological fluid (cf. 55). For adiabatic matter creation (Prigogine et al. 1989), in which the (total) entropy per particle remains constant, \( \alpha \) coincides with the particle enthalpy \( h \). More generally, (82) identifies the kinetic coefficient
\[ \alpha = \tau (\gamma - 1) \Theta h. \]  
(84)

To our knowledge, an explicit expression for the ‘creation coefficient’ of a particle-producing relativistic fluid, such as (84), has not previously been given. The factor preceding \( h \) in (84) is the ratio of two time-scales. Specifically, \( |T/T| = (\gamma - 1) / \Theta \) is the adiabatic cooling rate of the non-reacting fluid. Apparently particle production becomes adiabatic when \( \tau \) becomes equal to the cooling time (i.e. on the verge of reactive ‘freeze out’). Although \( \alpha \) contains the expansion rate, it is not in effect a dynamical quantity, since \( \Theta \) cancels in (83). However, taking \( \psi / \Theta \) as the effective thermodynamical force has been conventional in the literature, in line with \( \tau \sim \Theta^{-1} \).

For conserved total particle number, (72) reduces to an exchange pressure,
\[ \Pi_\epsilon = \tau (\gamma - 1) \sum k h_k \psi_k, \]  
(85)
proportional to the enthalpy defect of the reaction. The enthalpy defect, \( \Delta h = h_2 - h_1 \), takes a role similar to the total particle enthalpy \( h \) in (82),
\[ \Pi_\epsilon = -\tau (\gamma - 1)(\Delta h) \psi_1, \]  
(86)
with the forward (annihilation) rate \( \psi_1 \). As \( \Delta h \) and \( \psi_1 \) are both negative, the pressure is also negative.

The partial enthalpy is given by
\[ h_k / T = z_k K_k(z_k / K_k) / K_k(z_k), \]  
(87)
with modified Bessel functions of argument \( z_k \equiv m_k / T \), and includes the mass-energy of the particles. In the non-relativistic regime \( (z_k \gg 1) \) this relation becomes
\[ h_k / T = z_k / 2 + \mathcal{O}(z_k^{-1}), \]  
(88)
while for purely relativistic particles one has \( h_k = 4T \). At low energies \( \Delta h \) corresponds to the mass defect \( \Delta m \) of the reaction. A negative value of \( \Delta m \) constitutes an energy threshold below which the inverse reaction cannot occur, as in the decay of a massive particle into relativistic species. At sufficiently high energies, however, the exchange pressure will vanish in reactive equilibrium.

7.2 Cross-coupling

Transport phenomena of the same tensorial rank generally are subject to cross-coupling. According to Onsager’s relations, a symmetry then holds between the transport coefficients characterizing the cross-phenomena in the linear regime. In the case of the ‘scalar’ chemical phenomena and bulk viscosity the coupling conforms to an Onsager–Casimir reciprocity relation (De Groot & Mazur 1984), i.e. equal and opposite cross-coefficients. In Section 6, we have witnessed such cross terms occurring in the entropy production.

Antisymmetric cross-coupling of the phenomenological linear laws for bulk viscosity and reaction rate was formally demonstrated, both for molecular systems, with bulk viscosity due to the internal degrees of freedom (Kustova & Giordano 2011), and for reacting systems of relativistic point particles, where bulk viscosity is a relativistic phenomenon (Hermens et al. 1972). The coupled transport laws take the form
\[ \Pi = -\zeta \Theta - \xi \Delta \mu, \]  
(89)
\[ \Psi = -\chi \Delta \mu - \xi' \Theta. \]  
(90)
The volume rate \( \Psi \) characterizing the reaction is defined through \( \psi_k = v_k \Psi \), with ‘stoichiometric’ coefficient \( v_k = -1 \) for a reactant, and \( v_k = 1 \) for a reaction product. The ‘direct’ reactive driving force is the affinity \( \Delta \mu \equiv \sum_k v_k \mu_k \). The cross-coefficients obey the reciprocity relation \( \xi' = -\xi \). It is worth noting that this leads to cancellation of the cross terms in the entropy production,
\[ T \sigma = -\Psi \Delta \mu - \Pi \Theta = \chi (\Delta \mu)^2 + (\xi' + \xi) \Theta \Delta \mu + \xi \Theta^2. \]  
(91)

While in the foregoing we did encounter opposite cross terms in the entropy production, there is no cross-coupling through transport laws such as (89) and (90). The reactive bulk pressure (72) finds no dilatational counterpart in the particle production rate. In the underlying two-fluid model, the rates \( \psi_k \) are zero-order rates, corresponding to number density deviations \( \delta n_k = \delta n_k^{(0)} \), which occur as the thermal number densities \( n_k^{(0)} \) need not assume their reactive equilibrium (detailed balance) values. The zero-order rates are not expansion-driven, i.e. we have \( \xi' = 0 \). Onsager–Casimir reciprocity then demands that the cross-coefficient \( \xi' \) in the viscosity law should vanish as well. The reactive cross pressure (72), therefore, must not be identified with the reactive cross term in (89). Rather, it serves to express the intuitive viscous pressure model as in (71), consistent with entropy production.

The cross terms with coefficients \( \xi \) and \( \xi' \) stem from first-order number deviations which vanish in the case of the present two-fluid model. The implied condition \( \delta n_k^{(1)} = 0 \) is appropriate as a fitting condition for particle numbers conserved on the time-scale for thermal equilibrium. Non-vanishing number deviations then undergo a slower relaxation to reactive equilibrium. In the linear regime, the ‘straight’ production rates, \( \Gamma_k = \psi_k / n_k \), satisfy \( |\Gamma_k| = |(\delta n_k / n_k)| \tau^{-1} \), allowing \( |\Gamma_k| \sim \Theta \) to be ‘slow’ compared to the mean microscopic collision rate. The adiabatic character of the fluid components during the free time interval \( \tau \) is thus preserved.

7.3 Isentropic energy transfer

The reactive fluid pressure (72) differs in several respects from the result in the literature (Zimdahl 1997), where the energy source density \( \mathcal{E}_k = h_k \psi_k \) was assumed. This choice leads to temperature deviations which differ from (65) in their reactive part. The expansion part remaining the same, one still arrives at a reactive entropy production whose linear part is formally the same as in (66), and a reactive pressure of the form (67). However, the exchange term vanishes as the assumed energy transfer implies the rate relation \( \sum_k h_k \psi_k = 0 \). For the reactive pressure to be negative, it is necessary to retain the \( \mathcal{O}(\psi_k^2) \) terms.

Adopting the energy source term \( \mathcal{E}_k = h_k \psi_k \) is tantamount to assuming constant partial entropies per particle. From (73) it follows that entropy and particle production are then directly related by
\[ \sigma = \sum_k s_k \psi_k. \]  
(92)
With \( s_k \) depending on \( T_k \), one has
\[ s_k(T_k) = s_k(T) + c_{v_k} \delta T_k / T, \]  
(93)
which in (92) leads to the cross term (59). The validity of (59), however, does not rest on the condition of constant $s_k$. The condition restricts the focus on entropy production to reactive sources. It should be noted, however, that (92) represents neither the full reactive entropy production (57), nor the zero-order dissipation (58), but is given by

$$\sigma = \sigma_0 - \sum_k \psi_k c_{pk} \delta T_k / T = \sigma_0^{(0)} - \sum_k \psi_k (c_{pk} - c_{v_k}) \delta T_k / T.$$  \hfill (94)

We recall that, in contrast, the internal energy transfer (62) contributes a cross term involving $c_{vk}$ rather than $c_{Pk}$, opposite to $\sigma_0^{(1)}$. Isentropic energy (enthalpy) transfer produces the entropy surplus to keep $s_k$ constant. However, this entropy is not generated by collisional interactions between the fluid components. The enthalpy transfer does not correspond to the energy moment of the kinetic collision term. Therefore, an ‘outside’ particle source must be invoked. In this regard, one usually refers to creation of particles out of the gravitational field.

### 7.4 Non-collisional sources

While outside the kinetic realm, the source should couple to the particle distributions. In fact, linking it directly to the Juettner distribution (11),

$$H_k(x, p_k) = \lambda_k(x) f_k(x, p_k),$$  \hfill (95)

through a mean rate function $\lambda_k$, suffices to reproduce the enthalpy transfer considered in the previous subsection. Treating the effective source $H_k$ on the same footing as the usual collision terms, one obtains the particle number and energy rates

$$\psi_k = \int \rho \Delta H_k = \lambda_k n_k h_k^{-1},$$  \hfill (96)

$$\mathcal{E}_k = \int \rho \Delta (p_k^\mu U_\mu) H_k = \lambda_k n_k,$$  \hfill (97)

which directly combine into $\mathcal{E}_k = h_k \psi_k$. The rate coefficient follows as $\lambda_k = h_k \psi_k / n_k = h_k \Gamma_k$.

More generally, since the Juettner distribution is a two-parameter function, the source function could involve two independent rate coefficients,

$$\lambda_k = v_k + n_k (p_k^\mu U_\mu).$$  \hfill (98)

Given the general relation

$$\mathcal{E}_k = h_k \psi_k + n_k T_k \delta_k,$$  \hfill (99)

the coefficient $n_k$ may then be expected to be proportional to the entropy change $\delta_k$. Indeed, with (98), the combined moment integrals of $H_k = \lambda_k \delta_k$ yield

$$v_k = h_k (\Gamma_k + \delta_k), \quad n_k = -\delta_k.$$  \hfill (100)

Relation (98) is the two-fluid version of the ‘effective rate approximation’ introduced by Triginer, Zimdahl & Pavon (1996), for which we have here determined the rate coefficients.

Attributing the entropy cross rate in (94) to an effective isentropic particle source, we may proceed and write

$$\sum_k \psi_k (c_{pk} - c_{v_k}) \delta T_k / T = -\tau \sum_k (\gamma_k - \gamma) \psi_k + \mathcal{O}(\psi_k^2).$$  \hfill (101)

The linear cross term stems from the expansion part of the temperature deviations, which for $\sum_i h_i \psi_i = 0$ take the form

$$\delta T_k = -\tau (\gamma_k - \gamma) T \Theta + \tau \Theta (\psi_k / n_k c_k - \psi / n c).$$  \hfill (102)

The $\mathcal{O}(\psi_k^2)$ terms prove irrelevant as (102) yields the very same combination of quadratic rate terms in the thermal-viscous entropy production,

$$\sum_k n_k c_k T_k (\delta T_k / T)^2 = \zeta \Theta^2 / T - 2\tau \sum_k (\gamma_k - \gamma) \psi_k + \mathcal{O}(\psi_k^2).$$  \hfill (103)

Since (101) enters (94) with the opposite (negative) sign, the $\mathcal{O}(\psi_k^2)$ terms cancel, while the linear term emerges with a minus sign in the total entropy production,

$$\sigma = \zeta \Theta^2 / T - \tau \sum_k (\gamma_k - \gamma) \psi_k + \sigma_0^{(0)}.$$  \hfill (104)

The middle term on the right is now the opposite of how it would appear in (94). This sign reversal reflects the sign issue discussed in Section 6, from which the correct contribution here might have been anticipated. These essential details regarding the cross terms remain obscure by restricting the entropy production to reactive sources.

The solution (102) does satisfy condition (79), so that the thermal non-equilibrium pressure is given by

$$\sum_k n_k c_k (\gamma_k - \gamma) \delta T_k = -\tau \Theta + \tau T \sum_k (\gamma_k - \gamma) \psi_k.$$  \hfill (105)

Note that it is consistent with the entropy production (104). The cross contribution occurring here may be cast into the form

$$\tau T \sum_k (\gamma_k - \gamma) \psi_k = -\tau (\gamma - 1) \left[ h \psi - c_v T \sum_k \gamma_k \psi_k \right],$$  \hfill (106)

containing the creation pressure as obtained in the collision-time approach of Section 5.

### 8 NON-EQUILIBRIUM DECAY

The decay of a heavy particle species into relativistic particles (radiation) is a straightforward example of reactive non-equilibrium involving net particle production. To compare the associated creation pressure with the conventional bulk viscosity, we may conveniently employ expressions (82) and (51). This gives a relative magnitude

$$\frac{\Pi_k}{\Pi} = \frac{(\gamma - 1) \gamma \Gamma}{(\gamma_1 \gamma_2 (\gamma_2 - \gamma)) \Theta}.$$  \hfill (107)

For a fluid mixture consisting of a massive ($\gamma_1 = 5/3$) and a massless ($\gamma_2 = 4/3$) component, the value of $\gamma$ will be in the range $4/3 < \gamma < 5/3$.

It is evident that for $\gamma$ near the boundary values, i.e. if the fluid is either radiation- or matter-dominated, the ratio (107) may grow quite large. This is basically a consequence of the smallness of $\Pi$ in these regimes. For a production rate of the order of the expansion rate, the ratio is roughly given by the thermodynamic pre-factor which in the specified range assumes a minimum value $\approx 2.5$. On either side of the minimum the ratio increases rapidly, reaffirming the claim (Zimdahl 1997) that the reactive viscosity may by far dominate the conventional bulk viscosity.

Another example is the decay of an early cosmic vacuum state into radiation. This two-component model of a ‘deflational universe’ was presented in (Zimdahl & Pavon 2001). A homogeneous
and isotropic universe with particle production is described by the Einstein equations
\[ H = -4\pi G (\rho + p + \Pi_1), \quad H^2 = (8\pi G/3)\rho, \]  
(108)
with the gravitational constant \( G \), and Hubble expansion rate \( H = \Theta/3 \). The effects of particle production are accounted for by a reactive bulk-viscous pressure \( \Pi_1 \), for which the “adiabatic” creation pressure was adopted. With the usual equation of state, \( p = (\gamma - 1)\rho \), this may be written as \( \Pi_1 = -\gamma \rho \Gamma/3H \). In a reactive two-component fluid, however, the (total) entropy per particle is in general not constant, even if both components are isentropic. The production process, therefore, is essentially non-adiabatic.

To approach the problem afresh, we here apply the results of Section 7.4. The bulk viscosity and cross pressure of (105) both vanish for the overall relativistic equation of state \( (\gamma_1 = \gamma = 4/3) \).

A non-equilibrium pressure in the sense of (71) then leaves the (purely) reactive pressure \( \Pi_1 = -\tau_1 r \psi \) as a (near-adiabatic) particle production pressure, \( \Pi_1 \sim -\gamma (\gamma - 1)\rho \Gamma/3H \), on the reactive time-scale \( \tau_1 \sim \Theta^{-1} \). To be slightly more general, we adopt the form
\[ \Pi_1 = -f(\gamma)\rho \Gamma/3H, \]
(109)
with the coefficient function \( f(\gamma) \), which also includes the creation pressure (82). From (108) we then obtain the differential equation
\[ H = -(3\gamma/2)H^2[1 - \gamma^{-1}f(\gamma)\Gamma/3H]. \]
(110)
For fixed \( \gamma \) and an assumed rate \( \Gamma \propto H^2 \), the solution
\[ H(a) = H_0/(1 + a^{\gamma/2}) \]
(111)
is independent of the factor \( f(\gamma) \). The initial rate \( H_0 \equiv H(0) \) is twice the Hubble rate at the (arbitrarily set) value \( a = 1 \) of the scale factor at which \( \dot{a} = 0 \).

With \( \gamma = 4/3 \) the dependence of (111) on the scale factor is \( H \propto (1 + a^2)^{-1} \), as obtained in the case of adiabatic particle production. For \( a \ll 1 \) the expansion starts at a near-constant rate \( H_0 \), while eventually, for \( a \gg 1 \), it conforms to radiative scaling, \( H \propto a^{-2} \).

The corresponding energy density scales as \( \rho \propto (1 + a^2)^{-2} \), and can be split into ‘vacuum’ and ‘radiation’ components,
\[ \rho_1 \propto (1 + a^2)^{-3}, \quad \rho_2 \propto a^2(1 + a^2)^{-3}, \]
(112)
respectively. The vacuum density remains finite for \( a \to 0 \), while the radiation density vanishes. For \( a \gg 1 \) the densities decay as \( \rho_1 \propto a^{-6} \) and \( \rho_2 \propto a^{-4} \), the latter characteristic for relativistic species. The effective equation of state of the vacuum is
\[ P_v = \Pi_1 \equiv p_1 + \Pi_1 = -[(1 - a^2)/(1 + a^2)]\rho_1, \]
(113)
which becomes \( P_v \to -\rho_1 \) for \( a \ll 1 \). For further details concerning the two-fluid thermodynamics we refer to Zimdahl & Pavon (2001).

9 CONCLUDING REMARKS

In this paper, a heuristic two-fluid model of cosmological bulk viscosity, in which small deviations from thermal equilibrium account for the viscous pressure, is substantiated at the kinetic level, and further explored in relation to the radiative fluid model. We opted for a representative formulation in terms of specific heats and adiabatic exponents, quantities which have a direct connection to the degrees of freedom of the constituents of the system. In this manner, it becomes apparent that, while in classical (molecular) systems the bulk viscosity is connected to internal degrees of freedom, the bulk viscosity of the expanding and cooling binary mixture or radiative fluid can be understood in terms of the effective degrees of freedom represented by the different specific heats and adiabatic exponents, leading to different internal cooling rates. The microphysical picture was made precise in terms of the mean internal energy transfer due to the temperature differences induced.

Extending the original two-fluid model to allow for deviations from detailed balance furnishes an effective reactive ‘cross’ viscosity which includes, in particular, the creation pressure from net particle production. The two-fluid model, however, relying on a reactive disequilibrium which does not extend beyond the relativistic version of the chemical ‘law of mass action’, is lacking a counterpart coupling the effective viscous cross term to the reaction rate in the linear transport relations. Rather than holding its own as a dissipative mechanism, the reactive cross pressure remains inconsequential for the total entropy production, as the thermal and reactive cross terms cancel in the entropy source strength. A net non-vanishing cross-contribution to the entropy production may be modelled by an effective isentropic particle source. In both scenarios the reactive viscosity is linear in the production rates, and involves the same creation pressure. As a representative case for each, the non-equilibrium decay of heavy fluid particles into relativistic ones, and a cosmological vacuum ‘decaying’ into radiation, were briefly reviewed.

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APPENDIX A

In the monograph by De Groot et al. (1980), the first-order approximation to the bulk viscosity coefficient of a non-reactive binary gas mixture is given as

\[ \zeta = \frac{k_B T}{c \sigma(T)} x_i x_j \left( \frac{\alpha'_1}{A_{12}^1} \right) \]  
(A1)

Here, \( \sigma(T) \) is a characteristic cross-section for collisions between the species, whose number fractions are denoted by \( x_i = n_i / n \). The quantities \( \alpha'_1 \) are certain integral moments of the Jüttner distribution, given by

\[ \alpha'_1 = (\gamma_1 - \gamma)/(\gamma_2 - 1). \]  
(A2)

The denominator contains the so-called ‘collision bracket’

\[ A_{12}^{11} = [c n^2 \sigma(T)]^{-1} \times \int \, d\omega_1 d\omega_2 d\omega_1' d\omega_2' f_1^{(0)}(b)^{(0)} f_2^{(0)} W_{12} u_1(u_2 - u_2'), \]  
(A3)

whose upper indices indicate the (first) power of the microscopic energy variables in the integrand, while the lower indices label the colliding species. It has been made dimensionless by extracting the characteristic volume rate \( c n^2 \sigma(T) \). Therefore, although at least in the radiative case (A1) might suggest \( 1/c n^2 \sigma(T) = \tau \), the mean collision time, the cross-section \( \sigma(T) \) actually cancels out.

In view of microscopic energy conservation, the bracket (A3) may be rewritten as

\[ A_{12}^{11} = -[c n^2 \sigma(T)]^{-1} \times \int \, d\omega_1 d\omega_2 d\omega_1' d\omega_2' f_1^{(0)} f_2^{(0)} W_{12} u_1(u_2 - u_2'). \]  
(A4)

This contains the rate integral encountered in (32) and establishes the connection with the mean free time as

\[ c \sigma(T) A_{12}^{11} = -(x_1 c v_1 x_2 c v_2 / n c v) \tau^{-1}. \]  
(A5)

Inserting this into (A1) we have

\[ \zeta = -n c v \, T \, (\gamma_1 - \gamma)(\gamma_2 - \gamma), \]  
(A6)

which is the bulk viscosity coefficient as expressed by (51).

APPENDIX B

Observing first that the right-hand side of (54) corresponds to the deviation of the radiative component,

\[ n c v \delta T_m = -n_1 c v_1 \delta T_1, \]  
(B1)

we here note a difference with the COF (4), according to which one should have

\[ n c v \delta T_2 = -n_1 c v_1 (\delta T_1 - \delta T_2). \]  
(B2)

The term \( \delta T_2 \) on the right-hand side is lacking in (B1) because (54) was inferred from the non-equilibrium energy-momentum tensor of radiation hydrodynamics, which is due only to the radiation component. It is at this point that an apparent difference between the two matter temperatures enters, with respective deviations related by

\[ \delta T_m = \frac{n_2 c v_2}{n c v} \delta T_2. \]  
(B3)

For the matter-dominated system \( (n_1 c v_1 \ll n_2 c v_2 \simeq n c v) \) this implies

\[ \delta T_m \simeq \delta T_2 \ll \delta T_1, \]  
(B4)

i.e. the two matter temperatures and their deviations would become the same. Witness (B4), both deviations then vanish in the sense that they become negligible compared to \( \delta T_1 \). However, the two matter temperatures are a priori the same, and different from the Eckart temperature of the radiative fluid.

The distinction between matter and fluid equilibrium temperatures was preserved in Schweizer (1982), where they are denoted as ‘initial’ matter temperature \( T_i \) and ‘final’ observed temperature \( T_f \), respectively. They are related by the fitting condition \( a T_i^4 + \Pi = a T_f^4 + \Pi_f \). The quantity

\[ \Pi_i = (\partial \rho / \partial T)_\mu(T_f - T_i) \]  
(B5)

corresponds to (53), and yields the bulk viscosity\n
\[ \Pi = (4/3 - \gamma) \Pi_i. \]  
(B6)

In the notation of our paper, this is

\[ \Pi = -(\gamma_1 - \gamma)n c v \delta T_2, \]  
(B7)

with \( \gamma_1 = 4/3 \). This expression corresponds exactly to (51). Alternatively, in terms of

\[ \Pi_i = (\partial \rho_i / \partial T)_\mu(T_f - T_i) \]  
(B8)

the bulk viscosity becomes\n
\[ \Pi = (4/3 - \gamma_2) \Pi_i, \]  
(B9)

corresponding to\n
\[ \Pi = -(\gamma_1 - \gamma_2)n_2 c v_2 \delta T_2. \]  
(B10)

If (49) is applied to this latter expression, one recovers the bulk viscosity coefficient in the form (50).

Both (B7) and (B10) relate the bulk viscosity directly to a non-vanishing temperature difference \( \delta T_2 \), \( \delta T_m \). The final form of the radiative bulk viscosity is due to the identification (54) or (B1), made in the context of radiative transport theory, where all dissipative effects are due to the radiation. For instance, using (B1) in (B7), one has\n
\[ \Pi = (\gamma_1 - \gamma)n_1 c v_1 \delta T_1, \]  
(B11)

which corresponds to a form of Udey & Israel (1982), also leading to (52).

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