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# The method of regions and next-to-soft corrections in Drell–Yan production

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ABSTRACT

We perform a case study of the behaviour of gluon radiation beyond the soft approximation, using as an example the Drell–Yan production cross section at NNLO. We draw a careful distinction between the eikonal expansion, which is in powers of the soft gluon energies, and the expansion in powers of the threshold variable $1 - z$, which involves important hard-collinear effects. Focusing on the contribution to the NNLO Drell–Yan K-factor arising from real–virtual interference, we use the method of regions to classify all relevant contributions up to next-to-leading power in the threshold expansion. With this method, we reproduce the exact two-loop result to the required accuracy, including $z$-independent non-logarithmic contributions, and we precisely identify the origin of the soft-collinear interference which breaks simple soft-gluon factorisation at next-to-eikonal level. Our results pave the way for the development of a general factorisation formula for next-to-leading-power threshold logarithms, and clarify the nature of loop corrections to a set of recently proposed next-to-soft theorems.



## 1. Introduction

It is well known that singularities arise in perturbative scattering amplitudes due to low-energy (soft) emission of massless gauge bosons, and to collinear splittings of massless particles. These *infrared* (IR) singularities cancel for suitably defined inclusive cross sections, once real and virtual diagrams are combined [1–3]; more generally, they are known to factorise at the level of scattering amplitudes [4], and their general structure in the case of multi-parton non-abelian gauge amplitudes has been the subject of much recent activity (for a recent summary, see for example [5, 6], and the references therein).

Even for finite, infrared-safe cross sections, residual contributions persist after the cancellation of singularities, taking the form of potentially large kinematic logarithms at all orders in perturbation theory, which may need to be resummed. In the generic case of multi-scale processes, these logarithms can have a variety of arguments, such as transverse momenta which vanish at Born level, or event shape variables which vanish in the two-jet limit. In this note, we will concentrate on *threshold* logarithms, which arise in inclusive cross sections when real radiation is forced to be soft or collinear by the properties of the selected final state. Examples are: electroweak annihilation processes, such as Drell–Yan production or Higgs production via gluon fusion, where the threshold variables are $z = Q^2/\hat{s}$ and $z = M_H^2/\hat{s}$, respectively, with $\hat{s}$ the partonic center-of-mass energy; Deep Inelastic Scattering (DIS), where the threshold variable is the partonic version of Bjorken $x$; and $t\bar{t}$ production, where the threshold variable is $z = 4m_t^2/\hat{s}$. In all of these cases the cancellation of infrared singularities leaves behind logarithms of the general form $\alpha_s^n (1-z)^m \log^p(1-z)$, with $0 \le p \le 2n-1$, and $m \ge -1$.

Contributions with $m = -1$, which we describe as *leading power* (LP) threshold logarithms, have been extensively studied, and successfully resummed to very high logarithmic accuracy using a variety of formalisms [7–12]. It is however known that also logarithms accompanied by subleading powers of the threshold variable, most notably those with $m = 0$, which we call *next-to-leading power* (NLP) threshold logarithms, can give numerically significant contributions [13]. In recent years, a number of studies have ap-

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peared [14–27] developing our understanding of certain classes of NLP threshold logarithms. A full-fledged resummation formalism for NLP logarithms is however still not available.

An important class of NLP threshold logarithms, which is the best studied so far, is generated by contributions to scattering amplitudes that arise from the emission of soft gluons, at next-to-leading power in the soft gluon energy. We call these contributions *next-to-eikonal* (NE), or *next-to-soft*. It has been known for many years, at least in the abelian case [28–30], that next-to-soft emissions share many of the universal features that characterise leading-power soft radiation, which is described by the eikonal approximation. This understanding, to some extent, has been generalised to non-abelian theories in [15,21,31], where it was shown that the eikonal approximation can be generalised to take into account next-to-soft effects, while preserving many of the nice universality and factorisation properties which are present at leading power. Ultimately, however, in order to organise all NLP threshold logarithms, one must include also the effects of collinear emissions. The importance of collinear emissions is evident in the case of processes with final-state jets, for example DIS, where some threshold logarithms are directly associated with the mass of the current jet. It is crucial to realise, however, that collinear emissions will also contribute to NLP logarithms for processes, like Drell–Yan or Higgs production, where real radiation is forced to be soft by phase space constraints. In such cases the soft expansion breaks down because singularities arising from virtual hard collinear gluons interfere with the soft approximation. This issue was first tackled, in the abelian case, in Ref. [30], and similar effects were noted in Refs. [15,16]. The analysis of the present paper will precisely identify the origin of these interfering contributions in an example involving real–virtual interference for the Drell–Yan cross section at NNLO.

Quite interestingly, next-to-soft corrections to scattering amplitudes have been the focus of intense recent research also from a more formal point of view. It is well known that leading-power soft radiation can be studied with eikonal methods both in gauge theories and in gravity [32–36]. Recently, Ref. [37] conjectured that next-to-soft behaviour at tree-level is universal in gravity, based on the observation that the known universal soft behaviour [32] can be obtained via a Ward identity associated with the Bondi–Metzner–Sachs (BMS) symmetry at past and future null infinity [38]. Ref. [39] generalised this to Yang–Mills theory, and there have been a number of follow-up studies [40–52]. In particular, Ref. [53] pointed out the relationship between this body of work and the more phenomenological results of Refs. [15,21,28–30]. A key point of contention in the current literature is whether next-to-soft theorems receive corrections at loop level. As Ref. [52] makes clear, this is related to the sequential order in which the expansions in soft momentum and the dimensional regularisation parameter $\epsilon$ (in $4-2\epsilon$ dimensions) are carried out. The authors of Ref. [52] state that the soft expansion should be carried out first (with $\epsilon$ kept non-zero). Loop corrections were further explored in Refs. [45,48,51], with Ref. [48] advocating that the soft expansion be carried out *after* the $\epsilon$-expansion, which would correspond to how complete amplitudes are usually calculated.

Our aim in this Letter is to perform a case study of NLP threshold logarithms at loop level in Drell–Yan production, including in particular those that originate from next-to-soft corrections to the corresponding scattering amplitude. There are a number of motivations for doing so. First, our ultimate aim (building on the work of Refs. [15,21]), is to develop a fully general resummation prescription for NLP threshold logarithms. Our investigation here will provide crucial data in this regard, although we postpone a detailed discussion of factorisation at NLP accuracy to a subsequent paper [54]. Secondly, by explicitly characterising contributions in Drell–Yan according to their soft and/or collinear behaviour, we will be able to concretely examine the issue of loop corrections to next-to-soft theorems, including the interplay between the dimensional regularisation and soft expansions. We will verify explicitly that performing the $\epsilon$ expansion *before* the soft expansion correctly reproduces known results that are sensitive to this ordering. The reason is, as might be expected, the fact that there are collinear singularities arising from virtual exchanges of hard collinear gluons, which are not correctly taken into account if one performs a soft expansion before loop integrations.

More specifically, we will examine the $K$-factor for Drell–Yan production at NNLO, concentrating on those terms which arise from having one real and one virtual gluon emission, which are ideally suited to examine the questions posed above. Indeed, logarithms arising from double real emission were already understood from an effective next-to-soft approach in Ref. [21], using the fact that, for electroweak annihilation processes, real radiation near threshold is forced to be soft. Double virtual corrections, on the other hand, have a trivial dependence on the threshold variable $z$, and do not influence the present considerations. In this Letter, we will further concentrate on terms proportional to the colour prefactor $C_F^2$, which are the same as those that would be obtained in an abelian theory, as considered in the earlier work of [28–30]. This is sufficient to illustrate our main points, and a complete analysis will be given in forthcoming work [54]. Our task here will be to perform a detailed momentum-space analysis of the selected contributions, and trace the origin of all NLP threshold logarithms to hard, soft, or collinear configurations. To this end, we will use the *method of regions*, as developed in [55]. A similar analysis has recently been performed in the case of Higgs production via gluon fusion in the large top mass limit, to an impressive $\mathrm{N^3LO}$ accuracy [56], as part of the complete calculation of the soft and virtual contributions to the cross section at this order. In that case, the method of regions was used as an alternative technique to check the validity of the threshold expansion, and as a method to investigate the convergence properties of the expansion itself.[1] Our goal is different, namely to analyse the factorisation properties of various diagrammatic contributions to the cross section. As a consequence, in Ref. [56] the method of regions was applied after reduction to scalar master integrals, while here we apply it to complete diagrams, thus making it easier to trace various sources of next-to-soft behaviour in our chosen (Feynman) gauge. Furthermore, for the specific NNLO contributions we focus on, we will be able to show how the method of regions gives an exact account of threshold contributions also at next-to-leading power.

Our results will prove useful in the development of a factorisation formula for NLP threshold logarithms, which will generalise the well-known soft-collinear factorisation formula at leading power (see, for example, Ref. [58] for a review of the latter); work in this direction is in progress [54].[2] Interestingly, we find that our analysis with the method of regions is able to reproduce correctly all NLP threshold corrections, including terms with $m = 0$ and $p = 0$, which have no logarithms at all, and would correspond to terms of order $1/N$ in a Mellin-space analysis, with no $\log N$ enhancements. We think this gives evidence for the existence of a systematic organisation of threshold contributions to cross sections, order by order in $m$. The question then arises of how many terms a fully resummed approach would be able to control, given the progress already made in this regard by the physical evo-

[1] For a discussion of the limits of the threshold expansion in this process, see Ref. [57].

[2] Progress can also be made using effective field theory techniques [59]. One of the authors (CDW) is very grateful to Duff Neill for correspondence on this point, including sharing an early draft of Ref. [59].

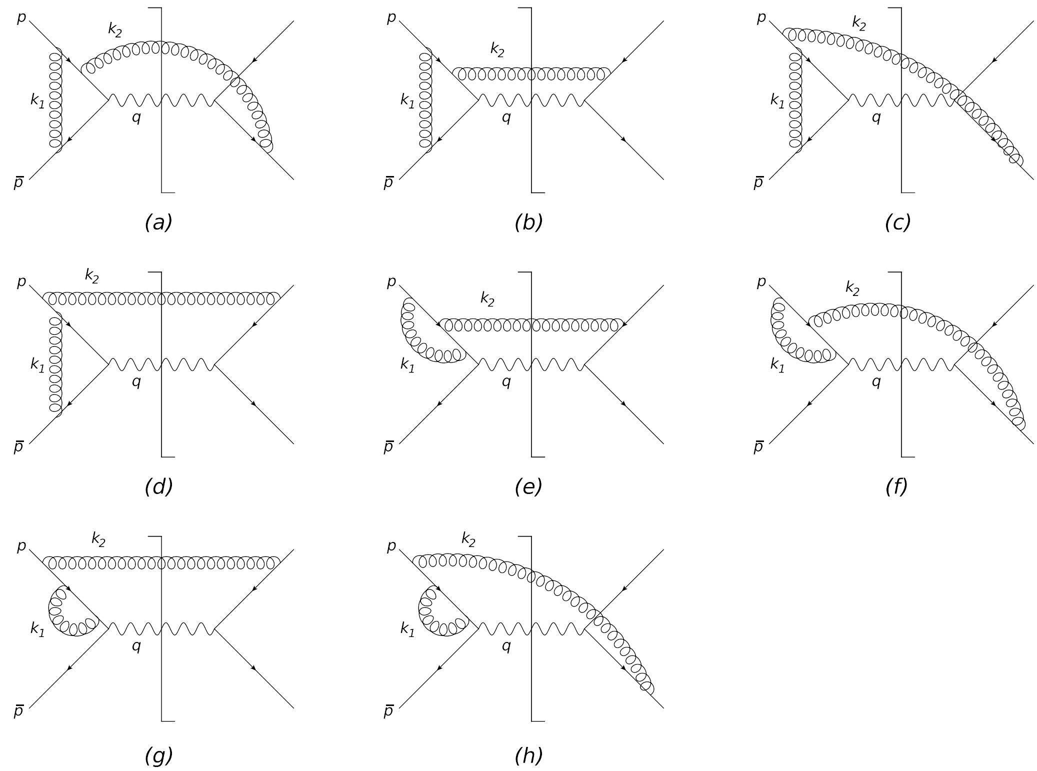


**Fig. 1.** Abelian-like cut diagrams contributing to the Drell–Yan cross section at NNLO, involving one real and one virtual gluon. Diagrams obtained by interchanging $p \leftrightarrow \bar{p}$ and/or complex conjugation are not shown.

lution kernel approach of Refs. [18–20,22,27]. The arguments of Refs. [28–30] (see also Refs. [15,49,50]) indicate that terms which are suppressed by more than one power of the threshold variable (NNLP logarithms) are not fully fixed by gauge invariance, and thus are not expected to be resummable in a factorisation-based approach. On the other hand, our paper provides evidence that, in principle, all threshold logarithms at NLP can be organised to all orders in perturbation theory.

The structure of the Letter is as follows. In Section 2 we review necessary information about Drell–Yan production. In Section 3 we apply the method of regions to classify all abelian-like terms in the real–virtual contribution to the NNLO Drell–Yan $K$-factor. In Section 4, we interpret our results in light of Refs. [45,48,51,52], focusing in particular on the required ordering of the soft and $\epsilon$-expansions. We discuss our results and conclude in Section 5.

## 2. Real–virtual interference in Drell–Yan at NNLO

As discussed in the introduction, we consider Drell–Yan production of a virtual vector boson [60], which at leading order proceeds via the process

$$q(p) + \bar{q}(\bar{p}) \to V^*(Q), \tag{1}$$

where we do not display flavor indices, so that the vector boson $V$ could be a photon, a $Z$ or a $W^\pm$ boson. The threshold variable in this case is $z = Q^2/\hat{s}$, with $Q = p + \bar{p}$. The Drell–Yan $K$-factor at $\mathcal{O}(\alpha_s^n)$ is defined by

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz}, \tag{2}$$

where $\sigma^{(n)}$ is the total cross-section including terms up to $\mathcal{O}(\alpha_s^n)$. The cross section has been calculated exactly up to $n = 2$ in Refs. [61–69], which allows scrutiny of threshold logarithms both at LP and at NLP accuracy. The relevant contributions take the form

$$\mathrm{LP}:\ \alpha_s^n \left[\frac{\log^m(1-z)}{1-z}\right]_+ \equiv \alpha_s^n \mathcal{D}_m(z); \qquad \mathrm{NLP}:\ \alpha_s^n \log^m(1-z),$$
$$0 \le m \le 2n-1. \tag{3}$$

Leading power logarithms, supplemented by terms proportional to $\delta(1-z)$, form the so-called 'soft + virtual' contribution, which has been recently computed to N$^3$LO in Ref. [70]. We see that NLP contributions show up as pure logarithms, integrably singular in the threshold region $z \to 1$. At NLO, such terms arise only through real emission contributions: these were analysed in Ref. [21], together with the double real emission contributions at NNLO, and shown to be reproducible from an effective next-to-eikonal approach. This is due to a lack of contamination in tree-level DY production from hard collinear effects, which is not true in more generic processes, or at loop level: beyond NLO, also for Drell–Yan kinematics, one must then differentiate between the expansion in emitted (soft) gluon momentum, and the *threshold* expansion which also includes collinear effects.

Following on from Ref. [15], the next milestone in understanding the structure of NLP threshold logs is to examine one-loop graphs at NNLO, involving one real and one virtual gluon. These were not considered explicitly in Refs. [15,21], due to the fact that hard collinear singularities were not accounted for. The interplay between (next-to) soft and collinear effects has been discussed at length in Ref. [30], at the price of neglecting discussion of double counting issues between gluon emissions that are simultaneously soft and collinear. In order to clarify these issues, we concentrate here on the abelian-like NNLO real–virtual interference contribution to the $K$-factor, corresponding to the (cut) Feynman diagrams shown in Fig. 1. As an example, to fix our notation, we note that diagram (a) contributes

$$F_a(z) = g_s^4 \int [dk_1][dk_2](2\pi)\delta(k_2^2)\theta(k_2^0)\delta\left(\frac{\omega}{2} - k_2^0\right)\frac{1}{k_1^2}$$

$$\times \mathrm{Tr}\left[\not{p}\gamma^\alpha \frac{\not{k}_2-\bar{\not{p}}}{(k_2-\bar{p})^2}\gamma^\mu \bar{\not{p}}\gamma^\rho \frac{\not{k}_1-\bar{\not{p}}}{(k_1-\bar{p})^2}\gamma_\alpha \right.$$
$$\left. \times \frac{\not{p}+\not{k}_1-\not{k}_2}{(p+k_1-k_2)^2}\gamma_\mu \frac{\not{p}+\not{k}_1}{(p+k_1)^2}\gamma_\rho\right], \tag{4}$$

where $\omega=\sqrt{\hat{s}}(1-z)$, and we have defined the integration measure

$$\int [dk_i] \equiv \frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\mu_{\overline{\rm MS}}^{2\epsilon}\int \frac{d^dk_i}{(2\pi)^d}, \tag{5}$$

with $d=4-2\epsilon$ and $\mu_{\overline{\rm MS}}=\mu e^{-\gamma_E/2}(4\pi)^{1/2}$. One must then add to Eq. (4), and to all other contributions from the diagrams depicted in Fig. 1, similar terms obtained by interchanging $p\leftrightarrow\bar{p}$ and by complex conjugation. Colour matrices have been neglected, given that we are focusing on the abelian-like part of the $K$-factor, which appears with an overall factor of $C_F^2$.

In order to reproduce the NLP threshold logarithms in the $K$-factor, one must now classify all next-to-soft and collinear contributions. This is the subject of the following section.

## 3. Method of regions analysis

The method of regions is a systematic procedure for expanding loop integrals about their singular regions [55], such that collinear and soft behaviours are disentangled. Whilst a general proof of its validity is not yet available (see for example [71,72]), it has been tested in a number of highly non-trivial examples, most recently in Higgs production via gluon fusion at N$^3$LO [56], a process closely related to Drell–Yan production. In what follows, however, we will apply the method of regions to identify all sources of NLP threshold logarithms, including next-to-soft contributions as well as collinear ones, going beyond the purely soft or collinear limits considered in Ref. [56].

We begin by defining the directions collinear to the incoming quark and antiquark by the light-like vectors $n_+$ and $n_-$, defined such that $n_+^2=n_-^2=0$ and $n_-\cdot n_+=2$. The physical momenta (in the centre of mass frame) are related to these vectors via

$$p^\mu=\frac{1}{2}(n_-p)n_+^\mu=\frac{\sqrt{\hat{s}}}{2}n_+^\mu,$$
$$\bar{p}^\mu=\frac{1}{2}(n_+\bar{p})n_-^\mu=\frac{\sqrt{\hat{s}}}{2}n_-^\mu, \tag{6}$$

where we introduced the short-hand notation $(n_\pm l)\equiv n_\pm^\mu l_\mu$. A generic momentum $l$ may then be decomposed into light-cone and transverse components according to

$$l^\mu=\frac{1}{2}(n_-l)n_+^\mu+\frac{1}{2}(n_+l)n_-^\mu+l_\perp^\mu. \tag{7}$$

We now distinguish different regions for the momentum $l^\mu$ by the different scalings of its components, defined according to a bookkeeping parameter $\lambda\sim\sqrt{E_{\rm soft}/E}$, where $E_{\rm soft}\sim\sqrt{\hat{s}}(1-z)$, and $E\sim\sqrt{\hat{s}}$ is the hard scale. More specifically, writing $l^\mu=\{l_+,l_-,l_\perp\}$, the relevant regions are defined as follows

$$\text{Hard:}\quad l\sim\sqrt{\hat{s}}\,(1,1,1)\,;$$
$$\text{Soft:}\quad l\sim\sqrt{\hat{s}}\left(\lambda^2,\lambda^2,\lambda^2\right);$$
$$\text{Collinear:}\quad l\sim\sqrt{\hat{s}}\left(1,\lambda,\lambda^2\right);$$
$$\text{Anticollinear:}\quad l\sim\sqrt{\hat{s}}\left(\lambda^2,\lambda,1\right). \tag{8}$$

In any given process, the external momenta are fixed. Here, for example, $p$ ($\bar{p}$) is by definition collinear (anticollinear), while $k_2$ is (next-to) soft. Different contributions to the $K$ factor then arise from various regions of the loop momentum $k_1$.

Our next task is to expand the propagators in Eq. (4) in the different regions. Focusing, as an example, on those associated with the $p$ leg, the most complicated case is

$$\frac{\not{p}+\not{k}_1-\not{k}_2}{(p+k_1-k_2)^2}. \tag{9}$$

Expanding to the second non-trivial order in $\lambda$ in the relevant momentum regions described above, this propagator becomes

Hard:

$$\frac{\sqrt{\hat{s}}\frac{\not{n}_+}{2}+\not{k}_1}{k_1^2+(n_+k_1)\sqrt{\hat{s}}}+\left[-\frac{\not{k}_2}{k_1^2+(n_+k_1)\sqrt{\hat{s}}}\right.$$
$$\left.+\frac{\left((n_+k_2)\sqrt{\hat{s}}+2(k_1k_2)\right)\left(\sqrt{\hat{s}}\frac{\not{n}_+}{2}+\not{k}_1\right)}{(k_1^2+(n_+k_1)\sqrt{\hat{s}})^2}\right]+\mathcal{O}(\lambda^4);$$

Collinear:

$$\frac{\left(\sqrt{\hat{s}}+(n_-k_1)\right)\frac{\not{n}_+}{2}}{k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)}$$
$$+\left[\frac{\not{k}_{1\perp}}{k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)}\right.$$
$$\left.+\frac{2(k_{1\perp}k_{2\perp})\left(\sqrt{\hat{s}}+(n_-k_1)\right)\frac{\not{n}_+}{2}}{\left(k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)\right)^2}\right]$$
$$+\left[\frac{(n_+k_1)\frac{\not{n}_-}{2}-\not{k}_2}{k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)}\right.$$
$$+\frac{2(k_{1\perp}k_{2\perp})\not{k}_{1\perp}+\left((n_+k_1)(n_-k_2)-k_2^2\right)\left(\sqrt{\hat{s}}+(n_-k_1)\right)\frac{\not{n}_+}{2}}{\left(k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)\right)^2}$$
$$\left.+\frac{4(k_{1\perp}k_{2\perp})^2\left(\sqrt{\hat{s}}+(n_-k_1)\right)\frac{\not{n}_+}{2}}{\left(k_1^2+(n_+k_1)\sqrt{\hat{s}}-(n_+k_2)\left(\sqrt{\hat{s}}+(n_-k_1)\right)\right)^3}\right]+\mathcal{O}(\lambda);$$

Anticollinear:

$$\frac{1}{\sqrt{\hat{s}}}\frac{\not{n}_-}{2}+\frac{1}{(n_+k_1)}\frac{\not{n}_+}{2}+\left[\frac{1}{(n_+k_1)}\frac{\not{k}_{1\perp}}{\sqrt{\hat{s}}}\right.$$
$$\left.+\left(-\frac{k_{1\perp}^2}{(n_+k_1)^2\sqrt{\hat{s}}}+\frac{(n_+k_2)}{(n_+k_1)^2}+\frac{(n_-k_2)}{(n_+k_1)\sqrt{\hat{s}}}\right)\frac{\not{n}_+}{2}\right]$$
$$+\left[\left(-\frac{k_1^2}{(n_+k_1)\hat{s}}+\frac{(n_+k_2)}{(n_+k_1)\sqrt{\hat{s}}}+\frac{(n_-k_2)}{\hat{s}}\right)\frac{\not{n}_-}{2}-\frac{\not{k}_2}{(n_+k_1)\sqrt{\hat{s}}}\right]$$
$$+\mathcal{O}(\lambda^3);$$

Soft:

$$\frac{1}{(n_+k_1)-(n_+k_2)}\frac{\not{n}_+}{2}+\left[\frac{1}{(n_+k_1)-(n_+k_2)}\frac{\not{k}_1-\not{k}_2}{\sqrt{\hat{s}}}\right.$$
$$\left.-\frac{1}{\left((n_+k_1)-(n_+k_2)\right)^2}\frac{(k_1-k_2)^2}{\sqrt{\hat{s}}}\frac{\not{n}_+}{2}\right]+\mathcal{O}(\lambda^2). \tag{10}$$

In order to clarify the power counting in Eq. (10), it may be useful to note that the expansion of the propagator given in Eq. (9)

in powers of $\lambda$ starts at $\mathcal{O}(\lambda^0)$ in the hard and in the anticollinear regions, while it starts at $\mathcal{O}(\lambda^{-2})$ in the collinear and soft regions. Moreover, the Taylor expansion is in powers of $\lambda^2$ in the hard and soft regions, while it is in powers of $\lambda$ in the collinear and anticollinear regions. In Eq. (10), different orders in $\lambda$ are enclosed within square brackets.

Similar expressions can be obtained for all other propagators, not all of which are independent (for example, the anticollinear region for the $p$ leg can be obtained from the collinear region on the $\bar{p}$ leg by relabelling $p \leftrightarrow \bar{p}$). After substituting all expanded propagators into Eq. (4), the integrals may be carried out in dimensional regularisation using standard techniques. One may then repeat this procedure for the remaining diagrams in Fig. 1. When this is done, it is useful to present results for two distinct sums of diagrams: those involving both quark legs, $p$ and $\bar{p}$, given in graphs (a)–(d) in Fig. 1, and those involving a single leg, given in graphs (e)–(h). Complete results to NLP accuracy are given below: for each region r, we write the $K$ factor as $K_{\rm r}(z) = K_{\rm E,r}(z) + K_{\rm NE,r}(z)$, separating the result into two parts, corresponding to leading and next-to-leading order in the eikonal (soft) expansion of the amplitude in powers of $k_2$, before phase space integration. The NLP logarithms in the eikonal contributions $K^{(2)}_{\rm E,r}(z)$ arise exclusively from corrections to the eikonal phase space, as discussed in Ref. [21]. Next-to-eikonal contributions $K^{(2)}_{\rm NE,r}$, on the other hand, consist of genuine corrections arising at the amplitude level.

### 3.1. Hard region

After integration over the loop momentum $k_1$, and the real radiation phase space for momentum $k_2$, we find that there is no contribution at LP or NLP arising from the hard region from diagrams (e)–(h). Diagrams (a)–(d), on the other hand, combine to give

$$\begin{aligned}
K^{(2)}_{\rm E,h}(z) = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[ & \frac{2\mathcal{D}_0(z)}{\epsilon^3} + \frac{-4 + 3\mathcal{D}_0(z) - 4\mathcal{D}_1(z)}{\epsilon^2} \\
& + \frac{-6 + 8\mathcal{D}_0(z) - 6\mathcal{D}_1(z) + 4\mathcal{D}_2(z) + 8\log(1-z)}{\epsilon} \\
& - 16 + 16\mathcal{D}_0(z) - 16\mathcal{D}_1(z) + 6\mathcal{D}_2(z) - \frac{8\mathcal{D}_3(z)}{3} \\
& + 12\log(1-z) - 8\log^2(1-z) \Bigg], \qquad (11)
\end{aligned}$$

$$\begin{aligned}
K^{(2)}_{\rm NE,h}(z) = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[ & -\frac{2}{\epsilon^3} + \frac{1 + 4\log(1-z)}{\epsilon^2} \\
& + \frac{-5 + 2\log(1-z) - 4\log^2(1-z)}{\epsilon} - 8 \\
& + 10\log(1-z) - 2\log^2(1-z) + \frac{8}{3}\log^3(1-z) \Bigg]. \qquad (12)
\end{aligned}$$

In writing our results for $K$ factors, we have chosen $\mu^2_{\overline{\rm MS}} = q^2$, we have omitted the overall factor of $C_F^2$, which is common to all our results, and we have also omitted, for brevity, terms involving logarithms multiplied by transcendental constants: these terms can easily be generated and do not carry any new information. Interestingly, we find that the plus distribution terms in Eq. (11) suffice to reproduce all corresponding terms in the exact NNLO Drell–Yan $K$-factor [67]. This means that the remaining regions may not contribute any further LP logarithms. We will briefly comment below on the interesting interplay between soft and hard regions which is suggested by this result.

### 3.2. Collinear and anticollinear regions

By symmetry, the collinear and anticollinear regions must give the same contribution, after summing over all graphs in Fig. 1, and including those obtained via complex conjugation and via the interchange $p \leftrightarrow \bar{p}$. The contribution from both regions from diagrams (a)–(d) is then

$$\begin{aligned}
K^{(2),\rm a-d}_{{\rm NE},c+\bar{c}}(z) = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[ & -\frac{1}{2\epsilon^2} + \frac{3\log(1-z)}{2\epsilon} + 1 \\
& - \frac{9}{4}\log^2(1-z) \Bigg]. \qquad (13)
\end{aligned}$$

As expected, we find only a contribution starting at NE level. Note however that it is not true that individual diagrams have only next-to-soft contributions in the collinear region. For example, diagrams (a), (c), (f) and (h) separately contain plus distribution terms. This, however, is an artifact of having used the Feynman gauge, and eikonal terms cancel when diagrams are summed. Likewise, the contribution from diagrams (e)–(h) read

$$\begin{aligned}
K^{(2),\rm e-h}_{{\rm NE},c+\bar{c}}(z) = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[ & -\frac{1}{2\epsilon^2} + \frac{-5 + 6\log(1-z)}{4\epsilon} - \frac{5}{2} \\
& + \frac{15}{4}\log(1-z) - \frac{9}{4}\log^2(1-z) \Bigg]. \qquad (14)
\end{aligned}$$

### 3.3. Soft region

In this region, all integrals are scaleless, and thus vanish in dimensional regularisation. This is consistent with the fact that eikonal terms have already been included in the hard region, according to its definition in Eq. (8). So far as divergent terms are concerned, this collocation of singular terms is not surprising: it is well known that one can shift singularities from the IR to the UV by using dimensional regularisation as we have just done, taking literally the vanishing of scaleless integrals without attempting to distinguish the ultraviolet and the infrared singularities they contain. It is interesting that, at least within the framework of a method-of-regions analysis, this mechanism appears to extend to finite, and even integrable, contributions to the cross section. Note finally that this result is compatible with the approach taken in Ref. [30], where the 'hard' function is taken to implicitly include the soft function, in order to extract the more interesting collinear contributions.

### 3.4. The complete abelian-like real–virtual NNLO K factor

Combining results from the preceding subsections, the complete $K$ factor arising from NNLO abelian-like real–virtual contributions, as computed by the method of regions, is given by

$$\begin{aligned}
& K^{(2)}_{\rm E+NE}(z) \\
& = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[ \frac{2\mathcal{D}_0(z) - 2}{\epsilon^3} \\
& \quad + \frac{-4\mathcal{D}_1(z) + 3\mathcal{D}_0(z) + 4\log(1-z) - 6}{\epsilon^2} \\
& \quad + \frac{16\mathcal{D}_2(z) - 24\mathcal{D}_1(z) + 32\mathcal{D}_0(z) - 16\log^2(1-z) + 52\log(1-z) - 49}{4\epsilon}
\end{aligned}$$

$$- \frac{8\mathcal{D}_3(z)}{3} + 6\mathcal{D}_2(z) - 16\mathcal{D}_1(z) + 16\mathcal{D}_0(z) + \frac{8}{3}\log^3(1-z) - \frac{29}{2}\log^2(1-z) + \frac{103}{4}\log(1-z) - \frac{51}{2}\Bigg]. \tag{15}$$

We find that Eq. (15) reproduces exactly the result obtained in Ref. [65], when the relevant diagrams are isolated,[3] including $z$-independent terms. This is an interesting fact: it reinforces the conjecture that one can carry out the calculation, either with the method of regions or in a factorised approach, as a systematic expansion in powers of the distance from threshold, $1-z$, including not only functions that are (integrably) singular at threshold, but also polynomial dependence.

Whilst fully integrated results are useful in obtaining the final NLP threshold logarithms in the $K$-factor, it is also useful to characterise what happens before the real emission integration is carried out. This is the subject of the following section.

## 4. Loop effects and the soft expansion

In this section, we examine our results in light of the recently proposed next-to-soft theorems of Ref. [37,39]. In particular, we focus on the issue, pointed out in Ref. [52], and further discussed in Refs. [45,48,51], that potential loop corrections to tree-level next-to-soft factors depend on the order in which the dimensional regularisation and soft expansions are carried out.

In Section 3 we presented results for the hard, collinear and anticollinear regions, after the integration over the phase space of the real gluon (with momentum $k_2$) had already been performed. Implicit in the above calculation, but not immediately visible in the final result, is the fact that the different regions are weighted by different scale-related factors. For example, after integration over $k_1$ (but before integration over $k_2$), the contribution from the *hard* region can be written schematically as

$$\text{Hard:} \quad \frac{(2p\cdot\bar{p})^{-\epsilon}}{\epsilon^2}\Big[\mathrm{E} + \mathrm{NE} + \ldots\Big] + \mathcal{O}\left(\epsilon^{-1}\right), \tag{16}$$

where with E and NE we denote terms at $\mathcal{O}(k_2^{-1})$ and $\mathcal{O}(k_2^0)$ respectively, and the ellipsis denotes higher-order terms in the soft expansion. Likewise, the collinear region contributes terms of the form[4]

$$\text{Collinear:} \quad \frac{(-2p\cdot k_2)^{-\epsilon}}{\epsilon}\Big[\mathrm{NE} + \ldots\Big] + \mathcal{O}\left(\epsilon^{0}\right), \tag{17}$$

while the anticollinear region is naturally obtained by replacing $p$ with $\bar{p}$. That these particular scales arise is not surprising: they are the only scales that survive in each given region. It is now clear why the eikonal terms are reproduced from the hard region in this formalism: these terms must arise from the soft-collinear factorisation formula, in which the relevant hard, soft and jet functions cannot depend on the scales $(p\cdot k_2)$ and $(\bar{p}\cdot k_2)$, as they are defined without reference to an extra emission. Interestingly, the collinear regions depends on $z$ through

$$(-2p\cdot k_2)^{-\epsilon} \sim (1-z)^{-\epsilon}, \tag{18}$$

and the same dependence arises in the anti collinear region, through

$$(-2\bar{p}\cdot k_2)^{-\epsilon} \sim (1-z)^{-\epsilon}. \tag{19}$$

This dependence is responsible for the pattern of NLP threshold logarithms in Eqs. (13) and (14), which is generated as follows. The phase space for the real gluon emission contains a further $z$ dependent factor $[(1-z)/z]^{1-2\epsilon}$ (see for example [65]), so that the $k_2$ integration leads to a result of the form

$$\frac{(1-z)^{-3\epsilon}}{\epsilon^2} = \frac{1}{\epsilon^2} - \frac{3}{2}\frac{\log(1-z)}{\epsilon} + \frac{9}{2}\log^2(1-z), \tag{20}$$

where the additional power of $\epsilon^{-1}$ arises after carrying out the phase space integration. Eq. (20) carries exactly the pattern of NLP threshold logarithms observed in Eqs. (13) and (14), after multiplying by the appropriate normalisation. It is clear that terms proportional to $(p\cdot k_2)^{-\epsilon}$ play a crucial role in order to correctly reproduce the known Drell–Yan $K$-factor at NLP accuracy.

The factor $(p\cdot k_2)^{-\epsilon}$ is very interesting from the point of view of the soft expansion in powers of $k_2$. Such a factor would be absent if one performed the soft expansion *before* the dimensional regularisation expansion, and it is clear from individual Feynman diagrams such as that of Eq. (4) why this is the case: carrying out the soft expansion before the $\epsilon$ expansion amounts to expanding the integrand before integration over the virtual momentum $k_1$. This, for example, replaces the mixed denominator according to

$$\frac{1}{(p-k_1-k_2)^2} \to \frac{1}{(p-k_1)^2}, \tag{21}$$

so that logarithmic dependence on $(p\cdot k_2)$ can no longer occur in the final result: only logarithmic dependence on $p\cdot\bar{p}$, which is still present as a scale in the denominator, can arise. This observation fixes the order in which these expansions must be carried out: to get the right answer, one must integrate over virtual momenta before expanding in soft momentum.[5] Note that the only terms which are "problematic" from the point of the view of the soft expansion (*i.e.* that depend on the sequential order of the soft and $\epsilon$ expansions) are those involving overall powers of $(p\cdot k_2)^{-\epsilon}$ or $(\bar{p}\cdot k_2)^{-\epsilon}$. These arise exclusively from the (anti-)collinear regions, which is not surprising: in the hard region, one may neglect the scales $(p\cdot k_2)$ and $(\bar{p}\cdot k_2)$ with respect to the hard scale $p\cdot\bar{p}$, leading to a power-like suppression of next-to-soft effects. That the collinear region leads to a breakdown of the Low–Burnett–Kroll theorem [28,29], due to the absence of a hard scale, is well-known, and was first pointed out by Del Duca [30]. It can also be understood from an effective field theory point of view [59]. Furthermore, the need to first perform the dimensional regularisation expansion has been recently discussed in Ref. [48]. Here, though, we see a concrete example of the impact of this effect on any systematic treatment of threshold corrections.

## 5. Discussion

In this paper, we have performed a case study of threshold effects in Drell–Yan production at next-to-leading power. We focused in particular on reproducing known logarithmic contributions to the real–virtual part of the NNLO $K$-factor, from the point of view of a threshold expansion: this is the first order at which there is an interplay between real and virtual gluons, so that collinear singularities may interfere with the soft expansion. As a consequence,

[3] Note that separate results for the double-real emission and for the real–virtual contribution to the NNLO Drell–Yan cross sections are not available in the literature: we have carried out an independent calculation of the relevant diagrams [54]. One may furthermore verify that combining Eq. (15) with the results of Ref. [21], and with the appropriate mass-factorisation counterterms, reproduces the complete result of Ref. [65].

[4] As mentioned above, the presence of NE terms only in Eq. (17) is the effect of a cancellation of eikonal terms which are present in individual diagrams.

[5] More precisely, one is allowed to neglect terms proportional to $k_2$ in the numerators of loop integrands. One must, however, keep denominators intact, since they can lead to logarithmic dependence.

our study allowed us also to investigate potential loop corrections to recently proposed next-to-soft theorems [37,39]. Our main goal, however, is to provide useful data for the development of a generally applicable resummation formalism for NLP threshold logarithms, building on previous efforts [14–17,21–23]. We used the method of regions [55,71,72] to separate out contributions from the hard, soft and (anti)-collinear momentum configurations. A first gratifying result is that a systematic application of this method beyond leading power allowed us to reproduce exactly all corresponding terms in the exact calculation, including $z$-independent contributions. This confirms that all threshold logarithms to this accuracy arise from soft or collinear singularities, and reinforces the idea of using the threshold expansion as a systematic tool for the analysis of QCD cross sections, both at finite orders [26,56] and in the context of threshold resummation. Our analysis also shows that collinear regions contribute logarithmic dependence on soft momenta, which affects the NLP threshold logarithms one obtains after integration over the real gluon phase space. This fixes the order in which the dimensional regularisation and soft expansions must be carried out, as was also discussed in Refs. [30,45,48,51, 52]. It is instructive and useful to see exactly how this mechanism operates in the familiar context of Drell–Yan production.

Our results will be instrumental in the construction of a systematic all-order treatment of threshold effects at NLP accuracy: they carry the basic information that the interplay between soft and collinear effects is considerably more intricate at NLP than it is in standard leading-power soft-collinear factorisation. A systematic treatment will require the introduction of new operator matrix elements, incorporating the effects of non-factorising soft radiation from collinearly enhanced configurations, as first suggested in Ref. [30]. Work to implement these considerations in a systematic way, beginning with the relatively simple case of electroweak annihilation processes, is in progress.

## 6. Note added

After the completion of this work, the calculation of the Higgs production cross section in the gluon fusion channel, in the large-top-mass approximation, at $N^3$LO and NLP, was completed and presented in Refs. [73,74].

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