Generalizations of damping for cavities with mirrors of finite transmittivity

van der Plank, R.W.F.; Suttorp, L.G.

DOI
10.1103/PhysRevA.53.1791

Publication date
1996

Published in
Physical Review A

PHYSICAL REVIEW A VOLUME 53, NUMBER 3 MARCH 1996

Generalization of damping theory for cavities with mirrors of finite transmittivity

R. W. F. van der Plank and L. G. Suttorp
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands
(Received 27 July 1995; revised manuscript received 17 October 1995)

Standard damping theory is generalized to incorporate the effects of finite mirror transmittivity. The correction to the standard Langevin equation for the quasimode annihilation operator is determined in first order in the transmittivity of the mirrors. From the Langevin equation an effective master equation is derived. As an example, we study the decay of a single two-level atom at a fixed position in a nonideal cavity. For this case we find a modification of the damped Rabi oscillations, which depends on the atomic position.

PACS number(s): 42.50.Ct, 42.50.Md

I. INTRODUCTION

There are mainly two approaches towards describing the photon loss through the mirrors of a Fabry-Pérot cavity. The first method is rather phenomenological. One quantizes the electromagnetic field inside the cavity while ignoring the fact that the mirrors are not perfectly reflecting. The coupling through the mirrors is then modeled by an interaction with an external bath. With the use of the Born and Markov approximations, one can obtain an effective master equation by taking the trace over the bath degrees of freedom [1]. In the resulting formalism, the field degrees of freedom are limited to a discrete set of quasimodes, of which usually only one is considered.

The second and more fundamental method is based upon the so-called universe-mode picture. In this approach, the cavity is embedded in a larger box, the "universe," the size of which is taken to infinity. The electromagnetic field inside the box is quantized and a dense set of universe modes is found. As derived by Lang *et al.* [2] for cavities with nearly perfect mirrors, it is possible to bunch these universe modes into quasimodes and to obtain an effective description. The field degrees of freedom are then reduced to a nondense, discrete set. The ensuing master equation has the same form as the one resulting from the phenomenological approach. However, the universe-mode picture gives more insight in the limits of validity of the formalism, especially with respect to the properties of the mirror [3].

An interesting example of a physical system for which a precise description of the photon loss is essential is an excited two-level atom in a cavity with imperfect mirrors. For that particular model a delay-differential equation has been derived from the universe-mode approach [4,5]. Unfortunately, this rather complicated equation has to be solved numerically if the reflectivity of the mirrors of the cavity is taken to be arbitrary. An analytical solution has been found only for cavities with nearly perfect mirrors. In that case the formalism can be shown to yield the same predictions as the (multi-)quasi-mode description of Lang *et al.* [2], at least for times much larger than the cavity round-trip time. To establish this connection several approximations have to be introduced. One of these is equivalent to replacing the spatial dependence of the universe modes in the cavity by that of the quasimode in which they participate. This approximation, which was also used by Lang *et al.* [2], becomes invalid if the transmittivity of the cavity is no longer very small.

The restriction to cavities with nearly perfect mirrors in the formalism of Lang *et al.* is a drawback when applying it to physical systems. In the present paper we want to show that the master equation obtained by Lang *et al.* [2] may be generalized so as to incorporate the effects of a finite mirror transmittivity. The correction that we will obtain is of first order in this transmittivity. Since its origin lies in the spatial dependence of the universe modes, the correction term is explicitly space dependent.

The theory that we shall present is one-dimensional, so that transverse effects are neglected. We assume that the electromagnetic field interacts with matter inside a thin slice in the cavity, with a thickness small compared to the relevant wavelengths. For simplicity we consider a cavity with a semitransparent mirror at one end and a perfectly reflecting one at the other. In the derivation of the master equation we shall deal with a single quasimode only. The generalization to more than one quasimode is straightforward, as our quasimodes are associated with strictly independent degrees of freedom.

In Secs. II and III we will derive the first-order correction to the standard Langevin equation and the associated master equation. In Sec. IV we will apply the formalism to the decay of a two-level atom in a nonideal cavity. The master equation will be solved with the use of the method of damping bases [6].

II. GENERALIZED DAMPING THEORY

In this section we will generalize the standard damping theory, as given by Lang *et al.* [2], to cavities with mirrors of small but finite transmittivity. The implications for the Langevin and master equations will be considered in Sec. III.

Let us consider a cavity enclosed in a large universe. The universe is bounded by two perfectly reflecting mirrors at $z=-L$ and l. The cavity is formed by the mirror at $z=l$ and an additional mirror at $z=0$. The latter mirror is semitransparent, with a small but finite transmittivity. The (real and positive) reflection and transmission coefficients r and t of this mirror satisfy the relation $r^2+t^2=1$. The boundary conditions at the outer mirrors restrict the allowed wave numbers to a discrete set with separation π/L for $L \gg l$. In the continuum limit, obtained when L approaches infinity, this set becomes dense. The modes of the universe are given by

$$U_k(z)=\begin{cases}\xi_k \sin k(z+L) & (z<0)\\ M_k \sin(kl\zeta) & (z>0),\end{cases} \tag{2.1}$$

where we defined

$$\zeta=1-z/l. \tag{2.2}$$

The coefficient ξ_k alternates between $+1$ and -1 for consecutive values of k. Furthermore, M_k is given by

$$M_k=\left[\sqrt{1+s^2}\,\frac{s}{s^2+\sin^2(\Delta k l)}\right]^{1/2}, \tag{2.3}$$

where the parameter $s=(1-r)/(2\sqrt{r})$ is a measure for the transmittivity of the semitransparent mirror. Finally, $\Delta k=k-k_0$ is the difference of the wave number of the mode and the nearest resonant wave number k_0, which satisfies the relation $\tan(k_0 l)=t/(r+1)$.

The universe modes can be grouped into quasimodes. Each quasimode is associated with a particular resonant wave number k_0. The (positive frequency part of the) electric field is a sum over all wave numbers k of the annihilation operators a_k of the corresponding universe modes, with the mode functions as coefficients. Using the concept of quasimodes, we write the electric field as a sum over resonant wave numbers k_0 of

$$\mathcal{E}_{k_0}(z)=\frac{1}{\sqrt{L}}\sum_k a_k U_k(z), \tag{2.4}$$

where the summation is restricted to k values with $|\Delta k|\leqslant \pi/(2l)$. We omit conventional prefactors with $\hbar$, c, or k that may occur in the definition of the electric field. In fact, we assume that $k_0 \gg \pi/(2l)$, so that all k in these prefactors may be replaced by the resonant wave number k_0. From now on we will restrict ourselves to a single quasimode and accordingly suppress the index k_0. The generalization to more quasimodes is straightforward, as different quasimodes correspond to strictly independent degrees of freedom [7].

For $z>0$ we can write the electric field as

$$\mathcal{E}(z)=\frac{1}{\sqrt{l}}\mathcal{N}(z)a(z). \tag{2.5}$$

Here we defined

$$a(z)=\sum_k \phi_k(z)a_k. \tag{2.6}$$

As before, the summation is restricted to k values with $|\Delta k|\leqslant \pi/(2l)$. The $\phi_k(z)$ are given by

$$\phi_k(z)=\sqrt{\frac{l}{L}}\,[\mathcal{N}(z)]^{-1}M_k \sin(kl\zeta), \tag{2.7}$$

with

$$\mathcal{N}(z)=\pm\left[\frac{l}{L}\sum_k M_k^2 \sin^2(kl\zeta)\right]^{1/2}. \tag{2.8}$$

Here the sign of $\mathcal{N}(z)$ is chosen according to the sign of $\sin(k_0 l\zeta)$. We do not consider z values for which $\sin(k_0 l\zeta)$ is equal to zero. For such cases the standard damping treatment cannot be expected to yield a meaningful description, as will be explained in Sec. III.

Note that $\phi_k(z)$ depends on z, since the dependence through the sine functions does not drop out in general. Therefore, the operator $a(z)$ will also depend on z. Nevertheless, it may be interpreted as a quasimode annihilation operator; it obeys the standard boson commutation relation $[a(z),a^\dagger(z)]=1$, as we have $\sum_k \phi_k^2(z)=1$.

We now want to derive an equation for the time evolution of the quasimode annihilation operator $a(z)$ in the Heisenberg picture. Before doing so, we have to specify the interactions in the system. Let us assume that the electric field interacts with matter that is situated in a thin slice inside the cavity at the position z. This is the case, for instance, if only a single atom is present in the cavity, with an interaction determined by the electric-dipole approximation. If the thickness of the slice is small compared to k_0^{-1}, the interaction Hamiltonian H_{int} depends on the universe-mode annihilation (and creation) operators through the electric-field operator (2.5) (and its Hermitian conjugate) and hence through $\mathcal{N}(z)a(z)$ (and its Hermitian conjugate). Thus the Heisenberg equation for the annihilation operator a_k is

$$\frac{d}{dt}a_k(t)=-i(\omega_0+\Delta\omega_k)a_k(t)+\frac{i}{\hbar}[H_{\mathrm{int}}(z,t),a(z,t)]\phi_k(z), \tag{2.9}$$

with $\omega_0=ck_0$ and $\Delta\omega_k=c\Delta k$. Note that the term $[H_{\mathrm{int}}(z,t),a(z,t)]$ contains a factor $\mathcal{N}(z)$.

The formal solution of the Heisenberg equation leads to the following expression for the quasimode annihilation operator $a(z,t)$:

$$\begin{aligned}a(z,t)=&\sum_k \phi_k(z)e^{-i(\omega_0+\Delta\omega_k)t}a_k(0)\\ &+\frac{i}{\hbar}\int_0^t dt'\,F(z,t-t')e^{-i\omega_0(t-t')}\\ &\times[H_{\mathrm{int}}(z,t'),a(z,t')],\end{aligned} \tag{2.10}$$

with the memory kernel

$$F(z,t)=\sum_k \phi_k^2(z)e^{-i\Delta\omega_k t}. \tag{2.11}$$

Differentiation with respect to t yields

$$\begin{aligned}\frac{\partial}{\partial t}a(z,t)=&-i\omega_0 a(z,t)+\frac{i}{\hbar}[H_{\mathrm{int}}(z,t),a(z,t)]\\ &-i\sum_k \phi_k(z)\Delta\omega_k e^{-i(\omega_0+\Delta\omega_k)t}a_k(0)\\ &+\frac{i}{\hbar}\int_0^t dt'\left(\frac{\partial}{\partial t}F(z,t-t')\right)e^{-i\omega_0(t-t')}\\ &\times[H_{\mathrm{int}}(z,t'),a(z,t')].\end{aligned} \tag{2.12}$$

In the next section we will discuss this equation under the assumption that we are only interested in the evolution of observables on a time scale that is slow compared to the cavity round-trip time $2l/c$, after the trivial time dependence given by the optical frequency ω_0 has been removed.

III. DERIVATION OF THE MASTER EQUATION

In the first part of this section we will discuss the standard damping theory, which is valid for nearly perfect cavities, and derive the master equation (19) for this case. In the second part we will extend the theory to cavities with small but finite transmittivity.

In the standard damping theory, first discussed by Lang *et al.* [2], the coefficients M_k are approximated by the formula

$$M_k=\left[\frac{c}{l}\,\frac{\Gamma}{\Gamma^2+\Delta\omega_k^2}\right]^{1/2}, \tag{3.1}$$

with the cavity decay constant [8]

$$\Gamma=s\frac{c}{l}. \tag{3.2}$$

Furthermore, the z dependence of the modes is approximated by making the replacement

$$\sin(kl\zeta)\rightarrow\sin(k_0l\zeta), \tag{3.3}$$

which renders $a(z,t)$ and $\phi_k(z)$ independent of z. As a consequence, the normalization factor $\mathcal{N}(z)$ defined in (2.8) will simply be given by $\sin(k_0l\zeta)$. Note that for z values for which $\sin(k_0l\zeta)=0$ (or close to 0), the replacement (3.3) obviously cannot be a good approximation. It is for this reason that we only consider z values for which $\sin(k_0l\zeta)$ differs from zero.

A further approximation is now used. The summation is no longer restricted to k values with $|\Delta k|\leqslant\pi/(2l)$, but is replaced (in the continuum limit) by an integration over k values from $-\infty$ to $+\infty$. The memory kernel can now be calculated with the use of the residue theorem. One obtains the simple expression $F(z,t)=e^{-\Gamma t}$ (for $t\geqslant 0$) and thus

$$\left(\Gamma+\frac{\partial}{\partial t}\right)F(z,t)=0. \tag{3.4}$$

Using (2.10) we can now rewrite (2.12) as

$$\frac{d}{dt}a(t)=-i\omega_0a(t)-\Gamma a(t)+\frac{i}{\hbar}[H_{\text{int}}(t),a(t)]+f(t), \tag{3.5}$$

with the abbreviation

$$f(t)=\sum_k \phi_k(\Gamma-i\Delta\omega_k)e^{-i(\omega_0+\Delta\omega_k)t}a_k(0). \tag{3.6}$$

One easily verifies that not only does $f(t)$ commute with $a(t)$, it also commutes, for $t\neq 0$, with both $a(0)$ and $a^\dagger(0)$. The latter property, which follows from (2.11) and (3.4), implies that the statistics of $f(t\neq 0)$ is completely independent of the statistics of the quasimode operators at $t=0$. For that reason $f(t)$ is usually called the stochastic force and (3.5) the (generalized) Langevin equation.

From the Langevin equation a master equation for the density operator ρ of the system can be derived. This is achieved by considering time derivatives of expectation values of normally ordered products of the quasimode operators a and $a^\dagger$ and rewriting these with the help of (3.5). Using the fact that $f(t)$ and $a(t)$ commute, one may reorder the operator products involving the stochastic force (or its Hermitian conjugate) in such a way that $f(t)$ is the last factor [and $f^\dagger(t)$ the first]. Rewriting the expectation values as traces of products of these operators and the density operator ρ, one finds that $f(t)$ and $f^\dagger(t)$ act directly on ρ. By demanding that the degrees of freedom associated with $f(t)$ are not excited at $t=0$, one obtains $f(t)\rho=0$ and $\rho f^\dagger(t)=0$. In contrast, the statistics of the quasimode operators at $t=0$ need not be specified, so that the dependence of ρ on the quasimode degrees of freedom at $t=0$ can be left arbitrary. The stochastic force drops out under the chosen circumstances, so that the time derivative of an expectation value of any normally ordered product of quasimode operators is found as the expectation value of a sum of products of these operators. After transforming to the Schrödinger picture one finally arrives at the standard master equation [1] for the density operator

$$\begin{aligned}\frac{d}{dt}\rho(t)=&-\frac{i}{\hbar}[H_{\text{int}}+\hbar\omega_0a^\dagger a,\rho(t)]+\Gamma[a,\rho(t)a^\dagger]\\&+\Gamma[a\rho(t),a^\dagger].\end{aligned} \tag{3.7}$$

As is well known [1], additional terms show up if the degrees of freedom associated with the stochastic force are assumed to be excited.

The Langevin and master equations derived above are valid only if the approximations regarding the space dependence and the mode structure are justified. We will now consider the more general case, in which we refrain from making these simplifying approximations. A direct consequence is that the memory kernel $F(z,t)$ will be a more complicated function of time, for which the property (3.4) certainly will not hold.

From (2.10) and (2.12) we derive quite generally

$$\begin{aligned}\frac{\partial}{\partial t}a(z,t)=&-i\omega_0a(z,t)-\Gamma a(z,t)+\frac{i}{\hbar}[H_{\text{int}}(z,t),a(z,t)]\\&+f(z,t)+\frac{i}{\hbar}\int_0^t dt'\left[\frac{\partial}{\partial t}F(z,t-t')\right.\\&\left.+\Gamma F(z,t-t')\right]e^{-i\omega_0(t-t')}\\&\times[H_{\text{int}}(z,t'),a(z,t')],\end{aligned} \tag{3.8}$$

where we defined

$$f(z,t)=\sum_k \phi_k(z)(\Gamma-i\Delta\omega_k)e^{-i(\omega_0+\Delta\omega_k)t}a_k(0) \tag{3.9}$$

and Γ is again given by (3.2).

For small values of s the memory kernel $F(z,t)$ and the normalization factor $\mathscr{N}(z)$ can be approximated in such a way that only terms up to first order in s are retained. As shown in the Appendix, the normalization factor, which has been defined in (2.8), then gets the form

$$\mathscr{N}(z)=\sin(k_0 l\zeta)\left[1-\frac{s}{2}v'(z)\right]+\mathscr{O}(s^2). \tag{3.10}$$

Similarly, the memory kernel $F(z,t)$, of which the general form has been given in (2.11), is found approximately as

$$F(z,t)=\begin{cases}1 & (\tau=0)\\ e^{-s\tau}[1+sv(z)]+\mathscr{O}(s/\tau)+\mathscr{O}(s^2) & (\tau>1).\end{cases} \tag{3.11}$$

Here we defined

$$v(z)=v'(z)+iv''(z), \tag{3.12}$$

with

$$v'(z)=-\frac{2\cos(2k_0 l\zeta)}{1-\cos(2k_0 l\zeta)}\left\{\zeta-\frac{1}{\pi}[2\zeta\beta(\zeta)-1]\sin(\pi\zeta)\right\}, \tag{3.13}$$

$$v''(z)=-2\zeta\frac{\sin(2k_0 l\zeta)}{1-\cos(2k_0 l\zeta)}. \tag{3.14}$$

Furthermore, we introduced the dimensionless time variable $\tau\equiv tc/l$. The function $\beta(\zeta)$ in (3.13) is related to the digamma function (see the Appendix).

Having found the memory kernel, we may derive a Langevin equation from (3.8). Indeed, since the combination of the terms within large square brackets can be shown to be of short range (see the Appendix), one may simplify the integral considerably by extracting the other factors. Evaluating the resulting expression, we arrive at the Langevin equation

$$\frac{\partial}{\partial t}a(z,t)=-i\omega_0 a(z,t)-\Gamma a(z,t)+f(z,t)+\frac{i}{\hbar}[H_{\rm int}(z,t),a(z,t)]\left(1+\Gamma\frac{l}{c}v(z)\right), \tag{3.15}$$

which contains terms that are linear in $s=\Gamma l/c$. The stochastic force $f(z,t)$ is given by (21) as before.

From the Langevin equation we can derive a master equation in a similar way as described in the first part of this section. To that end one starts again by considering the time derivative of the expectation value $\langle a^{\dagger p}(z,t)a^q(z,t)\rangle$ of a normally ordered product of quasimode creation and annihilation operators. From (9) and (20) we can easily prove the identity

$$f(z,t)+\frac{i}{\hbar}\int_0^t dt'\left[\frac{\partial}{\partial t}F(z,t-t')+\Gamma F(z,t-t')\right]\times e^{-i\omega_0(t-t')}[H_{\rm int}(z,t'),a(z,t')]=\sum_k\phi_k(z)(\Gamma-i\Delta\omega_k)a_k(t), \tag{3.16}$$

which shows that the combination at the left-hand side commutes with $a(z,t)$. Note that $f(z,t)$ itself may or may not commute with $a(z,t)$, depending on the interaction Hamiltonian. When evaluating the expectation value with the use of (3.8), we can bring the combination (3.16) through all factors $a(z,t)$ to the right [and its Hermitian conjugate through all factors $a^\dagger(z,t)$ to the left]. Subsequently, the integral term can be eliminated in a similar way as in deriving (3.15). The resulting expression for the time derivative of $\langle a^{\dagger p}(z,t)a^q(z,t)\rangle$ is then found to contain the contribution

$$p\langle f^\dagger(z,t)a^{\dagger p-1}(z,t)a^q(z,t)\rangle+q\langle a^{\dagger p}(z,t)a^{q-1}(z,t)f(z,t)\rangle. \tag{3.17}$$

Rewriting this expression as the trace of a sum of products of the relevant operators and the density operator ρ, one finds that the stochastic force acts directly on ρ, as before. In the standard damping theory one may choose ρ in such a way that operating with the stochastic force on ρ yields zero. The freedom to make that choice is a consequence of the fact that the stochastic force and the quasimode operators are associated with different degrees of freedom. In the presence case the situation is somewhat more subtle. In fact, the commutator of $f(z,t)$ and $a^\dagger(z,0)$ is given by

$$[f(z,t),a^\dagger(z,0)]=e^{-i\omega_0 t}\left[\frac{\partial}{\partial t}F(z,t)+\Gamma F(z,t)\right], \tag{3.18}$$

as follows from (2.6) and (3.9). As we have seen, the right-hand side approximately vanishes for $t\gg l/c$, so that for these values of t the degrees of freedom associated with the stochastic force are indeed independent of those connected to the quasimode operator $a(z,0)$. As before, we may therefore assume that the degrees of freedom associated with $f(z,t)$ are not excited, so that one gets once more $f(z,t)\rho=0$, at least for values of t that are much larger than l/c. Similarly, we may assume that $\rho f^\dagger(z,t)=0$ for t much larger than l/c.

By transforming to the Schrödinger picture, one finally arrives at the master equation

$$\begin{aligned}\frac{d}{dt}\rho(t)=&-\frac{i}{\hbar}[H_{\rm int}(z)+\hbar\omega_0 a^\dagger(z)a(z),\rho(t)]\\&+\Gamma[a(z),\rho(t)a^\dagger(z)]+\Gamma[a(z)\rho(t),a^\dagger(z)]\\&+\frac{i}{\hbar}\Gamma\frac{l}{c}v^*(z)[a(z),\rho(t)[a^\dagger(z),H_{\rm int}(z)]]\\&-\frac{i}{\hbar}\Gamma\frac{l}{c}v(z)[[H_{\rm int}(z),a(z)]\rho(t),a^\dagger(z)].\end{aligned} \tag{3.19}$$

As compared to the standard master equation, the equation derived here contains additional terms that are of the order of $s=\Gamma l/c$. They account, at least approximately, for the effects of a nonideal cavity mirror and the ensuing nontrivial space dependence of the quasimodes. The extra terms are different from what might have been expected naively from the form of the Langevin equation (3.15), as they cannot be found by multiplying the interaction Hamiltonian by a factor of $1+\Gamma v(z)l/c$. The reason for that is obvious from the derivation given above: the stochastic force $f(z,t)$ does not commute with $a(z,t)$ in general, but should be taken together with the correction term, as done in (3.16).

It remains to be checked whether the master equation conserves the Von Neumann conditions of Hermiticity and positive definiteness of the density operator. It is easily shown that Hermiticity is indeed conserved, but that the master equation may violate positivity in second order in s. Such a violation stems from the fact that only first-order terms in s were considered. In the example that we will consider in the following section, an additional (phenomenological) damping term ensures positivity in all orders in s.

Note that the generalization to several quasimodes is straightforward. This is a direct consequence of the fact that annihilation and creation operators associated with different quasimodes commute. The following multimode master equation is thus easily derived:

$$\begin{aligned}\frac{d}{dt}\rho(t)=&-\frac{i}{\hbar}[H_{\rm int}(z),\rho(t)]-i\sum_{k_0}\omega_{k_0}[a^\dagger_{k_0}(z)a_{k_0}(z),\rho(t)]\\&+\Gamma\sum_{k_0}[a_{k_0}(z),\rho(t)a^\dagger_{k_0}(z)]\\&+\Gamma\sum_{k_0}[a_{k_0}(z)\rho(t),a^\dagger_{k_0}(z)]\\&+\frac{i}{\hbar}\Gamma\frac{l}{c}\sum_{k_0}v^*_{k_0}(z)[a_{k_0}(z),\rho(t)[a^\dagger_{k_0}(z),H_{\rm int}(z)]]\\&-\frac{i}{\hbar}\Gamma\frac{l}{c}\sum_{k_0}v_{k_0}(z)[[H_{\rm int}(z),a_{k_0}(z)]\rho(t),a^\dagger_{k_0}(z)].\end{aligned}\tag{3.20}$$

IV. DECAYING ATOM IN A NONIDEAL CAVITY

To illustrate the consequences of the additional terms in the master equation, we will treat the decay of a single two-level atom in a cavity. This problem has been studied extensively before. For a cavity with perfect mirrors, it reduces to the Jaynes-Cummings model [9]. The effect of a finite probability for a photon to escape through nearly perfect mirrors was first discussed by Haroche [10]. The influence of a thermal reservoir on the atomic decay was considered by Sachdev [11]. A more general model, in which, moreover, the effects of atomic damping terms in transverse directions are taken into account, has been analyzed by Cirac *et al.* [12] and by Briegel and Englert [6]. The latter use the method of damping bases, which we will do as well. Numerical studies of atomic decay in a cavity with mirrors of arbitrary reflectivity have been carried out by Feng and Ujihara [5]. The relation of their work and ours will be discussed below. Some authors have considered the decay of an atom in a cavity by using Fermi's golden rule as a starting point and avoiding the introduction of quasimodes. This makes it possible to study three-dimensional generalizations [13] as well. However, in this way the Rabi-type oscillations are not retrieved.

We assume that there is no detuning between the cavity and the atom, so that the atomic transition frequency coincides with a quasimode resonant frequency. We use the electric-dipole approximation to describe the interaction of the atom at a position z with the electromagnetic field. We will neglect all nonresonant quasimodes.

Adopting the rotating-wave approximation, we may write the interaction Hamiltonian as

$$H_{\rm int}(z)=-\tfrac{1}{2}\hbar g(z)[a^\dagger(z)\sigma_-+a(z)\sigma_+].\tag{4.1}$$

Here $\sigma_\pm$ are the usual Pauli spin matrices often employed to describe a two-level atom. The (real) coupling constant $g(z)$ is given by $\bar{g}\mathcal{N}(z)$, where $\bar{g}$ contains the atomic dipole-moment matrix element and $\mathcal{N}(z)$ is given by (3.10).

The master equation, of which the general form has been derived above, reads for the present case

$$\frac{d}{dt}\rho=-\frac{i}{\hbar}[H_{\rm int},\rho]+\mathcal{L}_a\rho+\mathcal{L}_{a\sigma}\rho+\mathcal{L}_\sigma\rho,\tag{4.2}$$

with

$$\mathcal{L}_a\rho=\Gamma([a,\rho a^\dagger]+[a\rho,a^\dagger]),\tag{4.3a}$$

$$\mathcal{L}_{a\sigma}\rho=\frac{i}{2}\Gamma w[a^\dagger,\sigma_-\rho]-\frac{i}{2}\Gamma w^*[\rho\sigma_+,a],\tag{4.3b}$$

$$\mathcal{L}_\sigma\rho=\frac{\gamma_\parallel}{8}([\sigma_-,\rho\sigma_+]+[\sigma_-\rho,\sigma_+]).\tag{4.3c}$$

The first three terms in (4.2) follow directly from the master equation of Sec. III, while the last term is a standard atomic damping term [1], which accounts for the decay of the atom through all transverse field modes not included in the quasimode. The atomic damping term is of the form usually found when considering radiative decay only [1]. We have moved to the interaction picture, so that the optical frequency ω_0 has been transformed away.

The second term in (4.2) is the correction term to standard damping theory. It depends on the coupling parameter w, which is defined as

$$w(z)=w'(z)+iw''(z)=\frac{l}{c}g(z)v(z),\tag{4.4}$$

where $v(z)=v'(z)+iv''(z)$ has been given in Sec. III. Note that we will assume that w is small, as we have included first-order correction terms in the master equation only.

It can be shown that positivity is conserved by the master equation for parameter values satisfying $\gamma_\parallel\geqslant\frac{1}{2}\Gamma|w|^2$. In fact, the master equation can be written as

$$\begin{aligned}\frac{d}{dt}\rho=&A\rho+\rho A^\dagger+2\Gamma(a-\tfrac{1}{4}iw\sigma_-)\rho(a^\dagger+\tfrac{1}{4}iw^*\sigma_+)\\&+\tfrac{1}{4}(\gamma_\parallel-\tfrac{1}{2}\Gamma|w|^2)\sigma_-\rho\sigma_+,\end{aligned}\tag{4.5}$$

where A is an operator of which the precise form is not relevant here. From this form of the master equation it follows directly that a density operator that is positive definite at $t=0$ can never lose that property during its evolution if the coefficient of the last term is non-negative.

We will solve the master equation with the use of the damping bases introduced in [6]. We write the density matrix in the form

$$\rho=\rho_0\sigma_0+\rho_z\sigma_z+\rho_-\sigma_-+\rho_+\sigma_+\,, \tag{4.6}$$

with $\sigma_0=\frac{1}{2}(1-\sigma_z)$ and with the expansions

$$\rho_0=\sum_{n,k}\ \alpha_{n,k}(t)\rho_{n,k}\,, \tag{4.7a}$$

$$\rho_z=\sum_{n,k}\ \beta_{n,k}(t)\rho_{n,k}\,, \tag{4.7b}$$

$$\rho_-=\sum_{n,k}\ \gamma_{n,k-1}(t)\rho_{n,k}\,, \tag{4.7c}$$

$$\rho_+=\sum_{n,k}\ \eta_{n,k+1}(t)\rho_{n,k}\,, \tag{4.7d}$$

where the sums are taken over integer k and non-negative integer n. Here $\rho_{n,k}$ are the eigenoperators of $\mathscr{L}_a$. They satisfy the eigenvalue equation

$$\mathscr{L}_a\rho_{n,k}=-2\Gamma\left(n+\frac{|k|}{2}\right)\rho_{n,k} \tag{4.8}$$

and are given by the expressions

$$\rho_{n,k}=\begin{cases} a^{\dagger k}(-1)^{a^\dagger a+n}\begin{pmatrix} n+k \\ a^\dagger a+k\end{pmatrix} & (k\geqslant 0) \\ (-1)^{a^\dagger a+n}\begin{pmatrix} n+|k| \\ a^\dagger a+|k|\end{pmatrix}a^{|k|} & (k<0).\end{cases} \tag{4.9}$$

The master equation (4.2) leads to a set of coupled differential equations for the coefficients in the expansions (39). These coefficients can be taken together in four-dimensional vectors $X_{n,k}$, as in the standard damping theory [6]. For $k=0$ the vectors $X_{n,k}$ are

$$X_{n,0}=\begin{pmatrix}\alpha_{n+1,0}\\ \beta_{n,0}\\ -i\,\gamma_{n,0}\\ -i\,\eta_{n,0}\end{pmatrix}(n\geqslant 0), \tag{4.10}$$

$$X_{-1,0}=\begin{pmatrix}\alpha_{0,0}\\ 0\\ 0\\ 0\end{pmatrix}. \tag{4.11}$$

For $k\neq 0$ the vectors $X_{n,k}$ are nearly (but not completely) analogous [6]. They will not be needed in the following.

The coupled differential equations for the vectors $X_{n,k}$ have the general form [6]

$$\frac{d}{dt}X_{n,k}=M_{n,k}X_{n,k}+G_{n,k}X_{n+1,k} \tag{4.12}$$

for all integer k and n, with $n\geqslant -1$. Both $M_{n,k}$ and $G_{n,k}$ are 4×4 matrices. The matrix $M_{n,k}$ is the sum of a zeroth-order term that is independent of w and a first-order term linear in w:

$$M_{n,k}=M_{n,k}^{(0)}+M_{n,k}^{(1)}. \tag{4.13}$$

For $k=0$ and $n\geqslant 0$ one finds

$$M_{n,0}^{(0)}=\begin{pmatrix} -2\Gamma(n+1) & 0 & 2g(n+1) & -2g(n+1)\\ 0 & -2\Gamma n-\gamma_\parallel & -2g(n+1) & 2g(n+1)\\ -\frac{1}{2}g & \frac{1}{2}g & -\Gamma(2n+1)-\frac{1}{2}\gamma_\parallel & 0\\ \frac{1}{2}g & -\frac{1}{2}g & 0 & -\Gamma(2n+1)-\frac{1}{2}\gamma_\parallel\end{pmatrix}, \tag{4.14}$$

$$M_{n,0}^{(1)}=\begin{pmatrix} 0 & 0 & 2\,\Gamma w^*(n+1) & -2\,\Gamma w(n+1)\\ 0 & 0 & 0 & 0\\ 0 & \frac{1}{2}\Gamma w & 0 & 0\\ 0 & -\frac{1}{2}\Gamma w^* & 0 & 0\end{pmatrix}, \tag{4.15}$$

while for $k=0$ and $n=-1$ one gets $M_{-1,0}=0$. Analogous expressions can be derived for $M_{n,k}$ with $k\neq 0$. Finally, we have to specify the matrix $G_{n,k}$, which turns out to be independent of w. For $k=0$ and $n\geqslant 0$ it reads

$$G_{n,0}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & -g & 0 & 0 \end{pmatrix}, \tag{4.16}$$

whereas it vanishes for $k=0$ and $n=-1$. Similar results are obtained for $G_{n,k}$ with $k\neq 0$.

The differential equations (4.12) can be solved for given initial conditions. From the structure of the equations it follows that the time evolution is determined by the eigenvalues of the matrices $M_{n,k}$. In first order of w, these can be found by starting from the eigenvalue problem for $M^{(0)}_{n,k}$ and using perturbation theory. For $k=0$ and $n\geqslant 0$ we obtain in this way the set of eigenvalues

$$\lambda_{1,2}=-\Gamma(2n+1)-\frac{1}{2}\gamma_{\parallel}\pm 4g\Gamma w''(n+1)/S, \tag{4.17a}$$

$$\lambda_{3,4}=-\Gamma(2n+1)-\frac{1}{2}\gamma_{\parallel}\pm\frac{1}{2}iS\pm 4ig\Gamma w'(n+1)/S, \tag{4.17b}$$

with $S=[16g^2(n+1)-(2\Gamma-\gamma_{\parallel})^2]^{1/2}$. Here we assumed that the system is underdamped, that is, we took the argument of the square root to be positive. In the overdamped case the analytic continuation of the square root must be taken. Furthermore, it should be noted that in zeroth order two eigenvalues coincide. To evaluate the perturbed eigenvalues this degeneracy has been taken into account properly by employing degenerate perturbation theory. The above expressions for the eigenvalues are valid for $k=0$ only. The expressions for $k\neq 0$ are somewhat more complicated. Furthermore, we assumed that $S\neq 0$; the special case $S=0$, for which all four eigenvalues coincide in zeroth order, has to be treated separately.

To study the time evolution of the system in more detail we shall discuss a special case, with a particular initial condition. Let us consider the case that at $t=0$ no photon is present in the system, while the atom is in its excited state. This implies that for $t=0$ one has $\alpha_{0,0}=\beta_{0,0}=1$, whereas all other coefficients vanish. In other words, one starts with vectors $X_{n,k}(t=0)$ that are different from zero only for $(n,k)=(-1,0)$ or $(0,0)$. Due to the structure of (4.12), it follows that also for $t>0$ these values of (n,k) are the only ones for which $X_{n,k}$ differs from zero. Moreover, the equations for these two vectors are not coupled, since $G_{-1,0}$ vanishes. The time dependence of $X_{-1,0}$ is trivial, while that of $X_{0,0}$ is governed by the eigenvalues of the matrix $M_{0,0}$. These eigenvalues have been given above in first-order perturbation theory. It is a straightforward task to determine the associated eigenvectors up to first order. With the use of these, one gets the following explicit expressions for the coefficients in the density matrix:

$$\alpha_{0,0}(t)=1, \tag{4.18a}$$

$$\begin{aligned}\alpha_{1,0}(t)=&-\frac{8g}{S^2}\left(g+2\Gamma w'\frac{S^2-8g^2}{S^2}\right)\left[\cos\left(\frac{1}{2}St+4g\frac{\Gamma}{S}w't\right)\right.\\&\left.-\cosh\left(4g\frac{\Gamma}{S}w''t\right)\right]e^{-[\Gamma+(1/2)\gamma_{\parallel}]t},\end{aligned} \tag{4.18b}$$

$$\begin{aligned}\beta_{0,0}(t)=&\left[\frac{1}{S^2}\left(S^2-8g^2-8g\Gamma w'\frac{S^2-16g^2}{S^2}\right)\right.\\&\times\cos\left(\frac{1}{2}St+4g\frac{\Gamma}{S}w't\right)+\frac{8g}{S^2}\left(g+\Gamma w'\frac{S^2-16g^2}{S^2}\right)\\&\times\cosh\left(4g\frac{\Gamma}{S}w''t\right)+\frac{2\Gamma-\gamma_{\parallel}}{S^3}(S^2-8g\Gamma w')\\&\times\sin\left(\frac{1}{2}St+4g\frac{\Gamma}{S}w't\right)+\frac{8g}{S^3}(2\Gamma-\gamma_{\parallel})\Gamma w''\\&\left.\times\sinh\left(4g\frac{\Gamma}{S}w''t\right)\right]e^{-[\Gamma+(1/2)\gamma_{\parallel}]t},\end{aligned} \tag{4.18c}$$

$$\begin{aligned}\gamma_{0,0}(t)=[\eta_{0,0}(t)]^*=&\left\{-\frac{i}{S^2}(2\Gamma-\gamma_{\parallel})\left(g+i\Gamma w''\right.\right.\\&\left.+\frac{S^2-16g^2}{S^2}\Gamma w'\right)\left[\cos\left(\frac{1}{2}St+4g\frac{\Gamma}{S}w't\right)\right.\\&\left.-\cosh\left(4g\frac{\Gamma}{S}w''t\right)\right]+\frac{i}{S}\left(g+\Gamma w\frac{S^2-8g^2}{S^2}\right)\\&\times\left[\sin\left(\frac{1}{2}St+4g\frac{\Gamma}{S}w't\right)\right.\\&\left.\left.+i\,\sinh\left(4g\frac{\Gamma}{S}w''t\right)\right]\right\}e^{-[\Gamma+(1/2)\gamma_{\parallel}]t},\end{aligned} \tag{4.18d}$$

where S is now given by $S=[16g^2-(2\Gamma-\gamma_{\parallel})^2]^{1/2}$. As before, the solutions for the overdamped case follow by analytical continuation of the square root S. The effect of such an analytical continuation is easily seen. It basically interchanges the roles of the hyperbolic and trigonometric functions.

The explicit forms for the coefficients of the density matrix as given here yield direct information on the evolution of physical properties of the system, such as the average photon number or the average population inversion. Indeed, one easily proves the relations

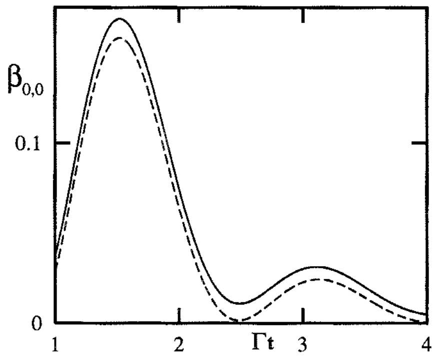

FIG. 1. Average atomic population $\beta_{0,0}$ as a function (—) of the dimensionless time Γt for $s=0.15$, $\Gamma=3\gamma_\parallel$, and $g_0=2\Gamma$, compared with the prediction (– –) of standard damping theory.

$$\langle a^\dagger(z)a(z)\rangle(t)=\alpha_{1,0}(t), \tag{4.19a}$$

$$\langle \sigma_z\rangle(t)=2\beta_{0,0}(t)-1, \tag{4.19b}$$

$$\langle a(z)\sigma_+\rangle(t)=[\langle a^\dagger(z)\sigma_-\rangle(t)]^*=4\gamma_{0,0}(t). \tag{4.19c}$$

As follows from the results presented here, the correction term in the master equation leads to several modifications in the time evolution of the density matrix. These modifications are determined by the nonideality parameter w. The results from standard damping theory are recovered by setting w equal to zero in all formulas.

The expressions (4.18) depend on the coupling constant $g(z)=\bar{g}\mathcal{N}(z)$, both directly and through the square root S. By using (3.10) and (4.4) we can rewrite $g(z)$ as $g_0(z)-\frac{1}{2}\Gamma w'(z)$ in first order of s. Here we introduced the zeroth-order coupling constant $g_0(z)=\bar{g}\sin(k_0 l\zeta)$. Correspondingly, $S(z)$ may be expanded up to terms linear in s; the result is $S_0(z)-8g_0(z)\Gamma w'(z)/S_0(z)$, where we defined $S_0(z)=[16g_0(z)^2-(2\Gamma-\gamma_\parallel)^2]^{1/2}$. With the use of these expansions the coefficients (4.18) can be simplified considerably. The coefficient $\beta_{0,0}$, for instance, which determines the average atomic population inversion, becomes

$$\begin{aligned}\beta_{0,0}(t)=\Bigg[&\frac{S_0^2-8g_0^2}{S_0^2}\cos\left(\frac{1}{2}S_0t\right)+\frac{8g_0^2}{S_0^2}\cosh\left(4g_0\frac{\Gamma}{S_0}w''t\right)\\&+\frac{2\Gamma-\gamma_\parallel}{S_0}\sin\left(\frac{1}{2}S_0t\right)+\frac{8g_0}{S_0^3}(2\Gamma-\gamma_\parallel)\Gamma w''\\&\times\sinh\left(4g_0\frac{\Gamma}{S_0}w''t\right)\Bigg]e^{-[\Gamma+(1/2)\gamma_\parallel]t}.\end{aligned} \tag{4.20}$$

As can be seen from this expression, the damped Rabi oscillations of the average atomic population inversion, which are characteristic for a decaying atom in a cavity, get modified if the mirrors are no longer nearly perfect. The changes in the time evolution of the population inversion are brought about by hyperbolic functions that depend on w''. Its counterpart w' has no effect on the evolution of the population inversion, although it does influence the average photon number.

The changes in the evolution of the atomic population due to the finite mirror transmittivity depend quite sensitively on the position of the atom. This fact has been observed before in the context of the numerical studies presented in [5]. Here it can be established analytically by considering the dependence of w'' on z, which follows from (3.14) and (4.4):

$$w''(z)=-2\zeta s\frac{g_0(z)}{\Gamma}\frac{\sin(2k_0l\zeta)}{1-\cos(2k_0l\zeta)}. \tag{4.21}$$

For an atomic position z at an antinode one has $w''(z)=0$, so that the damped Rabi oscillations remain unperturbed in that case. On the other hand, for values of z halfway between a node and a antinode one has $|w''(z)|=2\zeta s g_0(z)/\Gamma$, so that the hyperbolic functions do modify the oscillations in the atomic population inversion for such configurations. An example of such a modification of the time evolution is shown in Fig. 1, where the behavior of the population inversion is plotted for values of t that are of the order of Γ^{-1}. The influence of the finite mirror transmittivity is particularly manifest for those values of t where the oscillating atomic population inversion reaches a minimum. In fact, for a cavity with nearly perfect mirrors (or for atoms situated at an antinode) the population inversion really vanishes periodically, whereas for the nonideal case the population inversion remains finite at the minima and becomes zero only for infinite t. This feature has also been found in the numerical work by Feng and Ujihara [14]. In the overdamped case the time evolution of the atomic population inversion is modified as well. The corrections are determined then by trigonometric functions, which give oscillating contributions. Whether the system is overdamped or not depends on the magnitude of Γ as compared to g_0. It should be noted that the ratio of these parameters can be chosen independently of the value of $s=\Gamma l/c$. The latter has to be small in order that the present theory be valid.

V. CONCLUSION

The main results of the present paper are the master equation (3.19) and its multimode generalization (3.20). These equations describe the time evolution of the density matrix for a system inside a cavity with mirrors of a small but finite transmittivity. As compared to the standard master equation, which is valid for systems in cavities with vanishingly small transmittivity, the equations derived here contain additional terms. These terms arise from the interplay of the photon escape through the mirrors, on the one hand, and the interaction of the electromagnetic field inside the cavity, on the other hand.

To derive the master equation we had to introduce several simplifying assumptions. First of all, we neglected the spatial variation of the fields in the directions tranverse to the axis of the cavity, that is, we adopted a one-dimensional description of the system. Furthermore, we assumed that the fields interacts only at a single (one-dimensional) position in the cavity, so that only the field at that point enters the interaction Hamiltonian. Starting from these assumptions we could derive Langevin equations and master equations in which the influence of the transmittivity of the mirrors has been taken into account in a perturbative way. In deriving the master equation we supposed that the degrees of freedom of the electromagnetic field that are associated with the stochastic force are not excited. It is straightforward to generalize the theory so as to include the effects of excitation of these degrees of freedom.

The additional terms in the master equation have a rather simple structure. It is therefore relatively easy to determine the consequences of a finite mirror transmittivity in any specific model that satisfies the general assumptions described above. To show this we have evaluated the time dependence of the density operator that describes the decay of a single excited two-level atom in an otherwise empty cavity. From our results certain features, such as the modification of the damped Rabi oscillations brought about by the finiteness of the mirror transmittivity, could be studied analytically, whereas previous treatments had to depend on numerical methods.

It should be stressed that the example of the decaying atom was presented only as an illustration of the applicability of our master equations. For that reason, we confined ourselves in the discussion to the time dependence of the atomic population inversion starting from a simple initial condition. The generalization to different initial conditions is straightforward. Other models with effective nonlinear interactions of the fields inside the cavity may likewise be discussed on the basis of the master equations found here.

ACKNOWLEDGMENTS

We would like to thank Dr. A. J. van Wonderen for a stimulating discussion and for his valuable comments on an earlier version of the paper.

APPENDIX

In this appendix we will present some details of the derivation of the Langevin equation (3.15). The memory function defined in (2.11) can be written as

$$F(z,t)=\frac{1}{2[\mathcal{N}(z)]^2}\Big\{I(s,\tau)-\frac{1}{2}\cos(2k_0l\zeta)[I(s,\tau+2\zeta)+I(s,\tau-2\zeta)]+\frac{i}{2}\sin(2k_0l\zeta)[I(s,\tau+2\zeta)-I(s,\tau-2\zeta)]\Big\}, \tag{A1}$$

with $\tau=ct/l$, $\zeta=1-z/l$, and

$$[\mathcal{N}(z)]^2=\frac{1}{2}[I(s,0)-\cos(2k_0l\zeta)I(s,2\zeta)]. \tag{A2}$$

Here $I(s,\tau)$ is defined by

$$I(s,\tau)=\frac{\sqrt{1+s^2}}{\pi}\int_{-\pi/2}^{\pi/2}dx\,\frac{s}{s^2+\sin^2x}\cos(x\tau). \tag{A3}$$

The integral $I(s,\tau)$, which is even in τ, can be split in two parts by writing $s/(s^2+\sin^2x)$ as the sum of $s/(s^2+x^2)$ and a remainder. For small s and $\tau\geqslant 0$ the first part is

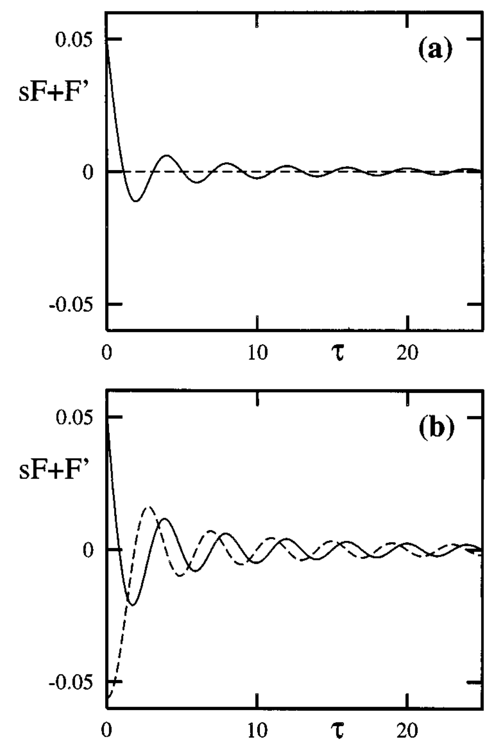

FIG. 2. Real (—) and imaginary (– –) parts of $(s+\partial/\partial\tau)F(z,t)$ as a function of $\tau=ct/l$, for $s=0.05$, $z=0.5l$, and either (a) $\sin[k_0(l-z)]=1$ or (b) $\sin[k_0(l-z)]=\cos[k_0(l-z)]=1/\sqrt{2}$.

$$\frac{\sqrt{1+s^2}}{\pi}\int_{-\pi/2}^{\pi/2}dx\,\frac{s}{s^2+x^2}\cos(x\tau)=e^{-s\tau}+s\tau\left[1-\frac{2}{\pi}\mathrm{Si}(\pi\tau/2)\right]-\frac{4s}{\pi^2}\cos(\pi\tau/2)+\mathcal{O}(s^2). \tag{A4}$$

The second part can likewise be calculated approximately for small s. For $\tau\geqslant 0$ it becomes

$$\frac{s}{\pi}\int_{-\pi/2}^{\pi/2}dx\,\frac{x^2-\sin^2x}{x^2\sin^2x}\cos(x\tau)+\mathcal{O}(s^2)=-s\tau\left[1-\frac{2}{\pi}\mathrm{Si}(\pi\tau/2)\right]+\frac{4s}{\pi^2}\cos(\pi\tau/2)+\frac{2s}{\pi}\sin(\pi\tau/2)[\tau\beta(\tau/2)-1]+\mathcal{O}(s^2), \tag{A5}$$

where [15] has been used. Here $\beta(x)$ is defined as $\frac{1}{2}\psi((x+1)/2)-\frac{1}{2}\psi(x/2)$, with $\psi(x)$ the digamma function.

Putting both parts of $I(s,\tau)$ together we have found, for $\tau\geqslant 0$,

$$I(s,\tau)=e^{-s\tau}+\frac{2s}{\pi}\sin(\pi\tau/2)[\tau\beta(\tau/2)-1]+\mathcal{O}(s^2). \tag{A6}$$

For $\tau=0$ one verifies that $I(s,0)=1$, while for $\tau>1$ one has $I(s,\tau)=\exp(-s\tau)+\mathscr{O}(s/\tau)+\mathscr{O}(s^2)$. Substitution of these results in (A1) and (A2) leads to (3.10) and (3.11).

The Langevin equation (3.15) is obtained by investigating the time behavior of the combination $(\Gamma+\partial/\partial t)F(z,t)$ of the memory kernel and its derivative, which occurs in (3.8). This combination is appreciably different from zero only for values of τ that are of the order of 1, so that t is of the order of l/c. Its time dependence is plotted in Fig. 2 in dimensionless units, for z in the middle of the cavity and either at an antinode or halfway between a node and an antinode. As we assumed that the time scale for the evolution of all observables is much larger than l/c (after the optical frequency has been transformed away), we can now calculate the last term in (3.8) for $t \gg l/c$ by the substitution

$$\frac{i}{\hbar}\int_0^t dt'\left(\frac{\partial}{\partial t}F(z,t-t')+\Gamma F(z,t-t')\right)e^{-i\omega_0(t-t')}[H_{\text{int}}(z,t'),a(z,t')]$$
$$\rightarrow \frac{i}{\hbar}\left[\int_0^\infty dt'\left(\frac{\partial}{\partial t'}F(z,t')+\Gamma F(z,t')\right)\right][H_{\text{int}}(z,t),a(z,t)]. \quad \text{(A7)}$$

Employing (A1) we obtain

$$\int_0^\infty dt'\left[\frac{\partial}{\partial t'}F(z,t')+\Gamma F(z,t')\right]=\frac{1}{2[\mathscr{N}(z)]^2}\left\{[1-\cos(2k_0 l\zeta)]\int_0^\infty d\tau\left(\frac{\partial}{\partial \tau}I(s,\tau)+sI(s,\tau)\right)\right.$$
$$\left.+\cos(2k_0 l\zeta)\int_0^{2\zeta} d\tau\frac{\partial}{\partial\tau}I(s,\tau)-i\ \sin(2k_0 l\zeta)\int_0^{2\zeta} d\tau\ I(s,\tau)\right\}. \quad \text{(A8)}$$

Inserting the approximate form (A6) and evaluating the integrals we finally get

$$\int_0^\infty dt'\left[\frac{\partial}{\partial t'}F(z,t')+\Gamma F(z,t')\right]=\Gamma\frac{l}{c}v(z)+\mathscr{O}(s^2), \quad \text{(A9)}$$

where we used (3.2) and (3.12)–(3.14). The Langevin equation (3.15) is found upon substitution of (A7) with (A9) in (20). As is obvious from the above derivation, the fast decay of the combination $(\Gamma+\partial/\partial t)F(z,t)$, with a decay time of the order of the cavity round-trip time l/c, is instrumental in establishing the Markov property of the ensuing Langevin equation.

[1] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1973).

[2] R. Lang, M. O. Scully, and W. E. Lamb, Jr., Phys. Rev. A **7**, 1788 (1973); R. Lang and M. O. Scully, Opt. Commun. **9**, 331 (1973).

[3] S. M. Barnett and P. M. Radmore, Opt. Commun. **68**, 364 (1988).

[4] R. J. Cook and P. W. Milonni, Phys. Rev. A **35**, 5081 (1987).

[5] X.-P. Feng and K. Ujihara, IEEE J. Quantum Electron. **25**, 2332 (1989); Phys. Rev. A **41**, 2668 (1990).

[6] H.-J. Briegel and B.-G. Englert, Phys. Rev. A **47**, 3311 (1993).

[7] In the articles by Feng and Ujihara [5] a different way of grouping the universe modes in quasimodes is used. Although this grouping has the nice effect of yielding Lorentzian linewidths, it leads to quasimodes that are associated with dependent degrees of freedom, as the overlap of different quasimodes is nonzero. This makes a multimode description somewhat more difficult. It is for this reason that we use the cutoff $|\Delta k|\leq\pi/(2l)$, which makes the quasimodes rigorously indepen-dent.

[8] In the article by Lang *et al.* [2], actually a different expression for Γ was employed. Instead of the factor $s=(1-r)/(2\sqrt{r})$, they used a factor of $(1-r^2)/(4r^2)$. The difference between these two factors is not important here.

[9] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).

[10] S. Haroche, in *New Trends in Atomic Physics*, Proceedings of the Les Houches Summer School of Theoretical Physics, Session XXXVIII, edited by G. Grynberg and R. Stora (North-Holland, Amsterdam, 1984), Vol. 1, p. 193.

[11] S. Sachdev, Phys. Rev. A **29**, 2627 (1984).

[12] J. I. Cirac, H. Ritsch, and P. Zoller, Phys. Rev. A **44**, 4541 (1991).

[13] F. De Martini *et al.*, Phys. Rev. A **43**, 2480 (1991).

[14] See Fig. 9a of the first paper in [5].

[15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed. (Academic, San Diego, 1992), Sec. 3.624. [Note that reprints 1 through 5 contain an error here. Refer to the sixth reprint.]