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Polynomial skew-products with wandering Fatou-disks

Han Peters[1] · **Liz Raquel Vivas**[2]

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Abstract Until recently, little was known about the existence of wandering Fatou components for rational maps in more than one complex variables. In 2014, examples of wandering Fatou components were constructed in Astorg et al. [1] for polynomial skew-products with an invariant parabolic fiber. In 2004 Lilov already proved the non-existence of wandering Fatou components for polynomial skew-products in the basin of an invariant super-attracting fiber. In the current work we investigate in how far his methods carry over to the geometrically attracting case. Lilov proved a stronger statement, namely that the orbit of any horizontal disk in the super-attracting basin must eventually intersect a fattened Fatou components. Here we give explicit constructions to show that this stronger result is false in the geometrically attracting case. However, we also prove that the constructed disks do not give rise to wandering Fatou components. Our construction therefore leaves open the existence of wandering Fatou components in the geometrically attracting case, showing however that the situation is significantly more complicated than in the super-attracting case.

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1 Introduction

We begin our introduction by describing the known results on wandering Fatou components for polynomial maps in two complex variables. A natural class of maps that has been of interest is the class of polynomial skew-products, as these maps lie on the boundary between one and two-dimensional dynamics.

A skew-product is a map of the form

$$F(t, z) = (f(t), g(t, z)) = (f(t), g_t(z)).$$

If $f(t_0) = t_0$ then the fiber $\{t = t_0\}$ is mapped to itself by the function g_{t_0}. Hence if $g_{t_0}(z)$ is a polynomial, then there are no one-dimensional wandering Fatou components in this fiber.

Consider the case where $|f'(t_0)| < 1$, so that all nearby fibers are attracted to the t_0-fiber. In this case each one-dimensional Fatou component of g_{t_0} in the t_0-fiber is contained in a two-dimensional Fatou component of F. One naturally wonders if all nearby Fatou components are eventually mapped onto one of these fattened (pre-) periodic Fatou components. Indeed, it was shown in the Ph.D. thesis of Lilov [12] that this is the case, under the additional assumptions that the t_0-fiber is super-attracting, i.e. that $f'(t_0) = 0$, and that the degree of $g_t(\cdot)$ is locally constant near the invariant fiber. In this article we investigate whether his results also hold in the geometrically attracting case, i.e. when $0 < |f'(t_0)| < 1$.

Lilov actually proved a much stronger result which implies the non-existence of wandering Fatou components. We denote by $\mathcal{A} \subset \mathbb{C}_t$ the immediate basin of the super-attracting fixed point t_0.

Theorem 1.1 (Lilov, 2003) *Let $t_1 \in \mathcal{A}$, and let D be an open one-dimensional disk lying in the t_1-fiber. Then the forward orbit of D must intersect one of the fattened Fatou components of g_{t_0}.*

An immediate corollary is that there are no wandering Fatou components in the set $\mathcal{A} \times \mathbb{C}_z$. On the other hand, it was recently shown in [1] that wandering Fatou components can exist in the basin of a *parabolic* invariant fiber $t_0 = f(t_0)$. We note that in the examples constructed in [1] the polynomial g_{t_0} also has a parabolic fixed point.

In this paper we prove that Theorem 1.1 does not hold in the geometrically attracting case. More precisely, we study explicit maps of the form

$$F(t, z) = (\alpha t, p(z) + q(t)),$$

where $\alpha < 1$ and p and q are polynomials.

Theorem 1.2 *There exist a triple (α, p, q) and a vertical holomorphic disk $D \subset \{t = t_0\}$ whose forward orbit accumulates at a point $(0, z_0)$, where z_0 is a repelling fixed point in the Julia set of p.*

In particular, it follows that the forward orbits of D will never intersect the fattened Fatou components of p. The disks D will be Fatou disks, i.e. the family $\{F^n|_D\}_{n\in\mathbb{N}}$ is normal. However, we will also show in Theorem 6.1 that the disks D are completely contained in the Julia set of the map F. We note that, were this not the case, then the existence of a wandering Fatou component would follow immediately from our construction. Therefore our construction shows that the geometrically attracting case is significantly more complicated than the super-attacting case. Whether wandering Fatou components can arise, as in the parabolic case, remains open.

In the next section we provide background on polynomial skew-products with a super-attracting fiber. In Sect. 3 we prove a Koenigs-type theorem for skew products with an attracting invariant fiber. In Sect. 4 we show that, for a smaller class of maps, we can obtain much faster convergence of the Koenigs-map. In Sect. 5 we use the convergence estimates to construct, again under additional assumptions, the vertical Fatou disks mentioned in Theorem 1.2. Although the maps for which we prove Theorem 1.2 have to satisfy several assumptions, we give explicit examples of such maps. In Sect. 6 we show that, again under additional assumptions, the constructed vertical Fatou disks lie entirely in the Julia set, and therefore do not give rise to wandering Fatou components.

2 Background

Let X be a complex manifold, and let $f : X \to X$ be a holomorphic endomorphism. We say that $z \in X$ lies in the *Fatou set* if there is a neighborhood $U(z)$ on which the family $\{f^n\}_{n\in\mathbb{N}}$ is normal. A connected component of the Fatou set is called a Fatou component. For rational functions of the Riemann sphere the Fatou components are well understood. It was proved in by Sulivan [17] that every Fatou component is (pre-) periodic, and the periodic Fatou components were classified earlier in the works of Fatou, Siegel and Herman.

In two complex dimensions much less is known about Fatou components. There has been considerable progress with respect to the classification of periodic Fatou components, see for example [2–4,7,13,18]. However, the existence question for wandering Fatou components has been until recently almost untouched. Only for a few specific cases, the non-existence of wandering Fatou components is known, for example for polynomial automorphism that act hyperbolically on their Julia sets, see [2], and for polynomial skew-products near a super-attracting invariant fiber.

A skew-product is a map $F : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$F(t, z) = (f(t), g(t, z)).$$

The holomorphic dynamics of polynomial skew-products was first studied by Heinemann in [9] and by Jonsson in [8]. The topology of Fatou components of skew products has been studied by Roeder in [14].

Since skew-products map vertical lines to vertical lines, often one-dimensional tools can be used to construct maps with specific dynamical behavior. For example in [6], Dujardin used skew-products to construct a non-laminar Green current, and in [5], Boc-Thaler, Fornæss and the first author constructed a Fatou component with a punctured limit set. In both articles polynomial skew-products were used to construct holomorphic endomorphisms of $\mathbb{P}^2$ with dynamical phenomena that were previously not known to occur. Hence in order to study the existence of wandering Fatou components for rational maps in two complex dimensions, it is worthwhile to first investigate whether wandering Fatou components can occur for polynomial skew-products. Recently examples of wandering Fatou components were found [1] for skew-product maps for which the fiber map $f(t)$ has a fixed point t_0 in which $f'(t_0) = 1$.

In his Ph.D. thesis, Lilov studied the dynamical behavior of polynomial skew-products near an invariant vertical fiber. Since his results were not published elsewhere, we will discuss Lilov's results in more detail. Let us assume that $f(0) = 0$, and that $|f'(0)| < 1$, so that the orbits of all nearby fibers converge to the invariant fiber $\{t = 0\}$.

Let us further assume that g is a polynomial in z, so that

$$g(t, z) = \alpha_0(t) + \alpha_1(t)z + \cdots + \alpha_d(t)z^d.$$

In this case the dynamics in the invariant fiber given by the polynomial

$$p(z) = g(0, z)$$

is very well understood. Each Fatou component U of p is (pre-)periodic, and every periodic Fatou component is either an attracting basin, a parabolic basin or a Siegel disk. For each of these cases, and hence for any Fatou component of p, the following holds.

Theorem 2.1 (Bulging of Fatou components) *Given a Fatou component U of p, there exists a Fatou component Ω of F for which $U \subset \Omega$.*

The proof of this theorem, which can be found in [12], relies upon the fact that the restriction p to the invariant fiber is a one-dimensional polynomial, whose Fatou components are (pre-)periodic. Every periodic Fatou components is either an attracting basin, a parabolic basin or a Siegel domain, and for each of these cases separately it can easily be shown that the component is contained in a larger two-dimensional Fatou component of F. For an attracting basin the set of points converging to the attracting periodic cycle is open. For the other two cases there exists a strong stable manifold through each point in the one-dimensional Fatou component, and the bulging Fatou component consist of the union of these manifolds.

Following Lilov we will refer to the Fatou components of F that contain a Fatou component of p as a *fattened* Fatou component. Recall that the one-dimensional Fatou component U must at some point be mapped onto either an attracting basin, a parabolic basin, or a Siegel disk. In his thesis Lilov studies the geometry of the fattened Fatou components in each of these three cases under the assumption that the horizontal dynamics is super-attracting, i.e. when $f'(0) = 0$.

The next Lemma was another important ingredient in the proof of Theorem 1.1 of Lilov. The proof makes use of the assumption that the degree of $g_t(\cdot)$ is locally constant, i.e. that $\alpha_d(0) \neq 0$.

Lemma 2.2 *There exist constants $\delta_1 > 0$, $c > 0$ such that if $|t| < \delta_1$ and $D(z, r) \subset \mathbb{C}_t$ is an arbitrary vertical disk of radius $r > 0$, then $F(t, D(z, r))$ contains a disk $D(g(t, z), r')$ of radius $r' \geq cr^d$.*

Applying this lemma repeatedly for the orbit of a vertical disk gives estimates from below for the radii of the images. On the other hand, by studying the geometry of the fattened Fatou components Lilov also obtains an upper bound on the largest possible vertical disk that can lie in the complement of the fattened Fatou components, depending on the distance to the invariant vertical fiber. A combination of these two estimates then leads to the conclusion that the orbit of any vertical disk must intersect one of the fattened Fatou components. Since the fattened Fatou components are (pre-) periodic, it follows that, in a neighborhood of the invariant fiber, all Fatou components of F are pre-periodic.

The question we study here is whether the argument used by Lilov can also be applied for the geometrically attracting case $0 < |f'(0)| < 1$. We note that the estimate in Lemma 2.2 is very rough. Indeed, this estimate is relevant only when p has a single critical point of order $d - 1$, and the t-coordinate of the vertical disk $D(z, r)$ lies in a small neighborhood of this critical point. If this is not the case then much better estimates can be obtained.

If the orbit of a vertical disk does not intersect the fattened Fatou components then this orbit must converge to the Julia set of p. If a critical point of p lies in the Fatou set then

the orbit of $D(z, r)$ will eventually stay away from that critical point, and the estimates are greatly improved. On the other hand, if a critical point lies in the Julia set, then it cannot be periodic, and the images $F^n(t, D(z, r))$, which must shrink, will usually be bounded away from that critical point. Hence if the orbit of a vertical disk does not intersect one of the fattened Fatou components then the radii of the images will shrink much more slowly than Lemma 2.2 suggests.

The above discussion leads to the main idea of this paper. In order to obtain a vertical disk whose orbit does not intersect the fattened Fatou component, one must construct a skew product F and a vertical disk D whose orbit frequently returns very closely to the critical set. We show that this can indeed be achieved for a large class of maps.

3 Linearization of unstable manifolds

Let us recall the following classical result.

Theorem 3.1 (Koenigs 1870) *Let $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with a fixed point at the origin. Suppose $\lambda = f'(0)$ is such that $|\lambda| \neq 0, 1$. Then there exists a local change of coordinates φ such that*

$$\varphi(\lambda \cdot \zeta) = f(\varphi(\zeta)).$$

for all ζ in a neighborhood of the origin. Moreover, the conformal map φ is unique up to a multiplicative constant.

When $|\lambda| > 1$ the linearization map φ can be defined by

$$\varphi(\zeta) = \lim_{n\to\infty} f^n(\lambda^{-n}\zeta).$$

We prove an analogue of Koenigs Theorem for saddle fixed points in $\mathbb{C}^2$, which will afterwards be applied to the skew product setting.

Theorem 3.2 *Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ with the origin a saddle fixed point. Write μ, λ for the eigenvalues of $DF(0)$, with $0 < |\mu| < 1 < |\lambda|$, and write $W_F^{u/s}$ for the unstable/stable manifold of F through the origin. Let V be the germ of an embedded Riemann surface passing through the origin, transverse to W_F^s, and let $\mathbf{v} \in T_0(\mathbb{C}^2) \cong \mathbb{C}^2$ a non-zero tangent vector not normal to V. Write $\mathbb{C}_\mathbf{v} = \{\zeta \cdot \mathbf{v} \in \mathbb{C}^2, \zeta \in \mathbb{C}\}$, and let $\pi_\mathbf{v}$ be the (local) projection from $\mathbb{C}_\mathbf{v}$ to V, which by assumption is locally a graph over $\mathbb{C}_\mathbf{v}$. Then the sequence of maps defined by*

$$\varphi_n : \zeta \mapsto F^n \circ \pi_\mathbf{v}(\lambda^{-n}\zeta\mathbf{v}).$$

converges locally uniformly to a map $\Phi : \mathbb{C} \to W_F^u$ satisfying the functional equation

$$\Phi(\lambda\zeta) = F(\Phi(\zeta)).$$

Before we start the proof of the theorem let us describe some known results. As before, let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a holomorphic germ with the origin as a saddle point, and assume that $DF(0)$ has eigenvalues λ and μ, with $0 < |\mu| < 1 < |\lambda|$. Without loss of generality we assume that the eigenvectors corresponding to λ and μ are respectively $(0, 1)$ and $(1, 0)$. We use the coordinates (t, z) as we did before. Locally the unstable manifold W_F^u is a graph over

the z-axis. Let π_u be the local projection from the z-axis to W^u_F. One usually constructs the linearization map φ as the limit for $n \to \infty$ of the maps:

$$\varphi_n(\zeta) = F^n \circ \pi_u(0, \lambda^{-n}\zeta).$$

However, when $|\lambda| > |\mu|$ we can omit the projection π as was shown in [10]:

Theorem 3.3 (Hubbard) *The maps*

$$\varphi_n(\zeta) = F^n(0, \lambda^{-n}\zeta) \tag{1}$$

converge to the linearization map $\varphi : \mathbb{C} \to W^u_F$.

In fact, for any vector $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$ with $v_2 \neq 0$ we can work with the sequence

$$\psi_n(\zeta) = F^n(\lambda^{-n}\zeta \cdot \mathbf{v}) = F^n(\lambda^{-n}\zeta v_1, \lambda^{-n}\zeta v_2).$$

by applying Theorem 3.2 with $V = \mathbb{C}_{\mathbf{v}}$.

Proof of Theorem 3.2 As before we assume $DF(0)$ has eigenvalues λ and μ, with $0 < |\mu| < 1 < |\lambda|$ and the eigenvectors corresponding to λ and μ are respectively $(0, 1)$ and $(1, 0)$, and we use coordinates (t, z). We first prove the result for the special case $v = (0, 1)$; we will see later that this implies the general result.

Write the Riemann surface V locally as a graph over the direction $(0, 1)$ by $V = \{(H(z), z), z \in \mathbb{C}\}$ and write the projection map $\pi_{\mathbf{v}} : \mathbb{C}_{\mathbf{v}} \to V$ as $\pi_{\mathbf{v}}(\zeta \mathbf{v}) = (H(\zeta), \zeta)$. Then we obtain

$$\varphi_{n+1}(\zeta) - \varphi_n(\zeta) = F^n(t_n, z_n) - F^n(u_n, v_n),$$

where

$$(t_n, z_n) = \left(\mu H\left(\frac{\zeta}{\lambda^{n+1}}\right) + O\left(\frac{\zeta^2}{\lambda^{2n}}\right), \frac{\zeta}{\lambda^n} + O\left(\frac{\zeta^2}{\lambda^{2n}}\right)\right)$$

and

$$(u_n, v_n) = \left(H\left(\frac{\zeta}{\lambda^n}\right), \frac{\zeta}{\lambda^n}\right).$$

Hence given $\epsilon > 0$ the equations

$$|t_n|, |z_n|, |u_n|, |v_n| \leq \epsilon |\lambda|^{-n}$$

and

$$|z_n - v_n| \leq \epsilon |\lambda|^{-2n}$$

hold for $|\zeta|$ sufficiently small.

Write $F^n = (F^n_1, F^n_2)$, and let $C > 0$ be such that

$$|F_1(t, z) - \mu t|, |F_2(t, z) - \lambda z| \leq C \max(|t|^2, |tz|, |z|^2)$$

for $|t|, |z| \leq 1$. By increasing C if necessary we can also make sure that

$$|F_1(t, z) - F_1(u, v)| \leq |\mu| \cdot |t - u| + C(\max(|t|, |z|, |u|, |v|) \max(|t - u|, |z - v|).$$

Given $\epsilon > 0$ we define $\epsilon_{n,0} := \epsilon$ and

$$\epsilon_{n,j} := \epsilon_{n,j-1}\left(1 + |\lambda|^{-n+2j}\right)$$

for $1 \leq j \leq \frac{n}{2}$, and

$$\epsilon_{n,j} := \epsilon_{n,j-1}\left(1 + |\lambda|^{-n+j}\right)$$

for $\frac{n}{2} < j \leq n$. Now we choose $\epsilon > 0$ sufficiently small so that $2C \cdot \epsilon_{n,n} < \max(1, |\lambda|^2 - |\mu|)$ for all $n \in \mathbb{N}$.

By induction on j, we see that for $j = 0, \ldots, n$ and $i = 1, 2$ one has

$$|F_i^j(t_n, z_n)|, |F_i^j(u_n, v_n)| \leq \epsilon_{n,j}|\lambda|^{-n+j}.$$

In fact, using $|\mu| < 1$ one obtains the following stronger estimate on the first coordinates:

$$|F_1^j(t_n, z_n)|, |F_1^j(u_n, v_n)| \leq \epsilon_{n,j} \max\left(|\lambda|^{-n}, |\lambda|^{-2n+2j}\right).$$

It then follows, again by induction on j, that

$$|F_1^j(t_n, z_n) - F_1^j(u_n, v_n)| \leq \epsilon_{n,j}|\lambda|^{-n}$$

while

$$|F_2^j(t_n, z_n) - F_2^j(u_n, v_n)| \leq \epsilon_{n,j}|\lambda|^{-2n+j}.$$

By taking $j = n$ it follows that $\|\varphi_{n+1} - \varphi_n\| \to 0$ as $n \to \infty$ for $|\zeta|$ sufficiently small, and from definition of φ_n it follows that $\|\varphi_{n+1} - \varphi_n\| \to 0$ locally uniformly on all of $\mathbb{C}$.

Now suppose that $\mathbf{v} \neq (0, 1)$ satisfies the stated hypotheses. We write

$$\psi_n(\zeta) = F^n \circ \pi_{\mathbf{v}}(\lambda^{-n}\zeta \cdot \mathbf{v})$$

and

$$\varphi_n(\zeta) = F^n \circ \pi_{(0,1)}(0, \lambda^{-n}\zeta).$$

Let us write $T = \pi_{(0,1)}^{-1} \circ \pi_{\mathbf{v}} : \mathbb{C}_{\mathbf{v}} \to \mathbb{C}_{(0,1)}$, and note that T is locally injective and fixes the origin. Writing $T(\zeta\mathbf{v}) = (0, a_1\zeta + O(\zeta^2))$ it follows that

$$\lambda^n T(\lambda^{-n}\zeta\mathbf{v}) \to (0, a_1\zeta)$$

locally uniformly as $n \to \infty$, and hence

$$\begin{aligned}\psi_n(\zeta) = F^n \circ \pi_{(0,1)}T(\lambda^{-n}\zeta\mathbf{v}) &= F^n \circ \pi_{(0,1)}(0, \lambda^{-n}a_1\zeta + O(\lambda^{-2n})) \\ &= \varphi_n(a_1\zeta) + O(\lambda^{-n}).\end{aligned}$$

Hence it follows that the sequence ψ_n converges to the map $\varphi(a_1\cdot)$, which satisfies the same functional equation. □

Remark 3.4 We note that Hubbard proved Theorem 3.3 above in $\mathbb{C}^m$ for any $m \geq 2$, as long as the eigenvalues $\{\lambda_i\}$ of $DF(0)$ satisfy $\lambda_1 \leq \cdots \leq \lambda_k < \lambda_{k+1} \leq \cdots \leq \lambda_n$, with $|\lambda_{k+1}| > 1$, and one considers the strong unstable manifold of dimension $n - k$. Here when $n - k > 1$ one does not work with a linear contraction, but with a *normal form*, see also [15,16]. Similarly, Theorem 3.2 holds in any dimension $m \geq 2$ as long as there is a single eigenvalue of maximal modulus. Just as above, the vector v should then be chosen not to lie in the subspace spanned by the (generalized) eigenvectors corresponding to all the eigenvalues of smaller modulus.

Going back to the set up that we are interested in.

Corollary 3.5 *Let F be a skew-product of the form*

$$F(t, z) = (\mu t, g(t, z)),$$

where $g(0, 0) = (0, 0)$ and let $p(z) = g(0, z)$. Assume that $\frac{\partial g}{\partial t}(0, 0) \neq 0$, and $\frac{\partial g}{\partial z}(0, 0) = \lambda$, where $|\lambda| > 1$ and $0 < |\mu| < |\lambda|$. Let $x \in \mathbb{C}$ and $k \in \mathbb{N}$ be such that $p^k(x) = 0$. Now assume that the eigenvector at $(0, 0)$ corresponding to the eigenvalue μ is not a tangent vector to the Riemann surface given locally by $F^k(\{z = x\})$. Then the holomorphic functions

$$\psi_n(\zeta) = \pi_2 F^n\left(\frac{\zeta}{\lambda^n}, x\right)$$

converge uniformly on compact subsets of $\mathbb{C}$ to a holomorphic function $\Phi : \mathbb{C}_\zeta \to \mathbb{C}_z$ which satisfies the functional equation

$$\Phi(\lambda \cdot \zeta) = p(\Phi(\zeta)). \tag{2}$$

Here and throughout the article we use π_i for the projection onto the i-th coordinate. Note that the Corollary is still valid if we remove this projection. The projection is there for the convenience in later sections.

Proof First, note that, due to the explicit form of F we can write:

$$F^n\left(\frac{\zeta}{\lambda^n}, x\right) = \left(\frac{\mu^n \zeta}{\lambda^n}, \psi_n(\zeta)\right)$$

The assumptions on the partial derivatives of g imply that the stable manifold of F and the vector $(1, 0)$ are transversal. Also $F(0, z) = (0, p(z))$ so the unstable manifold of F is precisely contained in the z-axis. We apply Theorem 3.2 to the Riemann surface given locally by $F^k(\{z = x\})$, taking $\mathbf{v} = (1, 0)$. Hence we obtain the following convergent sequence:

$$\varphi_n(\zeta) = F^n \circ \pi_{\mathbf{v}}(\lambda^{-n}\zeta, 0),$$

with limit φ. Applying F^k to the equation above, we obtain:

$$F^k\left(\frac{\mu^n \zeta}{\lambda^n}, \psi_n(\zeta)\right) = F^{n+k}\left(\frac{\zeta}{\lambda^n}, x\right) = F^n\left(F^k\left(\frac{\zeta}{\lambda^n}, x\right)\right) = \varphi_n(\mu^k \zeta).$$

For the last equality we note that the projection $\pi_{\mathbf{v}}$ leaves the first coordinate invariant.

Taking limits when $n \to \infty$, we obtain:

$$F^k\,(0, \Phi(\zeta)) = \varphi(\mu^k \zeta).$$

Since $F(0, z) = (0, p(z))$ the last equation is equivalent to

$$p^k\,(\Phi(\zeta)) = \varphi(\mu^k \zeta).$$

Hence the functional equation obtained in Theorem 3.2 and the fact that $(p^k)'(0) = \lambda^k \neq 0$ imply the desired result. □

Recall that the linearization map φ found in Koenigs Theorem is unique up to a multiplicative constant. Since the map Φ found in Corollary 3.5 maps $\mathbb{C}$ into $\mathbb{C}$ and satisfies the same functional equation, with the germ f replaced by the polynomial p, it must be a multiple of the Koenigs linearization map φ.

4 Faster convergence for degenerate resonant skew-products

Here we will study the rate at which the functions φ_n, introduced in the previous section in Corollary 3.5, converge to the linearization map Φ. The error $\|\varphi_n(\zeta) - \Phi(\zeta)\|$ is in general of order $\lambda^{(\epsilon-1)n}$, as can already been seen in the one-dimensional case. In this section we will only consider *resonant* germs, where $\mu = \lambda^{-1}$. Then for some degenerate maps the error will be of order λ^{-2n}. This occurs, for example, when the quadratic part of F vanishes.

For the maps that we will consider in this section the quadratic part will not vanish, but we will see that if we choose our quadratic part carefully, we can still obtain the rate of convergence λ^{-2n}. We will call the skew-products with this rate of convergence *degenerate*.

Again we consider skew-products of the form

$$F(t, z) = (\mu t, g(t, z)),$$

satisfying the assumptions in Corollary 3.5. Now we also assume that g is a polynomial, and that $\mu = \frac{1}{\lambda}$, where $\lambda = \frac{\partial g}{\partial z}(0, 0) \neq 0$ as before. Let us write

$$F(t, z) = \left(\frac{t}{\lambda}, p(z) + a_1(z)t + \cdots + a_m(z)t^m\right).$$

Let $x_0 \in \mathbb{C}$ satisfy $p^k(x_0) = 0$ for certain $k \in \mathbb{N}$, and define

$$\varphi_{n,j}(\zeta) = \pi_2 F^j\left(\frac{\zeta}{\lambda^n}, x_0\right),$$

and

$$\varphi_n = \varphi_{n,n}.$$

By our results from the previous section it follows that the functions φ_n converge to a holomophic function Φ, uniformly on compact subsets of $\mathbb{C}$.

Lemma 4.1 *The function Φ found in Corollary 3.5 depends only on the polynomial p, the point x_0 and the derivatives $\frac{\partial g}{\partial t}(0, p^j(x_0))$ for $j = 0, \ldots, k$.*

Proof Note that $\varphi_n(0) = 0$ and that $\frac{d}{d\zeta}\varphi_n(0)$ is independent of all other coefficients of g for every n. Hence it follows that $\Phi'(0)$ is also independent of the other coefficients of g. Since the Koenigs map Φ is unique up to a multiplicative constant, the statement follows. □

While the other coefficients have no effect on the limit function Φ, they will be important for us. Choosing them carefully will allow us to obtain a stronger rate of convergence.

For $j \geq k$, $\varphi_{n,j}$ is a holomorphic function with $\varphi_{n,j}(0) = 0$. Therefore, for $j \geq k$ we can write

$$\varphi_{n,j} = C_{n,j} + D_{n,j} + E_{n,j},$$

where the function $C_{n,j} = C_{n,j}(\zeta)$ contains all terms that are linear in ζ, $D_{n,j} = D_{n,j}(\zeta)$ contains all terms that are quadratic in ζ, and $E_{n,j} = E_{n,j}(\zeta)$ contains all higher order terms. Writing

$$F(t, z) = \left(\frac{t}{\lambda}, \lambda z + at + \gamma z^2 + \tau zt + bt^2 + O\left(z^3, z^2t, zt^2, t^3\right)\right), \tag{3}$$

we obtain for $j \geq k$:

$$\varphi_{n,j+1}(\zeta) = \lambda C_{n,j} + \lambda D_{n,j} + \frac{a\zeta}{\lambda^{n+j}} + \gamma C_{n,j}^2 + \frac{\tau C_{n,j}\zeta}{\lambda^{n+j}} + \frac{b\zeta^2}{\lambda^{2n+2j}} + E_{n,j+1}.$$

Hence we have

$$C_{n,j+1} = \lambda C_{n,j} + \frac{a\zeta}{\lambda^{n+j}},$$
$$D_{n,j+1} = \lambda D_{n,j} + \gamma C_{n,j}^2 + \frac{\tau C_{n,j}\zeta}{\lambda^{n+j}} + \frac{b\zeta^2}{\lambda^{2n+2j}}.$$

It follows that we can find a closed form for $C_{n,j}$, for $j \geq k$, of the form

$$C_{n,j} = \zeta \left(Y_1 \lambda^{-n+j} + Y_{-1}\lambda^{-n-j} \right),$$

Where Y_1 and Y_{-1} are independent of n and j. It follows that $C_{n,j}^2$ is of the form

$$C_{n,j}^2 = \zeta^2 \left(\tilde{Y}_2 \lambda^{-2n+2j} + \tilde{Y}_0 \lambda^{-2n} + \tilde{Y}_{-2}\lambda^{-2n-2j} \right),$$

Therefore there exist constants $X_2, X_1, X_0, X_{-1}, X_{-2} \in \mathbb{C}$ so that

$$D_{n,j} = \zeta^2 \left(X_2 \lambda^{-2n+2j} + X_1 \lambda^{-2n+j} + X_0 \lambda^{-2n} + X_{-1}\lambda^{-2n-j} + X_{-2}\lambda^{-2n-2j} \right),$$

again for $j \geq k$.

Note that the constant X_1 depends affinely on τ, b and γ. Hence for generic choices of p and x_0 we can find $b, \tau \in \mathbb{C}$ so that $X_1 = 0$.

Definition 4.2 If F is of the form (3) and $X_1 = 0$, then we say that F is a *degenerate resonant skew-product*.

Theorem 4.3 *Let F be a degenerate resonant skew-product, and let $\epsilon > 0$. Then we can find $\delta > 0$ such that*

$$|\varphi_{n+1}(\zeta) - \varphi_n(\zeta)| \leq |\lambda^{(\epsilon-2)n}|.$$

whenever $|\zeta| < \delta$.

Proof By our assumptions we have that

$$D_{n,j} = \zeta^2 \cdot \left(X_2 \lambda^{-2n+2j} + X_0 \lambda^{-2n} + X_{-1}\lambda^{-2n-j} + X_{-2}\lambda^{-2n-2j} \right).$$

Now notice first that

$$C_{n+1,j+1} - C_{n,j} = \hat{Y}_{-1}\lambda^{-n-j} + \hat{Y}_{-2}\lambda^{-n-j-2},$$

and due to our assumptions we have

$$D_{n+1,j+1} - D_{n,j} = \zeta^2 \left(\hat{X}_0 \lambda^{-2n} + \hat{X}_{-1}\lambda^{-2n-j} + \hat{X}_{-2}\lambda^{-2n-2j} \right).$$

Hence from the above equations we obtain for $|\zeta|$ sufficiently small that

$$|C_{n+1,n+1} + D_{n+1,n+1} - C_{n,n} + D_{n,n}| \leq |\lambda^{-2n}|.$$

What remains is to find similar estimates for $E_{n,j}$.

Lemma 4.4 *For $|\zeta|$ small enough we have that*

$$|E_{n,j}| \leq |\zeta|^2 |\lambda|^{-3n+3j},$$

for all $k \leq j \leq n$.

Proof We prove the statement by induction on j. Note that the required inequality holds for $j = k$ as long as $|\zeta|$ is sufficiently small. Now suppose that

$$|E_{n,j}| \leq |\zeta|^2 |\lambda|^{-3n+3j},$$

for certain $n > j \geq k$. Then we have that

$$E_{n,j+1} = \lambda E_{n,j} + \gamma \left(2C_{n,j} + D_{n,j} + E_{n,j}\right) \cdot \left(D_{n,j} + E_{n,j}\right) + \cdots. \tag{4}$$

Hence there exists a constant K so that for $|\zeta|$ sufficiently small we have

$$|E_{n,j+1}| \leq |\zeta|^2 |\lambda|^{-2} |\lambda|^{-3n+3j+3} + K|\zeta|^3 |\lambda|^{-3n+3j+3}.$$

By making ζ smaller if necessary we obtain

$$\left|E_{n,j+1}\right| \leq |\zeta|^2 |\lambda|^{-3n+3j+3}.$$

Lemma 4.5 *Let $\epsilon > 0$. Then for $|\zeta|$ small enough, we have that*

$$\left|E_{n+1,j+1} - E_{n,j}\right| \leq |\zeta|^2 |\lambda|^{-3n+(1+\epsilon)j},$$

for all $k \leq j \leq n$.

Proof We again prove the statement by induction. The first step

$$\left|E_{n+1,k+1} - E_{n,k}\right| \leq |\zeta|^2 |\lambda|^{-3n+(1+\epsilon)k}$$

follows directly from our previous lemma.

Suppose that $|E_{n+1,j+1} - E_{n,j}| \leq |\zeta|^2 |\lambda|^{-3n+(1+\epsilon)j}$ for certain $k \leq j \leq n$. Then it follows from Eq. 4, our induction hypothesis and our previous estimates that

$$\begin{aligned}
\left|E_{n+1,j+2} - E_{n,j+1}\right| &\leq |\lambda| \left|E_{n+1,j+1} - E_{n,j}\right| + K_2 |\zeta|^3 |\lambda|^{-3n+j} \\
&\leq |\lambda| \cdot |\zeta|^2 |\lambda|^{-3n+(1+\epsilon)j} + K_2 |\zeta|^3 |\lambda|^{-3n+j} \\
&\leq |\zeta|^2 |\lambda|^{-3n+(1+\epsilon)(j+1)},
\end{aligned}$$

where the last inequality holds (uniform over all j and n) for ζ sufficiently small. □

Completing the proof of Theorem 4.3: Plugging $j = n$ into the estimate of the last lemma gives

$$\left|E_{n+1,j+1} - E_{n,j}\right| \leq |\zeta|^2 |\lambda|^{(\epsilon-2)n},$$

which combined with our earlier estimates on $|C_{n+1,j+1} + D_{n+1,j+1} - C_{n,j} - D_{n,j}|$ gives the required estimate on $|\varphi_{n+1,j+1}(\zeta) - \varphi_{n,j}(\zeta)|$, and completes the proof of Theorem 4.3. □

5 Vertical Fatou disks

Again we consider skew-products of the form

$$F(t, z) = \left(\frac{t}{\lambda}, g(t, z)\right),$$

with $|\lambda| > 1$. Now we make the additional assumption that

$$g(t, z) = p(z) + q(t), \tag{5}$$

where $p(t) = \lambda t + h.o.t.$ and $q(t) = at + bt^2 + h.o.t.$ with $a \neq 0$. We assume that F is a degenerate resonant skew-product, so that Theorem 4.3 holds. Finally we assume that p has a critical point x_0 of order at least 3, for which $p^k(x_0) = 0$.

Let $w_0 \in \mathbb{C}$ be such that $\Phi(w_0) = x_0$, i.e. $\pi_2 \circ F^n(\frac{w_0}{\lambda^n}, x_0) \to x_0$ as $n \to \infty$. Such a w_0 might not be unique but it can be found since x_0 necessarily lies in the Julia set of p, and hence not in the exceptional set of p. Therefore x_0 is contained in the union

$$\bigcup_{n\in\mathbb{N}} p^n(U),$$

for any neighborhood U of the repelling fixed point $0 \in \mathcal{J}(p)$. The fact that w_0 exists now follows from the functional equation

$$\Phi(\lambda \cdot \zeta) = p(\Phi(\zeta)).$$

We will refer to the complex lines $\{t = \frac{w_0}{\lambda^n}\}$ as *critical fibers*.

Definition 5.1 We define the vertical disks D_n as follows:

$$D_n := \left\{ \left(\frac{w_0}{\lambda^n}, z \right) \mid |z - x_0| < |\lambda^{-3n/4}| \right\}.$$

We will prove that for n sufficiently large, the forward orbits of the disks D_n all avoid the bulged Fatou components of F.

Lemma 5.2 *For n sufficiently large we have that*

$$F^n(D_n) \subset D_{2n}.$$

Proof Since x_0 is a critical point of order at least 3 and we assumed that $g(t, z) = p(z) + q(t)$, it follows that $F(D_n)$ is contained in a vertical disk centered at the point $F(\frac{w_0}{\lambda^n}, x_0)$ of radius $M_1|\lambda^{-3n}|$, for some constant $M_1 > 0$ which is uniform over $n \in \mathbb{N}$.

For any $\epsilon > 0$ there exists an m independent of n so that for $j = k, \ldots, n - m$ the z-coordinates of the images $F^j(D_n)$ lie in the ϵ-neighborhood of the repelling fixed point $0 = p(0)$ with multiplier λ.

Hence by choosing $\epsilon > 0$ small enough we see that for n large enough the image $F^n(D_n)$ is contained in the disk centered at $F^n(\frac{w_0}{\lambda^n}, x_0)$ of radius

$$M_2|\lambda^{-3n}| \cdot |\lambda|^{\frac{5n}{4}} = M_2|\lambda|^{-\frac{7n}{4}}.$$

The first coordinate of $F^n(\frac{w_0}{\lambda^n}, x_0)$ is $\frac{w_0}{\lambda^{2n}}$, and the distance to x_0 of the second coordinate is at most $M_3|\lambda|^{(\epsilon-2)n}$ by Theorem 4.3. Hence by the definition of D_{2n} we have that for n large enough the image $F^n(D_n)$ is contained in the disk D_{2n}. □

Remark 5.3 The condition of resonance is needed in our proof, since we use Theorem 4.3, which is only valid in the resonant case. We have not been able to rule out Fatou discs in the non-resonant case.

Remark 5.4 An immediate consequence of Lemma 5.2 is that for sufficiently large $n \in \mathbb{N}$ we have that $F^{(2^\ell-1)n}(D_n) \to (0, x_0)$ as $\ell \to \infty$.

Our main result is now a quick consequence.

Theorem 5.5 *There exists an $N \in \mathbb{N}$ so that for every $n \geq N$ the forward images of the disks D_n never intersect the fattened Fatou-components of F.*

Proof Notice that $F^{ji}(D_n) \to (0, x_0)$ implies that the images $F^j(D_n)$ cannot intersect one of the bulged Fatou components of F. This follows from the fact that the Fatou components of p are (pre-)periodic, and the classification of periodic Fatou components. Each periodic Fatou component of p is either an attracting basin, a parabolic basin or a Siegel disk. In the attracting case, all orbits in the fattened Fatou converge to the attracting periodic point in the invariant fiber. In the other two cases, the fattened Fatou component consists of the strong stable manifolds through points in the one-dimensional Fatou component. It follows that for all three cases there can never be a subsequence for which an orbit in a fattened Fatou component converges to a non-periodic point in the Julia set. It follows immediately from the attraction in the horizontal direction that the same holds for the corresponding bulged Fatou components of F. □

Lemma 5.6 *For each sufficiently large $n \in \mathbb{N}$ the forward images $F^j(D_n)$ accumulate at a closed and invariant subset of the form*

$$\Omega = \{x_{-\ell}\}_{\ell=-k}^{+\infty},$$

where

$$p(x_{-\ell}) = x_{-\ell+1} \quad \text{and} \quad \lim_{\ell \to \infty} x_{-\ell} = 0. \tag{6}$$

Moreover, for each choice of a set Ω satisfying the conditions in (6) we can find a vertical disk with its ω-limit set equal to Ω.

Proof Recall that for each sufficiently large n the image $F^n(D_n)$ is contained in the disk D_{2n}. By the functional equation (2) it follows that

$$\pi_2 F^{2^j n - \ell}\left(D_{2^j n}\right)$$

converges to a point $x_{-\ell}$ as $j \to \infty$. Hence the ω-limit set contains the required sequence $\{x_{-\ell}\}$ and the point 0 which lies in the closure. To see that the ω-limit set contains no other points, notice that the images

$$\pi_2 F^{2^j n - \ell}\left(D_{2^j n}\right)$$

converge to 0 as $\ell \to \infty$, uniformly over all j as long as $2^{j-1} n > \ell$.

For the converse, let $\{x_n\}$ be an inverse orbit of p which converges to the repelling fixed point 0. Since $p'(0) \neq 0$, the function p is injective in a small disk $U = U(0)$. Let $N \in \mathbb{N}$ be such that $x_n \in U$ for $n \geq N$, which implies that x_{-N} determines the entire inverse orbit.

The holomorphic map Φ satisfies $\Phi(0) = 0$. Hence we can find $w \in \mathbb{C}$ so that $\Phi(\frac{w}{\lambda^N}) = x_{-N}$ and so that $\Phi(\frac{w}{\lambda^{N+j}}) \in U$ for all $j \in \mathbb{N}$. The claim follows. □

Remark 5.7 It is not hard to generate many explicit examples of degenerate resonant skew-products for which p has a critical point x_0 of order at least 3 with $p^k(x_0) = 0$. Consider for example the polynomials $p(z) = 2(z+1)^d - 2$, for $d \geq 4$ even. Here the multiple critical point $z = -1$ of order $d - 1$ is mapped in two steps to the repelling fixed point $z = 0$. Since the derivative $p'(0) = 2d$, we can consider skew products of the form

$$F(t, z) = \left(\frac{t}{2d}, 2(z+1)^d - 2 + at + bt^2\right).$$

Given $a \neq 0$ the value for b can be explicitly computed. For example if $d = 4$ and $a = 1$ then computation shows that we should take $b = -\frac{641}{4165}$ in order for F to be a degenerate resonant skew-product. One easily checks that the assumptions in Corollary 3.5 do not depend on

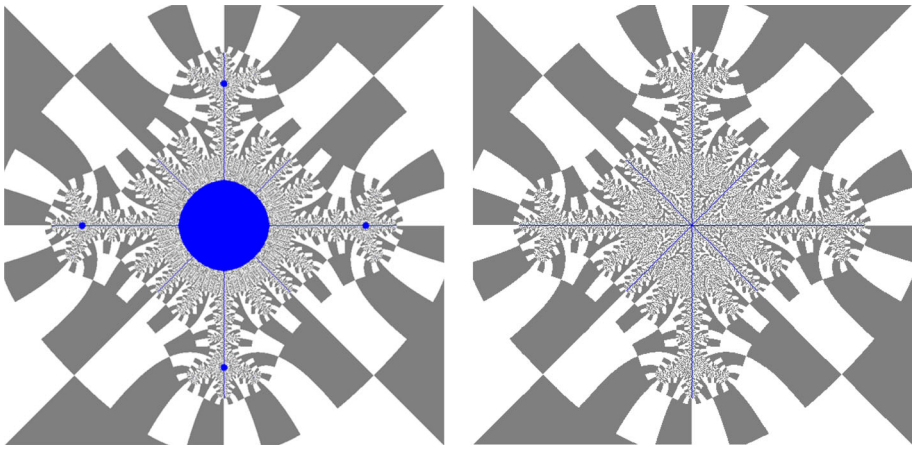

Fig. 1 A vertical Fatou disk and a nearby fiber

the value of b, as pointed out in Lemma 4.1, and are satisfied for this map. Two computer pictures of vertical fibers for this example are shown in Fig. 1. On the left a critical fiber $t \sim 0.1674669561$ with a clearly visible Fatou disk. On the right a nearby fiber $\{t = t_0\}$, for $t_0 = 0.16$, where the one-dimensional filled Julia set

$$\{z \in \mathbb{C} \mid \{F^n(t_0, z)\} \text{ bounded}\}$$

seems to have empty interior. This suggests that the Fatou disk lies in the Julia set of the two-dimensional map, which we prove in the next section.

The notion of a Fatou-map was introduced by Ueda in [19].

Definition 5.8 Let $f : X \to X$ be a holomorphic endomorphism of a complex manifold X. A holomorphic disks $D \subset X$ is a *Fatou disk* for f if the restriction of $\{f^n\}$ to the disk D is a normal family.

Lemma 5.9 *The disks D_n are Fatou disks.*

Proof The t-coordinates converge to zero. Note that there exist a radius of escape $R > 0$ that is valid for all $|t| < 1$, i.e. if (t, z) is such that $|z| > R$ and $|t| < 1$ then $\|F^n(t, z)\| \to \infty$. As the z-coordinates stay bounded for some subsequence, it therefore follows that the entire sequence of z-coordinates must stay bounded. Hence normality follows by Montel's Theorem. □

6 The disks lie in the Julia set

As in the previous section we consider a degenerate resonant skew-product of the form

$$F(t, z) = \left(\frac{t}{\lambda}, p(z) + q(t)\right).$$

As before, we assume that p has a critical point x_0 of order $s \geq 3$ which by p^k is mapped to the repelling fixed point $0 = p(0)$ with multiplier λ. In this section we make the additional assumption that p has no critical points besides x_0. Note that the examples mentioned in Remark 5.7 satisfy this assumption.

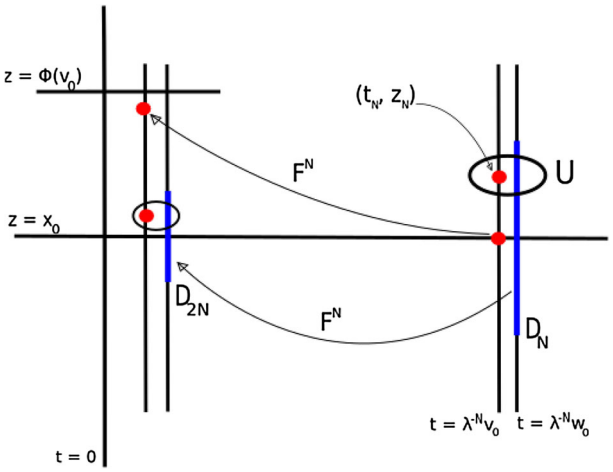

Fig. 2 The major players in the proof of Theorem 6.1

We let D_n be the vertical disk centered at $(\frac{w_0}{\lambda^n}, x_0)$ as defined in 5.1. Recall from Lemma 5.2 that there exists an $N \in \mathbb{N}$ for which $F^n(D_n) \subset D_{2n}$ for all $n \geq N$.

Theorem 6.1 *For $n \geq N$ the disks D_n lie entirely in the Julia set of F.*

Proof We will prove that D_N lies in the Julia set of F. As the condition that $F^n(D_n) \subset D_{2n}$ for $n \geq N$ is also satisfied when N is increased, it follows that D_n lies in the Julia set for all $n \geq N$.

Let us give an outline of the argument before diving into precise estimates. In Fig. 2 we sketch the role of the most important objects in the proof.

We suppose for the purpose of a contradiction that D_N does intersects some open set U which is a relatively compact subset of the Fatou set of F. Let $v_0 \neq w_0$ be such that U intersects $\{t = \lambda^{-N} v_0\}$. Since Φ is a nonconstant holomorphic function we may assume that $|v_0 - w_0|$ is sufficiently small so that $\Phi(v_0) \neq \Phi(w_0) = x_0$.

Let $(t_N, z_N) \in U \cap \{t = \lambda^{-N} v_0\}$ and write $(t_n, z_n) = F^{n-N}(t_N, z_N)$, with $F^n = (F_1^n, F_2^n)$ as before. We will show that

$$\left| \frac{\partial}{\partial z} F_2^{(2^l-1)N}(t_N, z_N) \right| \to \infty,$$

as $l \to \infty$, which contradicts the normality of the family $\{F^n\}$ in U. This estimate on the derivative follows in two steps. First we will find an estimate from below for

$$\left| z_{2^l N} - x_0 \right|.$$

For large l we obtain this lower bound because on the one hand the definition of Φ implies that $F_2^{2^l N}(t_{2^l N}, x_0)$ lies close to $\Phi(v_0) \neq x_0$, while on the other hand $F_2^{2^l N}(t_{2^l N}, z_{2^l N})$ lies close to x_0 since $F^{(2^l-1)N}(D_N) \subset D_{2^l N}$.

Since F is a skew product, we can express $\frac{\partial}{\partial z} F_2^{(2^l-1)N}(z_N, t_N)$ as a product of derivatives of $(2^l - 1)N$ one-dimensional polynomials. Combining the above lower bound on the distance to the critical point with the fact that the orbit (z_n) will spend most of its time near the repelling fixed point $0 = p(0)$ will give us the required estimate on

$$\left|\frac{\partial}{\partial z} F_2^{(2^l-1)N}(z_N, t_N)\right|.$$

Let us now go through the details of the argument outlined above. Recall that the ω-limit set of every disk D_n is contained in the Julia set of p, which has no interior. By the one-dimensional classification of Fatou components of p, the set U cannot be contained in a bulging Fatou component of p. Therefore orbits in the Fatou component containing U must converge to the Julia set of p as well. Since the Julia set of p has no interior and U is relatively compact in the Fatou set of F, it follows that the diameter of $F^j(U)$ must converge to 0 as $j \to \infty$.

Let us write $y = \Phi(v_0)$ and define $\epsilon > 0$ small enough so that

$$|y - x_0| > 3\epsilon.$$

Since the diameter of $F^j(U)$ converges to zero, there exists a $J_1 \in \mathbb{N}$ so that $\operatorname{diam}(F^j(U)) < \epsilon$ for $j \geq J_1$. Since $F^n(D_n) \subset D_{2n}$ for all $n \geq N$ and the disks D_n converge to the point $(0, x_0)$, it follows that there exists an $L \in \mathbb{N}$ so that for $\ell \geq L$ one has that $F^{(2^\ell-1)N}(D_N)$ is contained in the ball centered at $(0, x_0)$ with radius ϵ. It follows that for $\ell \geq L$ and $j = (2^\ell - 1)N \geq J_1$ we have that

$$d(F^j(U), (0, y)) > \epsilon.$$

Recall that $\pi_2 \circ F^j(\frac{v}{\lambda^j}, x_0)$ converges to y, hence there exists $J_2 \geq J_1$ so that for $j \geq J_2$ we have that

$$d\left(F^j\left(\frac{v}{\lambda^j}, x_0\right), (0, y)\right) < \frac{\epsilon}{2}.$$

It follows that for large $\ell \in \mathbb{N}$ we have

$$\left|z_{2^{\ell+1}N} - F_2^{2^\ell N}(t_{2^\ell N}, x_0)\right| > \frac{\epsilon}{2}, \tag{7}$$

where we recall that $t_{2^\ell N} = \lambda^{-2^\ell} v_0$ and

$$z_{2^{\ell+1}N} = F_2^{2^\ell N}(t_{2^\ell N}, z_{2^\ell N}).$$

Recall that x_0 is a critical point of p of order $s - 1$, which implies that

$$|p(z) - p(x_0)| \leq M\,|z - x_0|^s$$

for $|z - x_0|$ small and certain $M > 0$. Let $\mathcal{N}_0$ be a small disk centered at the point 0 so that

$$\lambda_- := |\lambda|^{1-\frac{1}{4s}} \leq \left|p'(z)\right| \leq |\lambda|^{1+\frac{1}{4s}} =: \lambda_+$$

for all $z \in \mathcal{N}_0$. Now note that for $2^\ell N < j < 2^{\ell+1}N$ the sequence (z_j) is bounded away from the critical point x_0, and is contained in $\mathcal{N}_0$ for all but an absolute number of iterates. Hence we can increase M if necessary so that

$$\left|z_{2^{\ell+1}N} - F_2^{2^\ell N}(t_{2^\ell N}, x_0)\right| \leq \lambda_+^{2^\ell N} \cdot M \cdot |z_{2^\ell N} - x_0|^s. \tag{8}$$

Combining Eqs. (7) and (8) we obtain

$$\left|z_{2^\ell N} - x_0\right| \geq \delta_0 \lambda_+^{-\frac{2^\ell N}{s}}, \tag{9}$$

for certain $\delta_0 > 0$.

Equation (9) induces an estimate from below on $|p'(z_{2^\ell N})|$. Since x_0 is a critical point of order $s-1$, there exists an $\delta_1 > 0$ so that

$$\left|p'(z)\right| \geq \delta_1 \cdot |z - x_0|^{s-1}$$

whenever $|z - x_0|$ is sufficiently small. It follows that

$$\left|p'(z_{2^\ell N})\right| \geq \delta_2 \cdot \lambda_+^{\frac{1-s}{s} 2^\ell N}.$$

Note that

$$DF^j(t_n, z_n) = \begin{bmatrix} f'(t_{n+j-1}) & 0 \\ q'(t_{n+j-1}) & p'(z_{n+j-1}) \end{bmatrix} \cdots \begin{bmatrix} f'(t_n) & 0 \\ q'(t_n) & p'(z_n) \end{bmatrix}.$$

It follows that

$$\frac{\partial F_2^j}{\partial z}(t_n, z_n) = \prod_{i=0}^{j-1} p'(z_{n+i}).$$

Since for $2^\ell N + 1 \leq j < 2^{\ell+1} N - 1$ we have that $z_j \in \mathcal{N}_0$ for all but an absolute number of iterates, and for all such j there is an absolute bound from below on $|z_j - x_0|$, and thus also on $|p'(z_j)|$, there exists an $\delta_3 > 0$ for which

$$\prod_{j=2^\ell N+1}^{2^{l+1}N-1} \left|p'(z_j)\right| \geq \delta_3 \lambda_-^{2^\ell N}.$$

Now notice that

$$\lambda_-^s \cdot \lambda_+^{1-s} \geq |\lambda|^{\frac{1}{2}}$$

which implies that

$$\left| \frac{\partial F_2^{2^\ell N}}{\partial z}(t_{2^\ell N}, z_{2^\ell N}) \right| \geq \delta_3 \lambda_-^{2^\ell N} \cdot \delta_2 \lambda_+^{\frac{1-s}{s} 2^\ell N} \geq \delta_5 |\lambda|^{2^\ell \frac{N}{2s}},$$

whose product over all ℓ converges to infinity. Therefore (F^j) cannot be a normal family in any neighborhood of (t_N, z_N), which contradicts our initial assumption. This completes the proof. □

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