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Effective action for the superfluid Fermi gas

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Abstract

In this paper we revisit the derivation of the Landau expansion for superfluid ^3He . Following Nijhoff, Capel and Den Breems we adopt a model interaction which contains an $L = 1$ pairing interaction and a contact term of the Hubbard type. We formulate the problem in the language of the path integral and construct the Gibbs free energy as a functional of the order parameter. In the mean-field approximation there is agreement with the exact results of Capel et al. Applying standard perturbation theory, we show that the contribution of the Hubbard interaction to the Landau coefficients is represented by a sum of 1PI diagrams.

Dedicated in good friendship to Hans W. Capel on the occasion of his 60th birthday

1. Introduction

In a series of papers [1–5], Capel together with Nijhoff and Den Breems, has investigated the phases of superfluid ^3He in the presence of a magnetic field. The starting point is a model Hamiltonian with two different types of interaction. One is a pairing interaction of the BCS type appropriate to P-wave pairing, that is for orbital momentum $L = 1$. This interaction is separable in the sense that the Hamiltonian can be reduced to a bilinear form. This property makes it possible to give an exact expression for the free energy of the system, as already shown in early work of Tindemans and Capel [6–8], and to calculate the coefficients of the Landau expansion with respect to the order parameter. Unfortunately, it is a well-known result of this theory that for $L = 1$ in zero magnetic field the isotropic Balian–Werthamer state (B-phase) is the unique stable phase with the lowest energy [9]. This is in conflict with the experimental observation of the anisotropic Anderson–Brinkman–Morel state (A-phase) in superfluid ^3He at high pressure.

In the approach of Nijhoff, Capel and Den Breems (NCB) to understand the occurrence of the A-phase, a contact term of the Hubbard type is added to the pairing

interaction. This model is used to investigate to which extent the phase diagram in the weak-coupling limit undergoes some qualitative change in the presence of the perturbations due to the Hubbard term. Their method of calculation is based on a theorem of Bogoliubov Jr [10], by which the free energy associated with their Hamiltonian can be expressed rigorously in terms of a free energy of an associated reference system. The associated reference Hamiltonian contains a bilinearization of the pairing interaction, as well as the Hubbard interaction. Assuming that the pairing interaction is dominant, the Landau coefficients have been calculated by NCB as a perturbation series in the Hubbard coupling constant.

In the present note we will approach the same problem from the point of view of the path integral, following the presentation in the thesis of Schakel [11]. Instead of making use of the theorem of Bogoliubov Jr we will follow the standard program of constructing the Gibbs free energy, called effective action in field theory, as a functional of the 18 real order parameter fields of the problem. In the mean-field approximation this effective action is identical to the Landau functional defined as the Legendre transform of the reference free energy by Capel, Den Ouden and Perk [12–14]. By minimizing this functional one may investigate which of the possible ordered phases will lead to the absolute minimum of the free energy functional.

The advantage of using the field theoretical approach is that the expansion coefficients of the effective action have a well-defined diagrammatic representation in terms of 1PI diagrams. In the present case the diagrammatic rules are specified by the 4×4 propagator in Nambu space, as in the Balian–Werthamer weak-coupling theory [15], and the four-point interaction vertex representing the Hubbard-type interaction term. Standard Feynman rules apply and it is a straightforward matter to derive the diagrammatic contribution of the Hubbard term to the effective action up to second order in the coupling constant. By systematizing the algebraic manipulations in this manner, higher-order calculations become feasible [16].

2. Generating functional

The superfluid phases of ^3He may be described by assuming the formation of Cooper pairs between neutral ^3He -atoms with angular momentum $L = 1$ and spin $S = 1$. The order parameter describing the pairing in this P-wave, spin triplet state is a complex tensor field, with 18 independent real components, which may be defined as the statistical average

$$m_{\sigma\rho}(x) = \langle \psi_{\sigma}(x) \mathbf{p} \psi_{\rho}(x) \rangle. \quad (1)$$

The dynamical variables $\psi_{\sigma}(x)$ are anti-commuting complex fields describing helium atoms with spin states $\sigma = \uparrow, \downarrow$. The momentum operator $\mathbf{p} = -\frac{1}{2}i(\vec{\nabla} - \overleftarrow{\nabla})$ transforms as a vector, and for each spatial component j , $j = 1, 2, 3$, the order parameter is a symmetric 2×2 matrix in spin space.

Our goal is to derive a Landau expansion for the free energy starting from the partition function

$$Z(T, \mu, V) = \int [D\psi D\psi^\dagger] e^{-S} = \text{Tr } e^{-S}, \quad (2)$$

which is the trace over all configurations of the fields in Euclidean space $x = (-i\tau, \mathbf{x})$, subject to the boundary condition

$$\psi_\sigma(0, \mathbf{x}) = -e^{\beta\mu} \psi_\sigma(-i\beta, \mathbf{x}), \quad (3)$$

with μ the chemical potential and $\beta = T^{-1}$ the inverse temperature. We use natural units in which $\hbar = k_B = 1$. The action $S = S_0 + S_{\text{int}}$ is the sum of a bilinear part

$$S_0[\psi^\dagger, \psi] = \int_0^\beta d\tau \int d^3x \psi_\sigma^\dagger(x) [\partial_\tau + \epsilon(\mathbf{p}) - \mu] \psi_\sigma(x), \quad (4)$$

where $\epsilon(\mathbf{p})$ is the particle kinetic energy, and an interaction part

$$S_{\text{int}}[\psi^\dagger, \psi] = \frac{1}{2} \int_0^\beta d\tau d\tau' \int d^3x d^3x' \psi_\sigma^\dagger(x) \psi_\rho^\dagger(x') U(x, x') \psi_\rho(x') \psi_\sigma(x), \quad (5)$$

which is specified by an instantaneous pair potential

$$U(x, x') = V(|\mathbf{x} - \mathbf{x}'|) \delta(\tau - \tau'). \quad (6)$$

Since we are interested in correlations at long distances, we expand the fields with respect to the relative distance $\mathbf{x} - \mathbf{x}'$. Keeping the first two terms in this expansion, we obtain

$$S_{\text{int}} = S_I + S_\lambda = \int_0^\beta d\tau \int d^3x [\mathcal{L}_I(x) + \mathcal{L}_\lambda(x)]. \quad (7)$$

The first term is a contact Hubbard-like interaction described by a Lagrangian density

$$\mathcal{L}_I(x) = I \psi_\rho^\dagger(x) \psi_\sigma^\dagger(x) \psi_\sigma(x) \psi_\rho(x) \quad (8)$$

and the second term a P-wave pairing interaction

$$\mathcal{L}_\lambda(x) = -\lambda \psi_\rho^\dagger(x) p^j \psi_\sigma^\dagger(x) \psi_\sigma(x) p^j \psi_\rho(x). \quad (9)$$

Following NCB we shall adopt this model interaction characterized by the two coupling constants I and λ which are taken both to be positive, corresponding to a repulsive Hubbard interaction and an attractive pairing interaction. The Hubbard interaction is considered to be small with respect to the pairing interaction.

The important point is that the pairing interaction (9) has a separable form, that is, the pairing interaction can be written as the product

$$\mathcal{L}_\lambda(x) = -\lambda \boldsymbol{\omega}_{\sigma\rho}^\dagger(x) \cdot \boldsymbol{\omega}_{\rho\sigma}(x) \quad (10)$$

of the bilinear combination

$$\omega_{\sigma\rho}(x) = \psi_{\sigma}(x) p \psi_{\rho}(x). \quad (11)$$

This dynamical variable is the microscopic order parameter for ^3He , whose average gives the macroscopic order parameter (1). The bilinear term of the pairing interaction (10) allows the functional integral over these degrees of freedom to be performed. We first introduce a source $\mathbf{J}^\dagger$ which couples to the microscopic order parameter (11) and define the generating functional

$$Z[\mathbf{J}, \mathbf{J}^\dagger] = \text{Tr} \exp(-S + \mathbf{J}^\dagger \cdot \boldsymbol{\omega} + \boldsymbol{\omega}^\dagger \cdot \mathbf{J}), \quad (12)$$

where the inner-product notation stands for

$$\mathbf{J}^\dagger \cdot \boldsymbol{\omega} = \int_0^\beta d\tau \int d^3x (J^\dagger)_{\rho\sigma}^j(x) \omega_{\sigma\rho}^j(x). \quad (13)$$

The order parameter is generated by taking the functional derivative

$$\left. \frac{\delta \log Z[\mathbf{J}, \mathbf{J}^\dagger]}{\delta (J^\dagger)_{\rho\sigma}(x)} \right|_{\mathbf{J}=\mathbf{J}^\dagger=0} = m_{\sigma\rho}(x). \quad (14)$$

A formal procedure to rewrite the generating functional (12) into an integral over order-parameter fields is to make use of the functional identity

$$\mathbf{1} = \int [D\mathbf{m} D\mathbf{m}^\dagger] \delta(\boldsymbol{\omega} - \mathbf{m}) \delta(\boldsymbol{\omega}^\dagger - \mathbf{m}^\dagger). \quad (15)$$

The delta-functions allow the replacement of $\boldsymbol{\omega}$ by $\mathbf{m}$ in (10). The insertion of the integral representation

$$\delta(\boldsymbol{\omega} - \mathbf{m}) = \int [D\mathbf{h}^\dagger] e^{\mathbf{h}^\dagger \cdot (\boldsymbol{\omega} - \mathbf{m})} \quad (16)$$

and a rearrangement of the order of integration leads to

$$Z[\mathbf{J}, \mathbf{J}^\dagger] = \int [D\mathbf{h} D\mathbf{h}^\dagger] \int [D\mathbf{m} D\mathbf{m}^\dagger] e^{\lambda \mathbf{m}^\dagger \cdot \mathbf{m} - \mathbf{h}^\dagger \cdot \mathbf{m} - \mathbf{m}^\dagger \cdot \mathbf{h}} \mathcal{Z}[\mathbf{h} + \mathbf{J}, \mathbf{h}^\dagger + \mathbf{J}^\dagger], \quad (17)$$

with a new generating functional

$$\mathcal{Z}[\mathbf{h}, \mathbf{h}^\dagger] = \text{Tr} \exp(-S_0 - S_I + \mathbf{h}^\dagger \cdot \boldsymbol{\omega} + \boldsymbol{\omega}^\dagger \cdot \mathbf{h}), \quad (18)$$

from which the pairing interaction S_λ has disappeared.

Integration of the Gaussian $\mathbf{m}$ -integral leaves an integration over the auxiliary fields $\mathbf{h}^\dagger$ and $\mathbf{h}$,

$$Z[\mathbf{J}, \mathbf{J}^\dagger] = \int [D\mathbf{h} D\mathbf{h}^\dagger] \exp\left(\frac{-1}{\lambda} (\mathbf{h}^\dagger - \mathbf{J}^\dagger) \cdot (\mathbf{h} - \mathbf{J}) - \beta H[\mathbf{h}, \mathbf{h}^\dagger]\right), \quad (19)$$

with the effective Hamiltonian

$$H[\mathbf{h}, \mathbf{h}^\dagger] = -\frac{1}{\beta} \log Z[\mathbf{h}, \mathbf{h}^\dagger]. \quad (20)$$

From expression (19) we may conclude that to leading order in the mean-field approximation the partition function is given by

$$Z = Z[0, 0] = \exp \left(\frac{-1}{\lambda} \bar{\mathbf{h}}^\dagger \cdot \bar{\mathbf{h}} - \beta H[\bar{\mathbf{h}}, \bar{\mathbf{h}}^\dagger] \right), \quad (21)$$

where $\bar{\mathbf{h}}$ and $\bar{\mathbf{h}}^\dagger$ are the solution of the extremum condition, also called the gap equation,

$$\frac{\delta H}{\delta \mathbf{h}^\dagger} + \frac{1}{\beta \lambda} \mathbf{h} = 0, \quad (22)$$

and its adjoint. In this approximation the order parameter (14) is given by

$$\mathbf{m} = \langle \boldsymbol{\omega} \rangle = \frac{1}{\lambda} \bar{\mathbf{h}}. \quad (23)$$

Since the mean-field approximation becomes exact in the thermodynamic limit, we recover the theorem of Bogoliubov Jr [10] that the free energy of the system is given by

$$F = H[\bar{\mathbf{h}}, \bar{\mathbf{h}}^\dagger] + \frac{1}{\beta \lambda} \bar{\mathbf{h}}^\dagger \cdot \bar{\mathbf{h}}, \quad (24)$$

in agreement with formulae [3, Eq. (1.8)] and [3, Eq. (1.9)]. In this limit the mean-field value $\bar{\mathbf{h}}$ is equal to the order parameter $\mathbf{m}$ apart from the multiplicative constant.

3. Perturbation expansion

The generating functional (18) is exactly of the form [3, Eq. (2.1)], which is the starting point for the perturbation expansion by NCB. To compactify the notation we rewrite the action in terms of the four-component Nambu fields

$$\Psi^T = (\psi_\uparrow, \psi_\downarrow, \psi_\uparrow^\dagger, \psi_\downarrow^\dagger), \quad \Psi^\dagger = (\psi_\uparrow^\dagger, \psi_\downarrow^\dagger, \psi_\uparrow, \psi_\downarrow), \quad (25)$$

as introduced by Balian and Werthamer [15]. We also define the adjoint field $\bar{\Psi} = \Psi^\dagger \gamma^0$ with the help of the Dirac matrix γ^0 . In the Dirac representation the γ -matrices have the form

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad (26)$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli spin matrices. For a summary of properties of the Dirac γ -matrices we refer to Appendix A2 of [17].

Fig. 1. Diagrams up to 2nd order in I .

In terms of the Balian–Werthamer fields, the bilinear Lagrange density in (4) may be written as

$$\mathcal{L}_0 = \frac{1}{2} \bar{\Psi} (-\gamma^0 p_0 + \epsilon(\mathbf{p}) - \mu) \Psi = \bar{\Psi} D_0^{-1} \Psi, \quad (27)$$

where $p_0 = -\partial_\tau$. This defines the free propagator D_0 . The Hubbard term (8) may be represented as

$$\mathcal{L}_I = \frac{1}{4} I (\bar{\Psi} \Psi) (\bar{\Psi} \Psi). \quad (28)$$

As a technical detail we note that this formula differs from [3, Eq. (2.8)] by the products ψ_σ^2 and $(\psi_\sigma^\dagger)^2$ which vanish for anti-commuting variables. The advantage of (28) is that it gives a simple vertex. However, compared to NCB a number of redundant terms are generated during calculations which cancel in the final result.

The bilinear terms involving the order parameter in Eq. (18) may be included in the non-interacting part by defining an inverse propagator

$$D^{-1}(\mathbf{h}, \mathbf{h}^\dagger) = \frac{1}{2} \begin{pmatrix} -p_0 + \epsilon(\mathbf{p}) - \mu & -2\mathbf{h} \cdot \mathbf{p} \\ 2\mathbf{h}^\dagger \cdot \mathbf{p} & p_0 + \epsilon(\mathbf{p}) - \mu \end{pmatrix}, \quad (29)$$

where the momentum operator $\mathbf{p}$ only acts on the fermion fields. Making use of the representation (26) of the γ -matrices, we may also write

$$D^{-1}(\mathbf{h}, \mathbf{h}^\dagger) = D_0^{-1} - \frac{1}{2}(1 + \gamma^0)\gamma^5 \mathbf{h} \cdot \mathbf{p} + \frac{1}{2}(1 - \gamma^0)\gamma^5 \mathbf{h}^\dagger \cdot \mathbf{p}. \quad (30)$$

Expanding out the interaction S_I in (18), one obtains for the effective Hamiltonian (20) a sum of connected diagrams. To second order in the coupling constant I the diagrams are shown in Fig. 1. The thick lines represent the propagator (29). Each four-point vertex contributes one factor $-\frac{1}{4}I$ from the Hubbard coupling.

In order to be able to determine the correct symmetry factors, it is useful to make a distinction between the ‘direct’ and the ‘exchange’ interaction, see Fig. 2. This enables one to keep track of the way in which the lines enter the vertex, and to assign the proper minus signs to the fermion loops. Each three-point vertex carries a factor $i\frac{1}{2}\sqrt{I}$.

Fig. 2. Four point Hubbard vertex.

4. Weak coupling

The single circle without vertex in Fig. 1 stands for the contribution without interaction terms

$$\beta H_0[\mathbf{h}, \mathbf{h}^\dagger] = - \int_0^\beta d\tau \int d^3x \operatorname{tr} \log D^{-1}(\mathbf{h}, \mathbf{h}^\dagger), \quad (31)$$

where the trace is taken over the Nambu indices. Using (29), the effective Hamiltonian (31) may be expanded with respect to the fields $\mathbf{h}$ and $\mathbf{h}^\dagger$ to obtain

$$H_0[\mathbf{h}, \mathbf{h}^\dagger] = H_0[0, 0] + \sum_N H_0^{(N)}[\mathbf{h}, \mathbf{h}^\dagger], \quad (32)$$

where

$$H_0^{(N)}[\mathbf{h}, \mathbf{h}^\dagger] = \frac{1}{\beta N} \int_0^\beta d\tau \int d^3x \operatorname{tr} \left[D_0 \begin{pmatrix} 0 & \mathbf{h} \cdot \mathbf{p} \\ -\mathbf{h}^\dagger \cdot \mathbf{p} & 0 \end{pmatrix} \right]^N. \quad (33)$$

For any odd value of N this is zero. Assuming constant fields, this expression can be worked out in momentum space, and one obtains

$$H_0^{(2N)}[\mathbf{h}, \mathbf{h}^\dagger] = \frac{V}{\beta N} \sum_n \int \frac{d^3p}{(2\pi)^3} \operatorname{tr} \left[\frac{\mathbf{h} \cdot \mathbf{p} \mathbf{h}^\dagger \cdot \mathbf{p}}{\omega_n^2 + \xi^2} \right]^N, \quad (34)$$

with $\omega_n = (2n + 1)\pi T$ the fermionic Matsubara frequencies, V the spatial volume, and $\xi(\mathbf{p}) = \epsilon(\mathbf{p}) - \mu$. The first three terms in (32) give a free energy density

$$\begin{aligned} f_0[\mathbf{h}, \mathbf{h}^\dagger] &= f_0[0, 0] + \frac{1}{2} A_0(T) \int \frac{d\Omega}{4\pi} \operatorname{tr}[\mathbf{h} \cdot \mathbf{p} \mathbf{h}^\dagger \cdot \mathbf{p}] \\ &+ \frac{1}{4} B_0(T_c) \int \frac{d\Omega}{4\pi} \operatorname{tr}[\mathbf{h} \cdot \mathbf{p} \mathbf{h}^\dagger \cdot \mathbf{p} \mathbf{h} \cdot \mathbf{p} \mathbf{h}^\dagger \cdot \mathbf{p}] + \dots, \end{aligned} \quad (35)$$

with $A_0(T) = N(0) \log(\epsilon_F/T)$ and $B_0(T) = N(0) 7\zeta(3)/8\pi^2 T^2$ the well-known weak-coupling coefficients for the free energy, where ϵ_F is the Fermi energy.

Minimizing the free energy (35) one finds that the precise form of the 3×3 complex matrices $\mathbf{h}^\dagger$ and $\mathbf{h}$ plays no role. This means that all minimal solutions all have the same minimal free energy. The only stable phase is the BW-state [9]. To lift this degeneracy one has to go beyond the weak-coupling limit.

5. Strong-coupling corrections

Let us now consider the strong-coupling corrections originating from the Hubbard interaction. We collect the two independent external source fields $\mathbf{h}^\dagger$ and $\mathbf{h}$ into the Nambu matrix

$$\mathbf{H} = \begin{pmatrix} 0 & \mathbf{h} \\ \mathbf{h}^\dagger & 0 \end{pmatrix}, \quad (36)$$

so that the inverse propagator (29) gets the single form

$$D^{-1} = D_0^{-1} + \mathbf{H}\gamma^0 \cdot \mathbf{p}. \quad (37)$$

In the generating functional (18) we write the coupling to the sources as

$$\mathbf{h}^\dagger \cdot \boldsymbol{\omega} + \boldsymbol{\omega}^\dagger \cdot \mathbf{h} = \mathbf{H} \cdot \boldsymbol{\Omega}, \quad (38)$$

which defines a hermitian matrix $\boldsymbol{\Omega}$ with off-diagonal components $\boldsymbol{\omega}^\dagger$ and $\boldsymbol{\omega}$ in analogy to (36).

The correlation functions that have to be calculated are defined by

$$\mathcal{G}^{(N)}(x_1, \dots, x_N) = \left. \frac{\delta^N \log \mathcal{Z}[\mathbf{H}]}{\delta \mathbf{H}(x_1) \cdots \delta \mathbf{H}(x_N)} \right|_{\mathbf{H}=0}. \quad (39)$$

In the diagrams shown in Fig. 1, the external field $\mathbf{H}$ occurs in the propagator lines. Therefore, the differentiation to consider is

$$\frac{\delta D}{\delta \mathbf{H}} = -D \frac{\delta D^{-1}}{\delta \mathbf{H}} D. \quad (40)$$

In diagrammatic language this operation corresponds to the insertion of a two-link vertex

$$\text{---}\bullet\text{---} = -\gamma^0 p \quad (41)$$

into a propagator line. In this manner the various contributions to the correlation functions (39) are generated from the diagrams in Fig. 1 by vertex insertions on the propagator lines in all possible ways. For example, in Fig. 3 the diagrams contributing to $\mathcal{G}^{(2)}$ have been drawn up to second order in I . The lines represent the free propagator D_0 given in (27). The dots denote the vertex (41). We note that actually the two middle diagrams in the upper row of Fig. 3 vanish because the anti-symmetric spin indices of the Hubbard term are contracted with the symmetric spin indices of the vertex (41).

The self-energy insertions (tadpoles) appearing in the second row of Fig. 3 can be resummed into the full propagator of the fermion fields

$$G^{(2)}(x, y) = \langle \Psi(x) \bar{\Psi}(y) \rangle. \quad (42)$$

Because the Green functions are calculated with respect to the symmetric state without external field, the Nambu (spin) indices can be extracted and we may write

$$\langle \Psi(x) \bar{\Psi}(y) \rangle = G_1^{(2)}(x, y) \frac{1}{2}(1 + \gamma^0) + G_2^{(2)}(x, y) \frac{1}{2}(1 - \gamma^0), \quad (43)$$

Fig. 3. Contributions to $\mathcal{G}^{(2)}$ up to second order in I . Fig. 4. Resummation of $\mathcal{G}^{(2)}$.

where $G_{1,2}^{(2)}$ are scalar functions.

Similarly, the vertex corrections add up to the corresponding irreducible four-point vertex functions $\Gamma^{(4)}$. The result of this resummation is depicted in Fig. 4, where lines now represent the full propagator. The contractions with the two external vertices project out the symmetric part of $\Gamma^{(4)}$ for which we may write

$$\text{sym } \Gamma_{\alpha\beta\gamma\delta}^{(4)} = \Gamma_1^{(4)} (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \quad (44)$$

where $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$ are Nambu indices. The subindex indicates that the $L = 1$ part of Γ is selected.

The correlation function $\mathcal{G}^{(4)}$ may be analyzed in a similar manner. Inserting vertices (41) in all possible ways, we find the set of diagrams shown in Fig. 5, where all lines are bare propagators. There is a one-to-one correspondence with the analytic expressions [3, Eq. (2.23)] and [3, Eq. (2.25)] of NCB. In our diagrammatic language we may do a selective resummation of diagrams into full propagators and vertex functions. This gives the diagrams in Fig. 6. To the order in which we are working this is the complete result. However, higher-order diagrams will give rise to higher-than-four-point vertex functions and additional contributions to $\mathcal{G}^{(4)}$.

6. Landau expansion

To obtain the coefficients in the Landau expansion of the free energy, we need the free energy as a functional of the order parameter. This thermodynamic potential (often called the Gibbs free energy) is obtained by the Legendre transformation from the external source $\mathbf{h}$ in (18) to the order parameter. Our route to the Legendre transformation starts from the functional integral (17). Interchanging the integration, we write for the generating functional

Fig. 5. Contributions to $\mathcal{G}^{(4)}$.

Fig. 6. Decomposition of $\mathcal{G}^{(4)}$.

$$Z[J, J^\dagger] = \int [D\mathbf{m} D\mathbf{m}^\dagger] e^{-\mathcal{F}[\mathbf{m}, \mathbf{m}^\dagger] + J^\dagger \cdot \mathbf{m} + \mathbf{m}^\dagger \cdot J}. \quad (45)$$

The effective action appearing herein is defined through

$$e^{-\mathcal{F}[\mathbf{m}, \mathbf{m}^\dagger]} = e^{\lambda \mathbf{m}^\dagger \cdot \mathbf{m}} \int [D\mathbf{h} D\mathbf{h}^\dagger] e^{-\mathbf{h}^\dagger \cdot \mathbf{m} - \mathbf{m}^\dagger \cdot \mathbf{h}} \mathcal{Z}[\mathbf{h}, \mathbf{h}^\dagger], \quad (46)$$

where the last factor is the generating functional (18). In the leading mean-field approximation this reduces to the Legendre transformation formula

$$\mathcal{F}[\mathbf{m}, \mathbf{m}^\dagger] - \lambda \mathbf{m}^\dagger \cdot \mathbf{m} = \mathbf{h}^\dagger \cdot \mathbf{m} + \mathbf{m}^\dagger \cdot \mathbf{h} - \log \mathcal{Z}[\mathbf{h}, \mathbf{h}^\dagger] \quad (47)$$

with $\mathbf{h}$ and $\mathbf{h}^\dagger$ the solution of the extremum condition

$$\frac{\delta \log \mathcal{Z}[\mathbf{h}, \mathbf{h}^\dagger]}{\delta \mathbf{h}^\dagger} = \mathbf{m} \quad (48)$$

and its adjoint.

The minima of the effective action obtained from the extremum condition

$$\frac{\delta \mathcal{F}[\mathbf{m}, \mathbf{m}^\dagger]}{\delta \mathbf{m}^\dagger} = 0 \quad (49)$$

and its adjoint, determine the possible superfluid phases of ^3He .

On account of (45) we may view the effective action as defining a theory with new fields $\mathbf{m}$ and $\mathbf{m}^\dagger$. In this language the partition function is represented as a sum of connected diagrams with vertex factors $\mathcal{F}^{(2N)}$ which are the coefficients in the Landau expansion of the effective action

$$\mathcal{F}[\mathbf{m}, \mathbf{m}^\dagger] = \mathcal{F}^{(0)} + \mathbf{m}^\dagger \cdot \mathcal{F}^{(2)} \cdot \mathbf{m} + \frac{1}{4} \mathbf{m}^\dagger \mathbf{m}^\dagger : \mathcal{F}^{(4)} : \mathbf{m} \mathbf{m} + \dots \quad (50)$$

The coefficient in the quadratic term is the inverse propagator of the theory

$$\mathcal{F}^{(2)} = [\mathcal{G}^{(2)}]^{-1} + \lambda, \quad (51)$$

with $\mathcal{G}^{(2)}$ given in diagrammatic form in Fig. 4. Hence we have to invert the correlation function $\mathcal{G}^{(2)}$. At this point it is convenient to change to the well-known vector representation for the order parameter by expressing the symmetric spin matrix $\mathbf{m}_{\sigma\sigma'}$ as a linear combination

$$m_{\sigma\sigma'}^i = \Delta(T) A_j^i (i\sigma_j \sigma_2)_{\sigma\sigma'} \quad (52)$$

of symmetric matrices with expansion coefficients A_j^i , which form a complex 3×3 matrix $\mathbf{A}$. The normalization is chosen such that

$$\text{Tr } \mathbf{A} \cdot \mathbf{A}^\dagger = 1. \quad (53)$$

Near the critical point, and in the absence of a magnetic field, general symmetry considerations lead to the Landau form of the Gibbs free energy density $f = \mathcal{F}/\beta V$, that is, the effective action (47) with a volume factor divided out. Expressed in terms of the matrix $\mathbf{A}$, this Landau form is

$$f = f^{(0)} + \alpha(T) \Delta^2 \text{Tr } \mathbf{A} \cdot \mathbf{A}^\dagger + \beta \Delta^4 \text{Tr } \mathbf{A} \cdot \mathbf{A}^\dagger \mathbf{A} \cdot \mathbf{A}^\dagger. \quad (54)$$

The quartic term stands for a sum of five independent invariants with Landau coefficients $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ [18]. The corrections to these coefficients arising from the Hubbard interaction can be extracted from the diagrams in Fig. 6. To second order in I , these contributions have been calculated by NCB. The results of the calculations of the third-order terms will be published elsewhere [16].

7. Critical temperature

The coefficient $\alpha(T)$ is of particular interest because its zero determines the critical temperature T_c . Comparing (54) with (50), we have the connection

$$\frac{1}{\beta V} \mathbf{m}^\dagger \cdot \mathcal{F}^{(2)} \cdot \mathbf{m} = \alpha(T) \Delta^2 \text{Tr } \mathbf{A} \cdot \mathbf{A}^\dagger. \quad (55)$$

Fig. 7. Bethe-Salpeter equation for $\Gamma^{(4)}$.

Using (52) in (55), it is obvious that we must have

$$-(i\sigma_2\sigma_k)_{\tau\tau'}\mathcal{F}_{\tau\tau',\rho\rho'}^{(2)ij}(i\sigma_1\sigma_2)_{\rho\rho'} = \beta V\alpha(T)\delta_{kl}\delta^{ij}. \quad (56)$$

The corresponding contractions applied to the correlation function $\mathcal{G}^{(2)}$ give a similar result with coefficient

$$-(i\sigma_2\sigma_k)_{\tau\tau'}\mathcal{G}_{\tau\tau',\rho\rho'}^{(2)ij}(i\sigma_1\sigma_2)_{\rho\rho'} = A(T)\delta_{kl}\delta^{ij}. \quad (57)$$

From (51) we then obtain the relation

$$\alpha(T) = \frac{1}{A(T)} - \lambda \quad (58)$$

between the coefficients in the conjugate expansions (35) and (54).

In weak coupling $A(T)$ reduces to the coefficient $A_0(T) = N(0)\log(\epsilon_F/T)$ in (35). In this approximation we recover the standard result

$$\alpha(T) \simeq \log\left(\frac{T}{T_c}\right), \quad (59)$$

where

$$T_c \simeq \epsilon_F \exp\left(\frac{-1}{N(0)\lambda}\right) \quad (60)$$

is the BCS expression for the critical temperature determined by the strength of the pairing interaction.

In this connection it may be of interest to mention that also a purely repulsive interaction ($\lambda = 0$) leads to an instability of the normal state of a fermionic system with respect to triplet P-wave pairing [19,20]. The critical temperature of this instability may be calculated by writing out a Bethe-Salpeter equation for the $L = 1$ part of the irreducible vertex function appearing in Fig. 6. The resummation of this class of diagrams gives an important contribution when the propagators carry the same momenta, the so-called Cooper loop. Using free propagators, one gets for the coefficient $A(T)$ defined by (57) the result

$$A(T) = \frac{A_0(T)}{1 - \tilde{F}_1 A_0(T)}, \quad (61)$$

where $A_0(T)$ is the weak-coupling value of $A(T)$. The coefficient $\alpha(T)$ which is the inverse of expression (61) then has a zero at

$$T_c \simeq \epsilon_F \exp\left(\frac{-1}{N(0)\tilde{F}_1}\right). \quad (62)$$

It has been shown that this transition is second order and that the state is the isotropic triplet phase [21].

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