Spontaneous emission from an atom in a nonindeal cavity: Application of a generalized master equation

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Spontaneous emission from an atom in a nonideal cavity: Application of a generalized master equation

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(Received 19 April 1996)

The decay of an excited two-level atom in a cavity with mirrors of finite transmittivity is studied with the help of a generalized multimode master equation that has been established recently. The time evolution of the decaying atom is obtained by accounting for the interaction with infinitely many quasimodes. The analytical and numerical results are compared with those of the standard multimode master equation and of a universe-mode approach. [S1050-2947(96)11109-4]

PACS number(s): 42.50.−p

I. INTRODUCTION

For systems with lossy cavities a master equation with standard damping terms [1] is often used. In the limit of an ideal cavity, with mirrors of vanishing transmittivity, this master equation can be derived from the so-called universe-mode picture, in which the evolution is governed by a Hamiltonian.

When the transmittivity becomes finite, the universe-mode description no longer leads to the standard master equation, but correction terms appear. In a recent article [2] we derived the corrections to the standard master equation in first order in the transmittivity, and thus obtained a "generalized" master equation.

Application of a master equation instead of a Schrödinger equation as employed in the universe-mode approach has the well-known advantage of any contracted description: by limiting the number of degrees of freedom that are taken into account, one ends up with a reduced system of equations, which facilitates the analysis of models. However, before making extensive use of the effective description it is important to assess its reliability, for instance by studying a simple model system for which both the master equation and the universe-mode equations can be solved.

In this article we analyze the decay of an initially excited two-level atom inside a nonideal cavity. The limiting case of an ideal cavity has been studied before [3,4]. In a previous paper [2] we have determined the general features of the atomic decay in a nonideal cavity by confining ourselves to a single cavity quasimode. However, as we will see in the following, it is necessary to include all quasimodes of the cavity if one wishes to determine the details of the time evolution of the atom and of the emitted fields. Even if the atom is at resonance with one of these modes, it is in general not sufficient to limit oneself to a single-mode description.

In Sec. II we describe the system and formulate the generalized multimode master equation. In Sec. III we solve both the generalized and the standard multimode master equations. Subsequently, in Sec. IV we use the universe-mode approach to study the system. Finally, we compare the results of the various approaches in Sec. V and draw some conclusions.

II. GENERALIZED MULTIMODE MASTER EQUATION

We consider a two-level atom inside a cavity. The cavity is formed by one semitransparent and one perfectly reflecting mirror, at a distance l. An additional perfectly reflecting mirror, which is placed at a distance L from the semitransparent one, closes the system and so forms the "universe." At a later stage we will let L approach infinity.

We adopt a one-dimensional model to describe the decaying atom. In other words, only the electromagnetic field modes that propagate parallel to the cavity axis and that fit in the universe are taken into account. Furthermore, we assume that the interaction is given by the standard electric-dipole term, and that we may use the rotating-wave approximation. For simplicity we take the atom to be in resonance with the cavity.

In the universe-mode description, the time evolution of the system is given by a Schrödinger equation. We will use the parameter τ to denote the time in units of the cavity round-trip time $t_c = 2l/c$, and consequently absorb a factor of t_c in the Hamiltonian.[1] The latter consists of two parts: a "free" term

$$H_0 = \hbar \sum_k (\omega_k - \omega_A) a_k^\dagger a_k, \tag{2.1}$$

where the a_k are annihilation operators of the universe modes, and an interaction term

$$H_{int} = -\frac{1}{2}\hbar \bar{g}\sqrt{l}\mathcal{E}\sigma_+ + \text{H.c.}, \tag{2.2}$$

where $\sigma_+ = (\sigma_-)^\dagger$ is twice the raising operator of the atom. The frequency ω_A is the atomic transition frequency. We have moved to a picture in which only the differences between the atomic frequency and the photonic frequencies occur. All frequencies are in units of $1/t_c$ and are thus dimensionless. The photonic wave numbers k are spaced at π/L, and the corresponding frequencies ω_k at $2\pi l/L$. The factor $\sqrt{l}$ in (2.2) has been inserted for convenience.

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[1]Note that the definition of τ differs by a factor of 2 from the definition in the Appendix of [2].

The electric field $\mathcal{E}$ at the position of the atom is given by

$$\mathcal{E}(\zeta) = \frac{1}{\sqrt{L}} \sum_k M_k \sin(\omega_k \zeta/2) a_k, \qquad (2.3)$$

where the dimensionless parameter ζ determines the atomic position with respect to the mirrors: for $\zeta=0$ the atom is at the perfect mirror, for $\zeta=1$ at the semitransparent mirror.

The frequency dependence of the amplitudes of the universe-mode functions is contained in the coefficients M_k. For the resonant case, which is considered here, we have

$$M_k^2 = \frac{1-R}{(1-\sqrt{R})^2 + 4\sqrt{R}\sin^2\left[\frac{1}{2}(\omega_k - \omega_A)\right]}$$

$$= \frac{\frac{1}{2}\Gamma\sqrt{1+\Gamma^2/4}}{\Gamma^2/4 + \sin^2\left[\frac{1}{2}(\omega_k - \omega_A)\right]}. \qquad (2.4)$$

Here Γ is given by $(1-\sqrt{R})R^{-1/4}$ with R the (intensity) reflectivity of the semitransparent mirror.

The generalized master equation follows by retaining those linear combinations of the annihilation operators that occur in the electric field, and integrating out the other field degrees of freedom. In Appendix A we give a brief outline of the basic ideas. A more complete discussion is given in [2].

The master equation contains a Hamiltonian contribution and several damping terms. As before, the Hamiltonian H consists of a "free" part and an interaction part. However, the former is replaced by

$$H_0^m = 2\pi\hbar \sum_q n_q b_q^\dagger(\zeta) b_q(\zeta), \qquad (2.5)$$

where the b_q are annihilation operators of the quasimodes sustained by the cavity. The quasimodes are labeled by the integers n_q, from which the quasimode frequencies follow as $\omega_A + 2\pi n_q$. The interaction term is found by rewriting Eq. (2.2) in terms of the b_q:

$$H_{\text{int}} = -\frac{1}{2}\hbar \sum_q g_q(\zeta)[b_q(\zeta)\sigma_+ + b_q^\dagger(\zeta)\sigma_-], \qquad (2.6)$$

where $g_q(\zeta)$ is a space-dependent and quasimode-dependent coupling constant that is proportional to $\bar{g}$ (see Appendix A). From now on we will suppress the explicit dependence on ζ.

Having defined the Hamiltonian we write the generalized master equation as [2]

$$\partial_\tau \rho = -\frac{i}{\hbar}[H,\rho] + \mathcal{L}_g \rho + \mathcal{L}_A \rho. \qquad (2.7)$$

Here two damping contributions appear: a quasimode damping $\mathcal{L}_g$ and an atomic damping $\mathcal{L}_A$. The former consists of two parts as well: the standard damping term

$$\mathcal{L}_s \rho = \Gamma \sum_q \; ([b_q, \rho b_q^\dagger] + [b_q \rho, b_q^\dagger]), \qquad (2.8)$$

and a correction term

$$\mathcal{L}_c \rho = \frac{i}{2}\Gamma \sum_q \; (w_q[b_q^\dagger, \sigma_- \rho] - w_q^*[\rho\sigma_+, b_q]), \quad (2.9)$$

with coefficients w_q that are functions of ζ (see Appendix A), and that are proportional to $\bar{g}$. The atomic damping term has the usual form

$$\mathcal{L}_A \rho = \frac{\gamma_\parallel}{8}([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+]). \qquad (2.10)$$

It has been added as a phenomenological term to account for atomic decay through transverse field degrees of freedom.

In the limit that Γ and $\bar{g}$ (and $\gamma_\parallel$) go to zero, the standard master equation is recovered after a rescaling of the time. This is because the correction term goes as $\Gamma\bar{g}$, which vanishes more quickly than the other damping terms and the Hamiltonian terms.

It should be noted that the derivation of the generalized master equation as given in [2] is valid only for small values of Γ: the correction terms include the effects of a finite mirror transmittivity up to first order in Γ. In the derivation we used, moreover, that the coupling between the field and the atom is not too strong on a scale set by the cavity round-trip time, that is, $\bar{g}$ is not too large. In Sec. V we shall see how strict these conditions are. In fact, it is one of the aims of the present paper to investigate precisely this issue.

To solve the master equation it is convenient to write the density operator as [5]:

$$\rho(\tau) = \rho_0(\tau)\sigma_0 + \rho_z(\tau)\sigma_z + \rho_-(\tau)\sigma_- + \rho_+(\tau)\sigma_+, \qquad (2.11)$$

with $\sigma_0 = \frac{1}{2}(1-\sigma_z)$ the projection operator on the ground state of the atom. The photonic operators ρ_0, ρ_z, ρ_- and ρ_+ are determined by the following expectation values (see Appendix A)

$$\alpha_{pq}(\tau) = \langle b_p^\dagger b_q \rangle_\tau, \qquad (2.12a)$$

$$\beta_0(\tau) = \frac{1}{2}\langle \sigma_z + 1 \rangle_\tau, \qquad (2.12b)$$

$$\gamma_q(\tau) = \frac{i}{2}\langle b_q \sigma_+ \rangle_\tau, \qquad (2.12c)$$

$$\eta_q(\tau) = -\frac{i}{2}\langle b_q^\dagger \sigma_- \rangle_\tau. \qquad (2.12d)$$

Application of the master equation yields

$$\partial_\tau \alpha_{pq} = [-2\Gamma + i2\pi(n_p - n_q)]\alpha_{pq} - G_p^*\gamma_q - G_q\eta_p, \qquad (2.13a)$$

$$\partial_\tau \beta_0 = -\gamma_\parallel \beta_0 + \sum_q g_q(\gamma_q + \eta_q), \qquad (2.13b)$$

$$\partial_\tau \gamma_q = \sum_p g_p \alpha_{pq} - G_q \beta_0 + (-\Gamma - \gamma_\parallel/2 - i2\pi n_q)\gamma_q, \tag{2.13c}$$

$$\partial_\tau \eta_q = \sum_p g_p \alpha_{qp} - G_q^* \beta_0 + (-\Gamma - \gamma_\parallel/2 + i2\pi n_q)\eta_q. \tag{2.13d}$$

The coefficients g_q and G_q are defined in Appendix A. In the following we only need the product of these two. In first order in Γ we have

$$g_q G_q = \frac{\bar{g}^2}{2}[1 - \cos(\omega_A \zeta + 2\pi n_q \zeta) - i\Gamma \zeta \sin(\omega_A \zeta + 2\pi n_q \zeta)]. \tag{2.14}$$

The coefficients α_{pq} and β_0 determine the dynamics of the field energy density and of the atomic upper-level population. In fact, the probability to find the atom in its upper level is given by the coefficient β_0:

$$P(\tau) = \frac{1}{2}\langle \sigma_z + 1 \rangle_\tau = \beta_0(\tau). \tag{2.15}$$

Furthermore, the expectation value of the electromagnetic energy density at the location of the atom is proportional to

$$E(\tau) = \langle \mathcal{E}^\dagger \mathcal{E} \rangle_\tau = \frac{1}{l}\sum_{pq} \frac{g_p g_q}{\bar{g}^2}\alpha_{pq}(\tau). \tag{2.16}$$

Both of these will be evaluated in the following.

III. SOLVING THE MASTER EQUATION

To solve the set of Eqs. (2.13a)–(2.13d) we accumulate the coefficients in a vector

$$\mathcal{X}(\tau) = (\beta_0, \{\eta_q\}; \gamma_1, \{\alpha_{q1}\}; \gamma_2, \{\alpha_{q2}\}; \dots)^T, \tag{3.1}$$

where T stands for transpose. The master equation is now equivalent to the matrix equation

$$\partial_\tau \mathcal{X}(\tau) = \mathcal{M} \cdot \mathcal{X}(\tau). \tag{3.2}$$

The matrix $\mathcal{M}$ can be written as

$$\mathcal{M} = M \otimes \mathbf{1} + \mathbf{1} \otimes M^*, \tag{3.3}$$

where we have

$$M = \begin{pmatrix} -\gamma_\parallel/2 & g_1 & g_2 & \cdots \\ -G_1 & -\Gamma - i2\pi n_1 & 0 & \cdots \\ -G_2 & 0 & -\Gamma - i2\pi n_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.4}$$

The solution of the eigenvalue problem associated with the matrix equation (3.2) is discussed in Appendix B. The eigenvalue problem for M leads to the equation

$$\lambda + \gamma_\parallel/2 + \sum_q \frac{g_q G_q}{\Gamma + \lambda + i2\pi n_q} = 0. \tag{3.5}$$

Having solved the eigenvalue problem it is straightforward to determine the expectation values of the atomic upper-level population and of the field energy density. In particular, it can be shown that the probability of finding the atom in its excited state is

$$P(\tau) = \left| \mathcal{L}^{-1}\left(\frac{1}{\lambda + \gamma_\parallel/2 + J_g(\lambda)}\right)\right|^2, \tag{3.6}$$

where we have

$$J_g(\lambda) = \sum_q \frac{g_q G_q}{\Gamma + \lambda + i2\pi n_q}, \tag{3.7}$$

and where $\mathcal{L}^{-1}$ indicates the inverse Laplace transform. Similarly, the average field energy density follows from the coefficients α_{pq}. These are found as

$$\alpha_{pq}(\tau) = \left[\mathcal{L}^{-1}\left(\frac{G_p}{\Gamma + \lambda + i2\pi n_p}\frac{1}{\lambda + \gamma_\parallel/2 + J_g(\lambda)}\right)\right]^* \\ \times \mathcal{L}^{-1}\left(\frac{G_q}{\Gamma + \lambda + i2\pi n_q}\frac{1}{\lambda + \gamma_\parallel/2 + J_g(\lambda)}\right). \tag{3.8}$$

Note that the factors $\Gamma + \lambda + i2\pi n_{p,q}$ in the denominators do not introduce extra poles. With (3.7) we get

$$E(\tau) = \frac{1}{\bar{g}^2 l}\left|\mathcal{L}^{-1}\left(\frac{J_g(\lambda)}{\lambda + \gamma_\parallel/2 + J_g(\lambda)}\right)\right|^2. \tag{3.9}$$

If only the resonant quasimode is retained, we obtain from (3.7)

$$J_g^{(1)}(\lambda) = \frac{g_1 G_1}{\Gamma + \lambda} = J_s^{(1)}(\lambda) + J_c^{(1)}(\lambda), \tag{3.10}$$

where

$$J_s^{(1)}(\lambda) = \frac{\bar{g}^2}{2}\frac{1 - \cos(\omega_A \zeta)}{\Gamma + \lambda} \tag{3.11}$$

is the contribution that follows from the standard master equation, and where

$$J_c^{(1)}(\lambda) = \frac{\bar{g}^2}{2}\frac{-i\Gamma \zeta \sin(\omega_A \zeta)}{\Gamma + \lambda} \tag{3.12}$$

is a term that is due to the corrections in the generalized master equation.

If all quasimodes are taken into account, we have

$$J_g^{(\infty)}(\lambda) = \sum_q \frac{g_q G_q}{\Gamma + \lambda + i2\pi n_q}$$

$$= \sum_{n=-\infty}^{\infty} \frac{\bar{g}^2}{2} \frac{1 - \cos(\omega_A \zeta + 2\pi n\zeta) - i\Gamma \zeta \sin(\omega_A \zeta + 2\pi n\zeta)}{\Gamma + \lambda + i2\pi n}. \qquad (3.13)$$

With the use of the summation formulas

$$\sum_{n=-\infty}^{\infty} \frac{\sin(nx+y)}{na+b} = \frac{\pi}{a} \frac{\sin(b(\pi-x)/a+y)}{\sin(b\pi/a)} \quad (0 < x < 2\pi), \qquad (3.14)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{1}{na+b} = \frac{\pi}{a} \frac{1}{\tan(b\pi/a)}, \qquad (3.15)$$

where the latter summation is done in a symmetric way, we get

$$J_g^{(\infty)}(\lambda) = J_s^{(\infty)}(\lambda) + J_c^{(\infty)}(\lambda), \qquad (3.16)$$

with

$$J_s^{(\infty)}(\lambda) = \frac{\bar{g}^2}{4(1 - e^{-\Gamma - \lambda})} (1 + e^{-\Gamma - \lambda} - e^{-\zeta(\Gamma + \lambda) + i\omega_A \zeta}$$

$$- e^{-(1-\zeta)(\Gamma + \lambda) - i\omega_A \zeta}), \qquad (3.17)$$

$$J_c^{(\infty)}(\lambda) = \frac{\bar{g}^2}{4(1 - e^{-\Gamma - \lambda})} (-\Gamma \zeta e^{-\zeta(\Gamma + \lambda) + i\omega_A \zeta}$$

$$+ \Gamma \zeta e^{-(1-\zeta)(\Gamma + \lambda) - i\omega_A \zeta}). \qquad (3.18)$$

A discussion of both the single-mode and the multimode case will be given in Sec. V.

IV. UNIVERSE-MODE APPROACH

In this section we will treat the system in the universe-mode approach. The spectrum of the universe modes is dense, so that, even when considering only a few quasimodes, infinitely many modes have to be taken into account. Nevertheless, it turns out to be feasible to deduce the dynamics of a decaying atom in this formalism, so that a comparison with our earlier results is possible. However, there is a restriction: additional atomic damping effects, like those introduced in the previous sections so as to account for the coupling to transverse modes, cannot be incorporated in the universe-mode approach without abandoning a purely Hamiltonian description. Hence, we have to set $\gamma_\parallel = 0$ before a comparison with the results of Sec. III is made.

In the universe-mode picture the time evolution is given by a Schrödinger equation with a Hamiltonian given by (2.1) and (2.2). We write the state vector as

$$|\Psi\rangle(\tau) = C(\tau)|e\rangle \otimes |0, \dots\rangle + \sum_k D_k(\tau)|g\rangle \otimes |1_k, \{0\}\rangle. \qquad (4.1)$$

Here $|g\rangle$ and $|e\rangle$ denote the ground and excited atomic states, respectively, and $|1_k, \{0\}\rangle$ denotes the state with one photon with wave number k. Solution of the Schrödinger equation leads to formal integral expressions for the coefficients $D_k(\tau)$:

$$D_k(\tau) = i\bar{g} \sqrt{\frac{l}{L}} M_k \sin(\omega_k \zeta/2) \int_0^\tau d\tau' e^{i(\omega_k - \omega_A)(\tau' - \tau)} C(\tau'). \qquad (4.2)$$

Insertion in the Schrödinger equation results in a closed equation for $C(\tau)$:

$$\partial_\tau C(\tau) = -\bar{g}^2 \frac{l}{L} \sum_k M_k^2 \sin^2(\omega_k \zeta/2)$$

$$\times \int_0^\tau d\tau' e^{i(\omega_k - \omega_A)(\tau' - \tau)} C(\tau'). \qquad (4.3)$$

This integro-differential equation can be transformed into a delay-differential equation for $C(\tau)$ [6,7]. However, we will not follow that path and use instead the technique of Laplace transformations to solve the equation.

After Laplace transformation the integro-differential equation (4.3) gets the form

$$\lambda C_{\mathcal{L}}(\lambda) - 1 = -\bar{g}^2 \mathcal{L} \left(\frac{l}{L} \sum_k M_k^2 \sin^2(\omega_k \zeta/2) e^{-i(\omega_k - \omega_A)\tau} \right)$$

$$\times C_{\mathcal{L}}(\lambda), \qquad (4.4)$$

where we have inserted the initial condition $C(0) = 1$. Upon taking the inverse Laplace transform we arrive at an expression for the probability $P(\tau)$ of finding the atom in the excited state:

$$P(\tau) = |C(\tau)|^2 = \left| \mathcal{L}^{-1} \left(\frac{1}{\lambda + J_u(\lambda)} \right) \right|^2. \qquad (4.5)$$

This expression is of the same form as Eq. (3.6), with $\gamma_\parallel = 0$. However, the function in the denominator is now found to be

$$J_u(\lambda) = \bar{g}^2 \frac{l}{L} \sum_k M_k^2 \sin^2(\omega_k \zeta/2) \frac{1}{\lambda + i(\omega_k - \omega_A)}. \qquad (4.6)$$

In a similar way we can also determine the field energy density at the position of the atom in terms of the coefficients $D_k(\tau)$. We find

$$E(\tau) = \frac{1}{L} \left| \sum_k M_k \sin(\omega_k \zeta/2) D_k \right|^2. \tag{4.7}$$

Substitution of Eq. (4.2) and of the solution of Eq. (4.4) yields

$$E(\tau) = \frac{1}{\bar{g}^2 l} \left| \mathcal{L}^{-1}\left(\frac{J_u(\lambda)}{\lambda + J_u(\lambda)} \right) \right|^2, \tag{4.8}$$

where we used the convolution theorem for Laplace transforms. Again, this expression has the same form as its counterpart, (3.9) with $\gamma_\| = 0$, albeit with a different function $J_u(\lambda)$.

In the continuum limit we may evaluate the function $J_u(\lambda)$ by making the replacement

$$\sum_k F(\omega_k) \rightarrow \frac{L}{2\pi l} \int dx F(x + \omega_A). \tag{4.9}$$

Then $J_u(\lambda)$ becomes

$$J_u(\lambda) = \frac{\bar{g}^2}{2\pi} \int dx \frac{\frac{1}{2}\Gamma \sqrt{1 + \Gamma^2/4}}{\Gamma^2/4 + \sin^2(x/2)} \sin^2(x\zeta/2 + \omega_A \zeta/2) \frac{1}{\lambda + ix}, \tag{4.10}$$

where we have substituted the expression (2.4) for the coefficients M_k. We now use the expansion

$$\frac{\frac{1}{2}\Gamma \sqrt{1 + \Gamma^2/4}}{\Gamma^2/4 + \sin^2(x/2)} = \sum_{n=-\infty}^{\infty} \left(\frac{i}{x + 2\pi n + i\Gamma'} - \frac{i}{x + 2\pi n - i\Gamma'} \right), \tag{4.11}$$

with $\Gamma' = 2\operatorname{arcsinh}(\Gamma/2) = -(1/2)\log R$. The integral in Eq. (4.10) can then be evaluated. Before treating the general case, we will consider the approximation in which only the resonant quasimode is retained. In this case the integration in Eq. (4.10) is limited to the interval $(-\pi, \pi)$. For small values of λ we can approximate the integral by evaluating it along a contour around the $n = 0$ poles. We then get

$$J_u^{(1)}(\lambda) = \frac{\bar{g}^2}{2} \frac{1 - \cos(\omega_A \zeta) - i\Gamma' \zeta \sin(\omega_A \zeta)}{\Gamma' + \lambda}. \tag{4.12}$$

As this expression is valid only in the neighborhood of $\lambda = 0$, we have to check under which condition it can be used in Eqs. (4.5) and (4.8). From these expressions it follows that one should require that the zeros of the equation $\lambda + J_u(\lambda) = 0$ are situated near the origin. This is indeed the case provided $\bar{g}$ is small compared to 1.

In the multimode case the integration over x is extended over the whole interval $(-\infty, \infty)$. The integration can then be performed exactly by contour integration around all poles, with the result

$$J_u^{(\infty)}(\lambda) = -\frac{\bar{g}^2}{4} \sum_{n=-\infty}^{\infty} \left(\frac{e^{-i\omega_A\zeta - \Gamma'\zeta - i2\pi n\zeta} + e^{i\omega_A\zeta - \zeta\lambda} - 2}{\lambda + \Gamma' + i2\pi n} \right. $$
$$\left. + \frac{e^{i\omega_A\zeta - \Gamma'\zeta + i2\pi n\zeta} - e^{i\omega_A\zeta - \zeta\lambda}}{\lambda - \Gamma' + i2\pi n} \right). \tag{4.13}$$

With the use of Eqs. (3.14) and (3.15), we finally get

$$J_u^{(\infty)}(\lambda) = \frac{\bar{g}^2}{4(1 - e^{-\Gamma'-\lambda})} (1 + e^{-\Gamma'-\lambda} - e^{-\zeta\lambda + i\omega_A\zeta} $$
$$- e^{-(1-\zeta)\lambda - \Gamma' - i\omega_A\zeta}). \tag{4.14}$$

Either Eq. (4.12) or Eq. (4.14) should be substituted into Eqs. (4.5) and (4.8). In Sec. V the resulting expressions for $P(\tau)$ and $E(\tau)$ will be compared to those following from the master equation.

V. DISCUSSION

In the previous sections we derived general expressions for the probability $P(\tau)$ of the atom to be in its excited state as well as for the average field energy density $E(\tau)$ at the location of the atom:

$$P(\tau) = \left| \mathcal{L}^{-1}\left(\frac{1}{\lambda + \gamma_\|/2 + J(\lambda)} \right) \right|^2, \tag{5.1a}$$

$$E(\tau) = \frac{1}{\bar{g}^2 l} \left| \mathcal{L}^{-1}\left(\frac{J(\lambda)}{\lambda + \gamma_\|/2 + J(\lambda)} \right) \right|^2$$
$$= \frac{1}{\bar{g}^2 l} \left| e^{-\gamma_\|\tau/2} \partial_\tau \left[e^{\gamma_\|\tau/2} \mathcal{L}^{-1}\left(\frac{1}{\lambda + \gamma_\|/2 + J(\lambda)} \right) \right] \right|^2. \tag{5.1b}$$

Here the explicit form of $J(\lambda)$ is different for the three theories considered, namely, the generalized and the standard master equations, and the universe-mode approach. Moreover, in the latter theory we had to assume $\gamma_\| = 0$.

In this section we will compare the results from the three theories, both analytically and numerically. In doing so we can make a further distinction, on the basis of the number of quasimodes that are taken into account. If only the resonant (or central) quasimode is incorporated, the expressions for the function $J(\lambda)$ are in the three approaches:

$$J_g^{(1)}(\lambda) = \frac{\bar{g}^2}{2} \frac{1 - \cos(\omega_A \zeta) - i\Gamma \zeta \sin(\omega_A \zeta)}{\Gamma + \lambda}, \tag{5.2a}$$

$$J_s^{(1)}(\lambda) = \frac{\bar{g}^2}{2} \frac{1 - \cos(\omega_A \zeta)}{\Gamma + \lambda}, \tag{5.2b}$$

$$J_u^{(1)}(\lambda) = \frac{\bar{g}^2}{2} \frac{1 - \cos(\omega_A \zeta) - i\Gamma' \zeta \sin(\omega_A \zeta)}{\Gamma' + \lambda}. \tag{5.2c}$$

It should be noted here that the difference between Γ and Γ' is of third order in Γ (or Γ'). Even for a cavity with 50% loss per round-trip ($R = 0.50$), the difference between Γ and Γ' is only about 1%. As both the standard and the

generalized master equation are derived for small Γ, with the neglect of terms of order higher than linear in Γ and $\Gamma\overline{g}$, one may replace Γ by Γ' in both Eqs. (5.2a) and (5.2b). If that is done, the generalized master equation and the universe-mode approach give the same result in the single-mode approximation, at least if one takes $\gamma_{\parallel}=0$. The standard master equation, however, coincides with the other two in a few special cases only, with the atom either near the ideal mirror ($\zeta=0$), or at an antinode $[\omega_A\zeta=\pi\,(\mathrm{mod}\,2\pi)]$. Other cases for which the theories coincide, namely, for $\Gamma=0$ or for atomic positions at a node $[\omega_A\zeta=0\,(\mathrm{mod}\,2\pi)]$, are trivial.

The single-mode master equations (the generalized as well as the standard) are easily applied to any model in which the electric field interacts with a source at a localized position inside a nonideal cavity. However, the restriction to one mode does lead to some loss of the details of the time evolution. This effect can be demonstrated for the decaying atom by making a comparison with the predictions of the multimode master equations.

In the multimode case the expressions for $J(\lambda)$ are

$$J_g^{(\infty)}(\lambda)=\frac{\overline{g}^2}{4(1-e^{-\Gamma-\lambda})}[1+e^{-\Gamma-\lambda}$$
$$-(1+\Gamma\zeta)e^{-\zeta(\Gamma+\lambda)+i\omega_A\zeta}$$
$$-(1-\Gamma\zeta)e^{-(1-\zeta)(\Gamma+\lambda)-i\omega_A\zeta}],\quad(5.3a)$$

$$J_s^{(\infty)}(\lambda)=\frac{\overline{g}^2}{4(1-e^{-\Gamma-\lambda})}(1+e^{-\Gamma-\lambda}-e^{-\zeta(\Gamma+\lambda)+i\omega_A\zeta}$$
$$-e^{-(1-\zeta)(\Gamma+\lambda)-i\omega_A\zeta}),\quad(5.3b)$$

$$J_u^{(\infty)}(\lambda)=\frac{\overline{g}^2}{4(1-e^{-\Gamma'-\lambda})}(1+e^{-\Gamma'-\lambda}-e^{-\zeta\lambda+i\omega_A\zeta}$$
$$-e^{-(1-\zeta)\lambda-\Gamma'-i\omega_A\zeta}).\quad(5.3c)$$

The inverse Laplace transformations in Eqs. (5.1a) and (5.1b) can be performed analytically in the single-mode approximation, with results that agree with those obtained earlier for the generalized [2] and standard master equation [2–4]. In the multimode case it is more convenient to carry out the inverse Laplace transformations numerically. To compare the predictions of the single-mode and the multimode approaches, we present in Fig. 1 the function $P(\tau)$ following from the generalized master equations for the same parameter values and about the same range of the time variable as in Figure 1 of Ref. [2]. We see that the qualitative behavior of $P(\tau)$ as a function of time, namely, as a damped oscillatory function, is conserved in going from the single-mode to the multimode theory. Quantitatively, however, the two approaches lead to different results: the multimode theory predicts more structure in the time evolution. It is interesting to note that neither of the two functions become zero for any finite time. The single-mode standard master
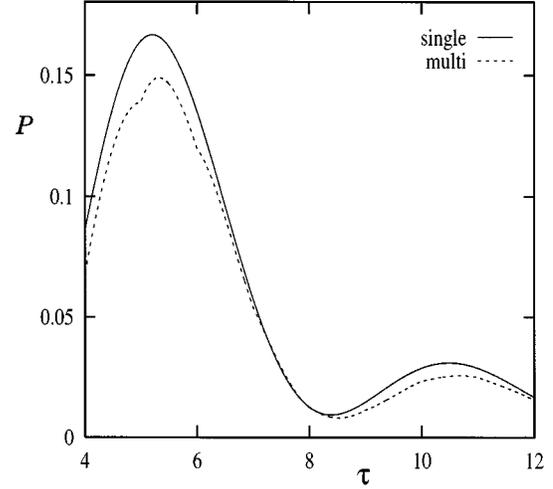

FIG. 1. Upper-level probability $P(\tau)$ as a function of the (dimensionless) time τ. Results follow from the single-mode and multimode generalized master equations for an initially excited atom in the middle of the cavity ($\zeta=0.5$), halfway between a node and an antinode of the resonant quasimode $[\omega_A\zeta=\pi/2\,(\mathrm{mod}\,\pi)]$, and for $\overline{g}^2=0.72$, $\Gamma=0.30$, and $\gamma_{\parallel}=0.10$.

equation predicts that $P(\tau)$ does vanish at regular time intervals, as had been noted in our earlier paper [2].

Having discussed the single-mode theories we will limit our discussion from now on to the multimode theories, with functions $J(\lambda)$ given in Eqs. (5.3a)–(5.3c). Neglecting again the difference between Γ and Γ', we find that the three approaches agree if the atom is near the ideal mirror ($\zeta=0$), as in the single-mode case. The three approaches no longer coincide for an atom at an antinode of the resonant quasimode. However, the master equations give identical results, different from those of the universe-mode approach, if the atom is situated in the middle of the cavity, either at a node or an antinode of the resonant quasimode $[\omega_A\zeta=0\,(\mathrm{mod}\,\pi)]$.

For arbitrary ζ and $\omega_A\zeta$ the generalized master equation and the universe-mode approach yield results that are equal up to second order in Γ. The standard master equation deviates already in first order in Γ. Hence, the generalized master equation gives a better description than the standard master equation for cavities with a nonideal mirror. This conclusion holds true even if the coupling constant $\overline{g}$ is of the order of unity. In deriving the generalized master equation we had to assume that $\overline{g}$ is not too large, but this turns out not to be a very strict condition, at least for the present physical example. Indeed, it looks as though the generalized master equation might well be adequate in a larger region than that implied by the derivation in [2]. The examples that will be discussed in the following corroborate this statement.

The time evolution of both $P(\tau)$ and $E(\tau)$ is determined by the eigenvalues λ that are solutions of the equation $\lambda+\gamma_{\parallel}/2+J(\lambda)=0$. Of course, the physics of the system imply that for infinite τ both functions should vanish, and hence that the real parts of the eigenvalues λ must be negative. For the universe-mode function (5.3c) this property can be proved rigorously, since the eigenvalues have to satisfy the equation

$$\lambda' + \gamma_\|/2 + \frac{\bar{g}^2}{4} \frac{\sinh(x+y) - \sinh(x)\cos(u) - \sinh(y)\cos(v)}{\cosh(x+y) - \cos(u+v)} = 0, \tag{5.4}$$

with $\lambda = \lambda' + i\lambda''$ and with $x = \zeta\lambda'$, $y = \Gamma' + (1-\zeta)\lambda'$, $u = (1-\zeta)\lambda'' + \omega_A\zeta$, $v = \zeta\lambda'' - \omega_A\zeta$. For $\lambda' > 0$ both x and y are positive. It follows that Eq. (5.4) does not allow for any solutions with $\lambda' > 0$, so that all contributions to $P(\tau)$ and $E(\tau)$ are damped. For the function (5.3b) a similar proof can be given. However, for Eq. (5.3a) an analogous reasoning cannot be followed. Here it should be kept in mind that any violation of the negativity condition on the eigenvalues in that case is due to terms of second order in Γ. Indeed, if we were to replace $1 - \Gamma\zeta$ by $(1+\Gamma\zeta)^{-1}$ in the last term of Eq. (5.3a), for instance, it could again be proved rigorously that all eigenvalues have a negative real part.

To gain additional insight in the behavior of $P(\tau)$ we expand its Laplace transform in powers of $\bar{g}^2/(4\lambda + 2\gamma_\| + \bar{g}^2)$. For the universe-mode case, with $\gamma_\| = 0$, one finds

$$\frac{1}{\lambda + J_u^{(\infty)}(\lambda)} = \sum_{k,m,n=0}^{\infty} C_{kmn} \frac{(\bar{g}^2/4)^k}{(\lambda + \bar{g}^2/4)^{k+1}}$$

$$\times \exp\{-\lambda[m\zeta + n(1-\zeta)]$$

$$+ i(m-n)\omega_A\zeta - n\Gamma'\}, \tag{5.5}$$

with

$$C_{kmn} = \sum_{p=max(m-k,n-k,0)}^{[(m+n-k)/2]} k(-1)^{m+n+k}$$

$$\times \frac{2^{m+n-k-2p}(k+p-1)!}{p!(k-m+p)!(k-n+p)!(m+n-k-2p)!}, \tag{5.6}$$

where we used $[x]$ in the upper limit of the sum to represent the largest integer smaller than or equal to x. The coefficients C_{kmn} can be expressed in (terminating) generalized hypergeometric functions $_3F_2$ with unit argument. However, a closed form for these functions does not seem to be available. Taking the inverse Laplace transformations we end up with

$$P(\tau) = \left| \sum_{m,n=0}^{\infty} e^{i(m-n)\omega_A\zeta - n\Gamma'} P_{mn}(\bar{g}^2\tau_{mn}/4) \right.$$

$$\left. \times e^{-\bar{g}^2\tau_{mn}/4} \theta(\tau_{mn}) \right|^2, \tag{5.7}$$

with $\tau_{mn} = \tau - m\zeta - n(1-\zeta)$ and with the polynomials

$$P_{mn}(x) = \sum_{k=|m-n|}^{m+n} C_{kmn} \frac{x^k}{k!}. \tag{5.8}$$

For the standard master equation the function $P(\tau)$ can be brought in a similar form with the same polynomials

$P_{mn}(x)$. For the generalized master equation this is true as well if second order terms in Γ are dropped.

The form (5.7) can be interpreted in terms of signals that are repeatedly reflected by the mirrors. In fact, the retarded time τ_{mn} corresponds to m reflections from one mirror and n reflections from the other. The step functions $\theta(\tau_{mn})$ account for the fact that it takes a finite amount of time for the signal to travel along a path determined by the precise number of reflections at each mirror. At first sight it might seem that $P(\tau)$ is discontinuous for $\tau_{mn} = 0$ as a consequence of these step functions. However, since one has $P_{mn}(0) = \delta_{m0}\delta_{n0}$, the discontinuity is suppressed for all positive τ. There is no such suppression for the derivative of $P(\tau)$: it is discontinuous at $\tau_{mn} = 0$. It should be noted that the interpretation in terms of repeated reflections could also be given by starting from the delay differential equation that has been formulated on the basis of the universe-mode approach [6,7].

To assess the validity of the three multimode approaches we will discuss a few numerical results for the function $P(\tau)$, which we have obtained for various values of the atomic position ζ within the cavity, and of the phase $\omega_A\zeta$ of the central quasimode at the atom. The mirror reflectivity R is taken to be 0.5 (or $\Gamma \approx 0.348$), so that the cavity is definitely nonideal. Of course, for small values of Γ the three approaches give nearly coinciding results. Furthermore, we have taken $\bar{g}^2 = 8$, and assumed the atomic decay parameter $\gamma_\|$ to be negligible.

In Fig. 2 we plotted $P(\tau)$ as a function of τ for an atom in the middle of the cavity ($\zeta = 0.5$), halfway between a node and antinode of the resonant quasimode
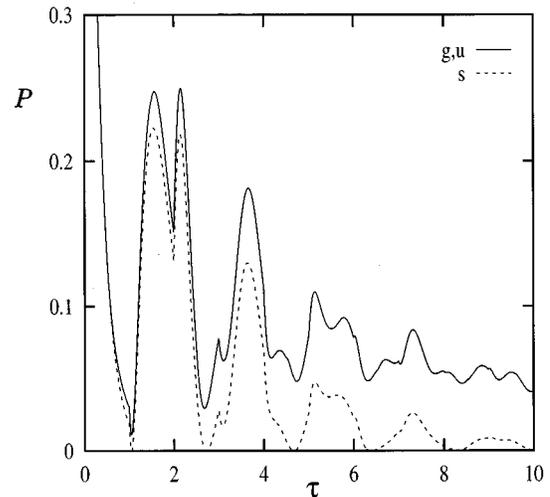

FIG. 2. Predictions for the upper-level probability $P(\tau)$ that follow from the generalized (g) or the standard master equation (s), and from the universe-mode approach (u). The atom is in the middle of the cavity ($\zeta = 0.5$), halfway between a node and an antinode of the resonant quasimode.
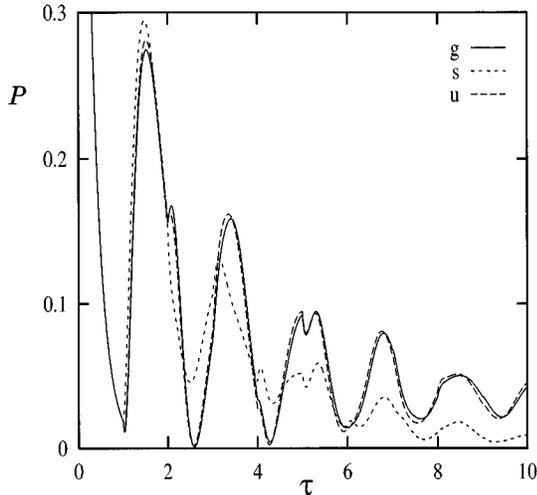
FIG. 3. Probability $P(\tau)$ for an atom near the semitransparent mirror ($\zeta=1$), halfway between a node and an antinode of the resonant quasimode.

[$\omega_A\zeta=\pi/2(\mathrm{mod}\,\pi)$]. In accordance with (5.7), we observe a conspicuous cusplike behavior at integer values of τ, which corresponds to $\tau_{mn}=0$ with $m-n$ even. On the scale used here the curve that results from the generalized master equation is indistinguishable from that following from the universe-mode approach. However, it differs quite substantially from that obtained with the help of the standard master equation.

As the generalized master equation and the universe-mode approach differ in second order in $\Gamma\zeta$, we expect that the difference between these theories becomes more visible for higher values of ζ. This is indeed the case, as can be seen in Fig. 3, where we plotted $P(\tau)$ for an atom situated near the semitransparent mirror ($\zeta=1$), again halfway between a node and an antinode. The curve following from the standard master equation has a larger deviation.

If the atom is situated precisely at an antinode of the resonant quasimode, we have a slightly different situation. If the atom is in the middle of the cavity, the two master equations yield identical results, and resemble the results from the universe-mode approach. However, for an atomic position near the semitransparent mirror, the pattern is the same as before: the results from the generalized master equation resemble those from the universe-mode approach, while the standard master equation again fails (see Fig. 4).

We conclude that the generalized master equation gives a reasonable description of cavities with fairly high loss (up to some 50% per round-trip), whereas the standard master equation clearly fails for those cavities. The results for the decaying atom suggest that the generalized master equation might be a good description even for large coupling constants. Extremely lossy cavities, with $R\ll1$, cannot be described accurately with the generalized master equation. Of course this is no surprise, as for these cavities there is *a priori* no reason why any quasimode description should be qualitatively correct.

APPENDIX A: DERIVATION OF MASTER EQUATION

To derive the generalized master equation one starts from the expression (2.3) for the electric field at the position of the
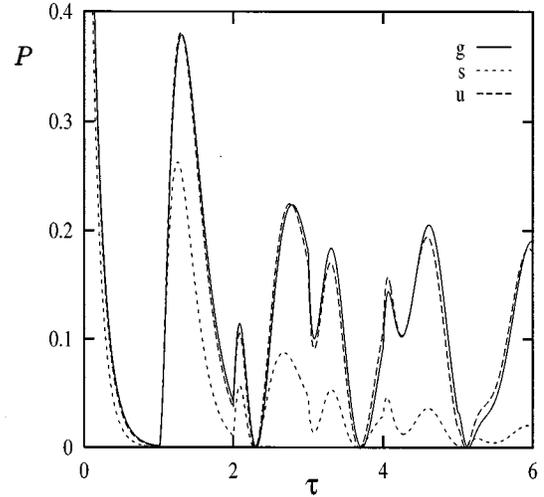

FIG. 4. Probability $P(\tau)$ for an atom near the semitransparent mirror ($\zeta=1$), at an antinode of the resonant quasimode.

atom. It contains a certain linear combination of the annihilation operators a_k of the universe modes. In view of the frequency dependence of the coefficients M_k given in Eq. (2.4) this linear combination may conveniently be split into contributions from limited ranges of the frequency, by defining (dimensionless) frequency intervals through the relation $|\omega_k-\omega_A-2\pi n_q|<\pi$, with integer n_q. With each of these contributions to the electric field one may associate a quasimode of the cavity, by defining a quasimode annihilation operator b_q as

$$b_q(\zeta)=\sum_k^{(q)}\phi_k(\zeta)a_k,\qquad\text{(A1)}$$

where the summation is over all k values with $|\omega_k-\omega_A-2\pi n_q|<\pi$. The coefficients ϕ_k are

$$\phi_k(\zeta)=\sqrt{\frac{l}{L}}\frac{M_k}{\mathcal{N}_q(\zeta)}\sin(\omega_k\zeta/2).\qquad\text{(A2)}$$

The factor l/L is due to the different quantization volume, and is directly related to the different densities in k space of the degrees of freedom in the universe-mode and quasimode pictures. Furthermore, a normalization factor

$$[\mathcal{N}_q(\zeta)]^2=\frac{l}{L}\sum_k^{(q)}M_k^2\sin^2(\omega_k\zeta/2)\qquad\text{(A3)}$$

is introduced in such a way that the standard commutation relation for the annihilation operator b_q and its Hermitian conjugate is preserved.

The electric field (2.3) can be rewritten in terms of the quasimode annihilation operators as

$$\mathcal{E}(\zeta)=\frac{1}{\sqrt{l}}\sum_q\mathcal{N}_q(\zeta)b_q(\zeta).\qquad\text{(A4)}$$

As a consequence, the interaction Hamiltonian (2.2) can be brought in the form (2.6), with

$$g_q(\zeta) = \bar{g}\mathcal{N}_q(\zeta). \tag{A5}$$

If one uses the fact that different quasimodes are strictly independent degrees of freedom, one may derive a generalized multimode master equation, as has been established in [2]. One finds the damping terms (2.8) and (2.9) with

$$w_q(\zeta) = \frac{g_q}{2}(v'_q + iv''_q), \tag{A6}$$

$$v'_q = -\frac{2\cos(\omega_A\zeta + 2\pi n_q\zeta)}{1 - \cos(\omega_A\zeta + 2\pi n_q\zeta)}\left\{\zeta - \frac{\sin(\pi\zeta)}{\pi}\{\zeta\psi[(\zeta+1)/2]\right.$$

$$\left. - \zeta\psi(\zeta/2) - 1\}\right\}, \tag{A7}$$

$$v''_q = -\frac{2\zeta\sin(\omega_A\zeta + 2\pi n_q\zeta)}{1 - \cos(\omega_A\zeta + 2\pi n_q\zeta)}. \tag{A8}$$

Here $\psi(x)$ is the digamma function.

To solve the master equation we write the density operator in the form (2.11), where we can expand the photonic operators in terms of the basis set [5]

$$\rho_{n,k}^{(q)} = \begin{cases} (b_q^\dagger)^k(-1)^{b_q^\dagger b_q + n}\begin{pmatrix} n+k \\ b_q^\dagger b_q + k \end{pmatrix} & (k \geq 0) \\ (\rho_{n,-k}^{(q)})^\dagger & (k < 0). \end{cases} \tag{A9}$$

In fact, since there is only one excitation in the system we may write

$$\rho_0(\tau) = \sum_{pq} \alpha_{pq}(\tau)\rho_{0,1}^{(q)}\rho_{0,-1}^{(p)}\prod_{r \neq p,q} \rho_{0,0}^{(r)}$$

$$+ \left(1 - \sum_q \alpha_{qq}(\tau)\right)\prod_p \rho_{0,0}^{(p)}, \tag{A10a}$$

$$\rho_z(\tau) = \beta_0(\tau)\prod_q \rho_{0,0}^{(q)}, \tag{A10b}$$

$$\rho_-(\tau) = -\frac{i}{2}\sum_q \gamma_q(\tau)\rho_{0,1}^{(q)}\prod_{p \neq q} \rho_{0,0}^{(p)}, \tag{A10c}$$

$$\rho_+(\tau) = \frac{i}{2}\sum_q \eta_q(\tau)\rho_{0,-1}^{(q)}\prod_{p \neq q} \rho_{0,0}^{(p)}. \tag{A10d}$$

It should be noted that we use a convention for the coefficients that is slightly different from that employed in [2].

The operators $\rho_{n,k}^{(q)}$ have the nice property

$$\text{Tr}_q[(b_q^\dagger)^n(b_q)^{n+k}\rho_{m,l}^{(q)}] = \delta_{nm}\delta_{kl}(n+k)! \quad (k \geq 0), \tag{A11}$$

where Tr_q is a trace over the degree of freedom associated with the quasimode q. This identity enables us to find the relations (2.12a)–(2.12d).

From the master equation one easily derives the set of differential equations (2.13a)–(2.13d). The coefficients at the

right-hand sides of these equations contain the functions g_q and G_q, which in first order of Γ read

$$g_q = \bar{g}\sin(\omega_A\zeta/2 + \pi n_q\zeta)\left(1 - \frac{\Gamma}{4}v'_q(\zeta)\right), \tag{A12}$$

$$G_q = \bar{g}\sin(\omega_A\zeta/2 + \pi n_q\zeta)\left(1 + \frac{\Gamma}{4}v'_q(\zeta) + \frac{i}{2}\Gamma v''_q(\zeta)\right). \tag{A13}$$

Their product is given by Eq. (2.14), again in first order of Γ.

APPENDIX B: EIGENVALUE PROBLEM FOR THE MASTER EQUATION

As shown in Sec. III the master equation can be reduced to a matrix equation of the form (3.2), with a matrix $\mathcal{M}$ that is a sum of two tensor products. The convention for the tensor product that is used in Eq. (3.4) is best shown with the help of an example; for a tensor product of two 2×2 matrices A and B we define

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}. \tag{B1}$$

To solve the eigenvalue problem for $\mathcal{M}$ it is sufficient to determine the eigensystem of M. Indeed, from the (right) eigenvalue equation

$$M \cdot X_i^R = \lambda_i X_i^R, \tag{B2}$$

it follows that

$$(M \otimes \mathbf{1} + \mathbf{1} \otimes M^*) \cdot [X_i^R \otimes (X_j^R)^*] = (\lambda_i + \lambda_j^*) \cdot [X_i^R \otimes (X_j^R)^*]. \tag{B3}$$

The left eigenvector problem can be treated similarly.

The eigenvalue equation for the matrix M leads to Eq. (3.5). Upon writing the right and left eigenvectors as

$$X^R = \begin{pmatrix} x_0^R \\ x_1^R \\ \vdots \end{pmatrix}, \quad X^L = (x_0^L, x_1^L, \ldots), \tag{B4}$$

respectively, we find for their components

$$x_p^R = \frac{-G_p x_0^R}{\Gamma + \lambda + i2\pi n_p}, \quad x_p^L = \frac{g_p x_0^L}{\Gamma + \lambda + i2\pi n_p}, \tag{B5}$$

with $p \geq 1$. Note that the inner product of a left and a right eigenvector with eigenvalues λ_i and λ_j is zero for $\lambda_i \neq \lambda_j$.

Now that the eigenvalue problem for the matrix M is solved, it is straightforward to write down the time dependence of the coefficients contained in $\mathcal{X}$. If we introduce the vectors

$$e_0^L = (1, 0, \ldots), \quad e_0^R = (e_0^L)^T = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, \qquad \text{(B6)}$$

we can write, for instance,

$$\beta_0(\tau) = (e_0^L \otimes e_0^L) \cdot \exp[(M \otimes \mathbf{1} + \mathbf{1} \otimes M^*)\tau] \cdot (e_0^R \otimes e_0^R). \tag{B7}$$

With the help of the relation

$$\frac{e_0^L \cdot X_i^R X_i^L \cdot e_0^R}{X_i^L \cdot X_i^R} = \left[1 - \sum_q \frac{g_q G_q}{(\Gamma + \lambda_i + i 2\pi n_q)^2} \right]^{-1}$$

$$= \left[\partial_\lambda \left(\lambda + \gamma_\parallel/2 + \sum_q \frac{g_q G_q}{\Gamma + \lambda + i 2\pi n_q} \right) \bigg|_{\lambda = \lambda_i} \right]^{-1}$$

$$\tag{B8}$$

one easily arrives at the expression (3.6) for the probability of the atom being in its excited state.

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