Essays on auctions

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Risk Aversion and Optimal Reserve Prices

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2.1 Introduction

Most of the studies in the auctions literature begin with the assumption that the bidders are risk neutral for income (or that they have a utility function). A well-known result obtained under this assumption is the revenue equivalence theorem (e.g., Vickrey, 1961; Myerson, 1981; Riley and Samuelson, 1981; Krishna and Maenner, 2001; and Milgrom and Segal, 2002), which predicts that when bidders have independent private information (or signals) the payment rules do not matter under the same allocation rule. In other words, as long as the lowest bidder is indifferent between winning and losing, the allocation rule uniquely determines the expected payoffs of all participants, including the buyers as well as the seller.

None of the above predictions is robust to changes in the bidders' risk preferences, however. Existing studies show (e.g., Holt,1980; Riley and Samuelson, 1981; Harris and Raviv, 1981; Milgrom and Weber, 1982; Matthews, 1983, 1987; Maskin and Riley, 1984; Cox et al. ,1982, 1988) that in symmetric independent private values settings, bidders submit higher bids in the first-price auction when they are risk averse rather than risk neutral. This implies that the expected revenue is higher in the first-price auction than it is in the second-price auction, as the bids in the second-price auction are unaffected by the risk attitudes of the bidders. This unambiguous result has two immediate consequences. First, it suggests that a risk neutral seller would prefer the Dutch or first-price auctions (henceforth, FPA) to the English or second-price auctions (henceforth, SPA) when bidders are risk averse and have symmetric independent private values.[1] Second, even if the bidders' values are interdependent and signals affiliated, by a continuity argument the seller's preference for the

[1] We use the term FPA for both the first-price sealed-bid auction and the strategically equivalent Dutch (descending) auction. We use the term SPA for both the second-price sealed-bid (Vickrey) auction and the "button" model of the English ascending-bid auction, as they have the same dominant strategy equilibria in our private values setting (Milgrom and Weber, 1982).

FPA may continue to hold, as long as the interdependence of the bidders' information and valuation is not "too" strong.[2]

The above-mentioned results have been obtained assuming that the same reserve price is exogenously given in all auctions. However, the reserve price in most real auctions is set by the seller. To the extent that it influences bidding behavior and depends on the type of auction, the endogeneity of the reserve price should be taken into account. In particular, the comparative statics of the optimal reserve price are of direct interest because they bear on ex post efficiency. Lowering the reserve price decreases the probability of the inefficient event in which no sale occurs because the maximum value of the bidders exceeds the seller's value but not the reserve price.

In auction models with private values and independent signals, it is well-known that the seller tends to fix a reserve price strictly higher than his own value for the object for sale.[3] This commitment that maximizes the seller's expected revenue (or utility) ex ante entails potential inefficiency ex post, as it excludes some buyers from purchasing the good even though they are willing to pay a price higher than the seller's value. Indeed, exclusion inefficiency of this kind frequently emerge from the studies of various economic problems such as credit rationing (e.g., Stiglitz and Weiss (1981)), monopoly pricing (e.g., Armstrong (1996)), and so on. The general message from these results is that under incomplete information, a monopolistic seller or creditor typically engages in mechanisms that exclude some low value buyers or high risk borrowers – despite the fact that it incurs inefficiency. As such, these results help identify the sources of market failure, offer plausible explanations of otherwise puzzling economic

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[2] This and related results are established, e.g., by Holt (1980), Riley and Samuelson (1981), Harris and Raviv (1981), Milgrom and Weber (1982), Matthews (1983, 1987), Maskin and Riley (1984), Cox et al. (1982, 1988), Smith and Levin (1996), and Eso and White (2004).

[3] This result was independently found in, e.g., Matthews (1980), Myerson (1981), Riley and Samuelson (1981), and Maskin and Riley (1984). See also Hu, Matthews, and Zou (2010) for a more recent treatment.

phenomena, and suggest possible solutions in terms of relevant regulatory policies.

This chapter focuses on the effects of buyer and seller risk aversion on the seller's optimal reserve price in standard first and second-price auctions. Sharp results are obtained by restricting attention to the otherwise simplest setting, that of symmetric and independent private values. The main results are Theorems $1-3$.

Theorem 1 establishes that if the seller and/or the buyers are risk averse, then the seller sets a lower reserve price in the FPA than in the SPA. This is in contrast to when all parties are risk neutral, in which case the revenue equivalence theorem implies that the seller's optimal reserve price is the same in both auctions. Risk aversion thus makes the FPA more ex post efficient than the SPA. The result hinges on how the FPA equilibrium bid function is affected by a marginal increase in the reserve price. Risk averse bidders increase their bids less than do risk neutral bidders, and a risk averse seller values the increase in the bids of the high bidders relatively less than does a risk neutral seller because of diminishing marginal utility. Both forces lower the seller's marginal incentive to raise the reserve price.

Theorem 2 establishes that in either auction, a more risk averse seller sets a lower reserve price. Thus, the more risk averse the seller, the more ex post efficient are both auctions. The intuition is straightforward: a more risk averse seller values more (on the margin) a decrease in the risk of not selling the object. The proof, however, is surprisingly intricate.[4]

Theorem 3 establishes that in two fairly general cases, the seller sets a lower reserve price in the FPA if the bidders are more risk averse. (Bidder risk aversion does not affect the SPA equilibrium.) In case (a) the reverse hazard rate function of the bidders' values is decreasing, and either the more risk averse or the less risk averse group of bidders (or both) exhibit nonincreasing absolute risk aversion. In case (b) the more risk averse bidders are strictly more risk averse, in the sense that the minimum of their

[4] Theorem 3 in Waehrer et al. (1998) is our Theorem 2 for the case of risk neutral bidders (and a more general information structure). Our proof takes a different approach.

Arrow-Pratt measure of risk aversion exceeds the maximum of that of the less risk averse buyers. In either case the rate at which the FPA bid function increases in the reserve price is smaller when the bidders are more risk averse. This gives the seller less incentive to raise the reserve price. This effect is stronger if the seller is also risk averse, as then the fact that more risk averse bidders bid higher than less risk averse bidders implies that the seller has a lower marginal utility for the increase in their bids caused by an increase in the reserve price.

The chapter begins with the model in Section 2.2. Useful technical results are in Section 2.3. The FPA equilibrium is studied in Section 2.4. The seller's preferences over auctions with the same reserve price are determined in Section 2.5, and his optimal reserve prices are examined in Section 2.6. Section 2.7 concludes.

2.2 Model

An indivisible object is to be possibly sold to one of $n \geq 2$ potential buyers through either a FPA or a SPA with a reserve price. Each buyer $i \in N = \{1, ..., n\}$ has a private value for the object, v_i, which is unknown to the others. Ex ante, these values are independently distributed on an interval $[L, H]$ according to the same distribution function F, which has a density function $f = F'$ that is strictly positive and continuously differentiable on $[L, H]$. Some of our results are obtained under the assumption of a decreasing reverse hazard rate function:

$$\text{(DRH)} \quad \frac{f(v)}{F(v)} \text{ strictly decreases on } (L, H).$$

Each participant maximizes expected utility. Each buyer has the same utility function, $u_B : \mathbb{R} \to \mathbb{R}$. If a buyer with value v wins and pays a price b, his utility is $u_B(v - b)$; his utility is $u_B(0)$ if he loses.[5] We assume u_B is twice continuously differentiable, with $u_B' > 0$ and $u_B'' \leq 0$.

The seller has a value $v_0 \in [L, H)$ for the object, and a twice continuously differentiable utility function, $u_S : \mathbb{R} \to \mathbb{R}$, satisfying $u_S' > 0$ and $u_S'' \leq 0$. The seller's utility is $u_S(b)$ if a sale occurs at price b, and it is $u_S(v_0)$ otherwise.

We consider first and second-price auctions with a reserve price $r \in [L, H]$.[6] In an equilibrium of either auction, a buyer with a value $v < r$ abstains from bidding. In a SPA, the dominant strategy of a buyer with $v \geq r$ is to submit a bid equal to v. We restrict attention to this equilibrium of the SPA.

Turning to the FPA, it is useful to define $G \equiv F^{n-1}$. If a buyer has value v, then $G(v)$ is the probability that every other buyer has a lower value.

[5] A more general formulation would have $u_B = u(v, -b)$ as the winning bidder's payoff. Under appropriate assumptions, as in Maskin and Riley (1984) or Matthews (1987), our main results extend to this generalization.

[6] This is without loss of generality, since in either auction the equilibrium for $r < L$ is the same as it is for $r = L$ (Maskin and Riley, 1984, Remark 2.1), and for $r > H$ there is zero probability of a sale in any equilibrium.

Let $g \equiv G'$ be the associated density, and let $\ell(v) = g(v)/G(v)$. Lastly, define the function $\gamma = [u_B - u_B(0)]/u'_B$. Then the unique symmetric equilibrium bidding function of the FPA, $b(\cdot, r)$, is equal to the solution on $[r, H]$ of the differential equation,

$$b_1(v,r) = \frac{g(v)\left[u_B(v-b) - u_B(0)\right]}{G(v)u'_B(v-b)} = \ell(v)\gamma(v-b), \tag{2.1}$$

that satisfies the initial condition $b(r,r) = r$ (e.g., Maskin and Riley, 1984). We restrict attention to this equilibrium of the FPA.[7] Observe that for $r \in (L, H)$, $b_1(r,r) = 0$ and $b_2(r,r) = 1 - b_1(r,r) = 1$. For $r = L$ we have $b_1(L,L) = \frac{n-1}{n}$,[8] and $b_2(L,L)$ is undefined (see footnote 11 below).

Let $R_i = -u''_i/u'_i$ denote the Arrow-Pratt measure of absolute risk aversion for $i = B, S$. The case in which the bidders have constant absolute risk aversion (CARA) provides a benchmark, as then (2.1) can be solved explicitly. When $R_B \equiv a$ for some $a \geq 0$, the FPA equilibrium is

$$b^a(v,r) = \frac{1}{a}\ln\left(e^{av} - a\int_r^v \left(\frac{G(y)}{G(v)}\right)e^{ay}dy\right) \quad \text{for } v \geq r. \tag{2.2}$$

The explicit solution for the CARA bidders allows us to conveniently visualize the behavior of the bid functions. For instance, Figure 2.1 shows how risk aversion causes the bidders to submit higher bids in comparison with the risk neutral bids.

2.3 Technical Preliminaries

It will be repeatedly useful to note that the function γ is related to the risk aversion measure R_B by $\gamma' = 1 + R_B\gamma$. For $t \geq 0$ we have $\gamma(t) \geq 0$,

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[7] It is the only equilibrium if $r > L$ and the buyers have nonincreasing absolute risk aversion (Maskin and Riley, 2003).

[8] We obtain $b_1(L,L) = \frac{n-1}{n}$ from (2.1) and L'Hospital's rule:

$$\begin{aligned} b_1(L,L) &= \lim_{v \downarrow L} \frac{(n-1)f(v)\left[u_B(v - b(v,L)) - u_B(0)\right]}{F(v)u'_B(v - b(v,L))} \\ &= \lim_{v \downarrow L}(n-1)(1 - b_1(v,L)) = (n-1)(1 - b_1(L,L)). \end{aligned}$$

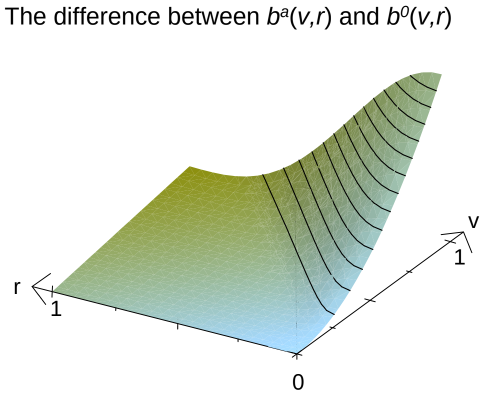

FIGURE 2.1. The FPA bid function increases in the bidders' risk aversion. The figure depicts the difference $b^a(v,r) - b^0(v,r)$ (where $a = 2$ and b^0 is the risk neutral bid). This difference is strictly positive for $a > 0$, and it increases in v and decreases in r.

and so $\gamma'(t) > 0$. If $\hat{u}_B$ is another utility function such that $\widehat{R}_B > R_B$, then $\hat{\gamma}(t) > \gamma(t)$ for $t > 0$ (Pratt, 1964, Theorem 1).

We will also use the following two lemmas (their proofs are in the Appendix). The first lemma is a variation of the "Ranking Lemma" of Milgrom (2004).

Lemma 1 *For $c < d \leq \infty$ and $h : [c,d] \to \mathbb{R}$ differentiable, if $h(c) \geq 0$ then*

(i) $h > 0$ *on* $(c,d]$ *if* $[\forall t \geq c,\ h(t) = 0 \Rightarrow h'(t) > 0]$,

(ii) $h > 0$ *on* $(c,d]$ *if* $[\forall t > c,\ h(t) \leq 0 \Rightarrow h'(t) > 0]$.

Lemma 2 *For $c < d \leq \infty$ and $i = 1, 2$, let the functions $h_i : [c, d] \to \mathbb{R}$ be differentiable and satisfy $h_1' < h_2'$ on (c, d). Let t_i maximize h_i on $[c, d]$. If $t_i \in (c, d)$ for $i = 1$ or $i = 2$, then $t_1 < t_2$.*

2.4 Properties of the FPA Equilibrium

The FPA equilibrium is well known to satisfy $b(v, r) < v$ and $b_1(v, r) > 0$ for any $L \leq r < v$.[9] Our first proposition provides an expression for b_2 that shows how the equilibrium varies with the reserve price.

Proposition 1 *For $r \in (L, H)$ and $v \in [r, H]$,*

$$b_2(v, r) = \frac{G(r)}{G(v)} \exp\left(-\int_r^v b_1(y, r) R_B(y - b(y, r)) dy\right), \tag{2.3}$$

and hence

$$0 \underset{(i)}{<} b_2(v, r) \underset{(ii)}{\leq} \frac{G(r)}{G(v)}.$$

Inequality (ii) is an equality if $R_B = 0$; it is a strict inequality if $v > r$ and $R_B > 0$.[10] *Lastly, $b_2(v, L) = 0$ for all $v \in (L, H]$.*[11]

Proof. Because the right-hand side of (2.1) is continuously differentiable in b, v, and r (which does not appear explicitly), the solution $b(v, r)$ is continuously differentiable in v and r for $r \in (L, H)$ and $v \in [r, H]$ (e.g., Hale, 2009; Chapter 1, Theorem 3.3). This implies, in turn, that b_{12} exists: by differentiating (2.1) with respect to r and using $\gamma' = 1 + R_B \gamma$, we obtain

$$b_{12}(v, r) = -\ell(v) (1 + \gamma R_B) b_2. \tag{2.4}$$

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[9] Fix $r \in [L, H)$, and consider $h(v) = v - b(v, r)$ on $[r, H]$. Note that $h(r) = 0$. Suppose $h(v) \leq 0$ for some $v > r$. Then $h'(v) = 1 - \ell(v)\gamma(h(v)) > 0$. Hence, by Lemma 3 (ii), $v - b(v, r) = h(v) > 0$ for $v \in (r, H]$. This and (2.1) imply $b_1(v, r) > 0$ for $v \in (r, H]$.

[10] Here and below, $R_B > 0$ is a functional inequality, meaning that $R_B(y) > 0$ for all y in the relevant interval, which is $[0, H - L]$.

[11] The derivative $b_2(L, L)$ cannot be defined because $b(L, r)$ is undefined for $r > L$.

Fix $r \in (L, H)$. Since $b_2(r, r) \equiv 1$, the continuity of $b_2(\cdot, r)$ on $[r, H]$ implies the existence of $\bar{v} \in (r, H]$ such that

$$\bar{v} = \max\{v \in [r, H] : b_2(y, r) > 0 \text{ for all } r \leq y < v\}.$$

Note that $b_2(\bar{v}, r) = 0$ if $\bar{v} < H$. Now, $\ln b_2(y, r)$ is well defined for $y \in [r, \bar{v})$, and from (2.4) we have

$$\frac{\partial}{\partial y} \ln b_2(y, r) = -\ell(y)\left(1 + \gamma R_B\right).$$

Integrating this on $[r, v]$ for $v \in (r, \bar{v})$ yields, since $\ell(y) = d \ln G$ and $\ln b_2(r, r) = 0$,

$$\begin{aligned} \ln b_2(v, r) &= -\int_r^v d \ln G - \int_r^v \ell(y) \gamma R_B dy \\ &= \ln \frac{G(r)}{G(v)} - \int_r^v b_1 R_B dy, \end{aligned}$$

using (2.1) in the last step. Hence, (2.3) holds at any $v \in (r, \bar{v})$. It also holds at $v = \bar{v}$, by the continuity of $b_2(\cdot, r)$. As this implies $b_2(\bar{v}, r) > 0$, we have $\bar{v} = H$. This proves that (2.3) holds for any $r \in (L, H)$ and $v \in [r, H]$. Inequality (i) is now immediate from $r > L$. Inequality (ii) also follows from (2.3), since $\gamma R_B = 0$ if $R_B = 0$, and $\gamma R_B > 0$ for $v > r$. Lastly, fix $v \in (L, H]$. The right side of (2.3) then converges to 0 as $r \downarrow L$. Hence, $b_2(v, L) = \lim_{r \downarrow L} b_2(v, r) = 0$. ■

Our second proposition establishes the intuitive property that a bidder's profit conditional on winning, $v - b(v, r)$, increases in v, provided that the reverse hazard rate is decreasing.

Proposition 2 *If* (DRH) *holds, then* $b_1(v, r) < 1$ *for* $L \leq r < v$.

Proof. We apply Lemma 1(i) to $1 - b_1(\cdot, r)$. Recall $1 - b_1(r, r) > 0$. Suppose $1 - b_1 = 0$ at some $v \geq r$. Then $v > r$. Differentiating (2.1) with respect to v, and evaluating the result at this (v, r), yields

$$b_{11} = \ell' \gamma + \ell \gamma' \times (1 - b_1) = \ell' \gamma.$$

We have $\gamma(v-b)>0$ because $b<v$, and $\ell'(v)<0$ by (DRH). Hence, at this (v,r), $\partial\left[1-b_1\right]/\partial v=-b_{11}>0$. Lemma 1$(i)$ now implies $1-b_1>0$ for $v\geq r$. ■

Our third proposition determines the effects of the bidders becoming more risk averse. Part (i) shows that the bid function increases in their risk aversion, generalizing the well-known result that bids are higher when the bidders are risk averse than when they are risk neutral. The remainder of the proposition establishes more surprising results, assuming that (DRH) holds, and the seller and/or the buyers exhibit nonincreasing absolute risk aversion. Parts (ii) and (iii), respectively, show that then, the more risk averse are the bidders, the more rapidly the bid function increases in a bidder's value, but the more slowly it increases in the reserve price. The latter property is largely why the seller's optimal reserve price decreases in the risk aversion of the bidders, as we shall see.

Proposition 3 *Let $\hat{u}_B$ be another utility function satisfying the same assumptions as u_B, with an absolute risk aversion measure satisfying $\widehat{R}_B>R_B$ on $[0,H-L]$. Let $\hat{b}$ be the FPA equilibrium when the buyers have utility $\hat{u}$. Then*

(i) $\hat{b}(v,r)>b(v,r)$ *for* $v>r$.

If (DRH) *holds, and R_B and/or $\widehat{R}_B$ is nonincreasing, then*

(ii) $\hat{b}_1(v,r)>b_1(v,r)$ *for* $v>r$, *and*

(iii) $\hat{b}_2(v,r)<b_2(v,r)$ *for* $v>r$.

Proof. (i) We apply Lemma 1(ii) to $\hat{b}(\cdot,r)-b(\cdot,r)$. We have $\hat{b}(r,r)=b(r,r)$. Suppose $\hat{b}\leq b$ for some $v>r$. Then $\hat{\gamma}(v-\hat{b})\geq\hat{\gamma}(v-b)$, since $\hat{\gamma}$ is increasing on $\mathbb{R}_+$. Since $\widehat{R}_B>R_B$ on $[0,H-L]$, we have $\hat{\gamma}(v-b)>\gamma(v-b)$. Hence, $\hat{\gamma}(v-\hat{b})>\gamma(v-b)$. This and (2.1) yields

$$\hat{b}_1-b_1=\left[\hat{\gamma}(v-\hat{b})-\gamma(v-b)\right]\ell(v)>0.$$

Lemma 1(ii) now implies $\hat{b}>b$ for all $v>r$.

(*ii*) We apply Lemma 1(*i*) to $\hat{b}_1(\cdot, r) - b_1(\cdot, r)$ on intervals of the form $[\xi_k, H]$, where $\xi_k \downarrow r$ as $k \to \infty$. We will show that $\hat{b}_1(\cdot, r) > b_1(\cdot, r)$ on each interval, and hence on $(r, H]$. To obtain ξ_k, let $\{v_k\}$ be a sequence such that $v_k \downarrow r$. Since $\hat{b}(r, r) = b(r, r)$ and $\hat{b}(v_k, r) > b(v_k, r)$, the mean value theorem implies $\xi_k \in (r, v_k)$ exists such that $\hat{b}_1(\xi_k, r) > b_1(\xi_k, r)$. Note that $\xi_k \downarrow r$. Now, suppose $\hat{b}_1(v, r) = b_1(v, r)$ for some $v \geq \xi_k$. Since R_B or $\widehat{R}_B$ is nonincreasing and $\hat{b} > b$ at (v, r), we have

$$\widehat{R}_B(v - \hat{b}) > R_B(v - b). \tag{2.5}$$

Because $\hat{b}_1 = b_1$ at (v, r), from (2.1) we obtain $\hat{\gamma}(v - \hat{b}) = \gamma(v - b)$. Hence, using (2.1) to differentiate $\hat{b}_1$ and b_1 yields

$$\begin{aligned} \hat{b}_{11} - b_{11} &= \left[\ell'\hat{\gamma} + \ell\left(1 + \widehat{R}_B\hat{\gamma}\right)(1 - \hat{b}_1)\right] \\ &\quad - \left[\ell'\gamma + \ell\left(1 + R_B\gamma\right)(1 - b_1)\right] \\ &= \left[\widehat{R}_B(v - \hat{b}) - R_B(v - b)\right]\hat{b}_1(1 - \hat{b}_1) > 0, \end{aligned}$$

where the inequality follows from (2.5), $\hat{b}_1 > 0$, and $\hat{b}_1 < 1$ (by Proposition 2, since we have (DRH) here). Lemma 1(*i*) now implies $\hat{b}_1(\cdot, r) > b_1(\cdot, r)$ on each $(\xi_k, H]$.

(*iii*) We apply Lemma 1(*ii*) to $b_2(\cdot, r) - \hat{b}_2(\cdot, r)$. We have $b_2(r, r) = \hat{b}_2(r, r)$. Suppose $b_2 \leq \hat{b}_2$ for some $v > r$. As (2.1) holds for both b_1 and $\hat{b}_1$, differentiating $b_1 - \hat{b}_1$ with respect to r yields

$$\begin{aligned} b_{12} - \hat{b}_{12} &= -(1 + R_B\gamma)\ell b_2 + (1 + \widehat{R}_B\hat{\gamma})\ell\hat{b}_2 \\ &= -(\ell + R_B b_1)b_2 + (\ell + \widehat{R}_B\hat{b}_1)\hat{b}_2 \\ &= \left(\hat{b}_2 - b_2\right)\ell + \widehat{R}_B\hat{b}_1\hat{b}_2 - R_B b_1 b_2. \end{aligned}$$

Thus, because $\ell > 0$, the hypothesis $b_2 \leq \hat{b}_2$ implies

$$b_{12} - \hat{b}_{12} \geq \left(\widehat{R}_B\hat{b}_1 - R_B b_1\right)\hat{b}_2. \tag{2.6}$$

Since (DRH) holds and $\widehat{R}_B > R_B$, Proposition 3(*ii*) implies $\hat{b}_1 > b_1$. Thus, since $\hat{b}_2 > 0$ by Proposition 1, from (2.6) we obtain $b_{12} - \hat{b}_{12} > 0$. Lemma 1(*ii*) now implies $b_2 > \hat{b}_2$, for $v > r$. ■

2.5 Seller Preferences over Auctions with the Same Reserve Price

Let $V_I(r)$ and $V_{II}(r)$ denote the seller's equilibrium expected utility in the FPA and SPA auctions, respectively, as a function of the reserve price. The revenue equivalence theorem establishes $V_I(r) = V_{II}(r)$ if all participants are risk neutral.

As shown by Maskin and Riley (1984), risk aversion on the part of the seller and/or the buyers causes the seller to prefer the FPA to the SPA if both have the same reserve price.[12] This is due to two effects. The first is a direct "revenue effect": buyer risk aversion causes them to bid more in the FPA. The second is a "risk effect": the high bid in a FPA is a less risky random variable than it is in a SPA, and so preferred by a risk averse seller.

For future reference we record this result as part (i) of the following proposition. Part (ii) records the result that in a FPA, the seller prefers the buyers to be more risk averse, a consequence of the fact that they then bid more.

Proposition 4 (i) *If R_B and/or R_S is positive, then $V_I(r) > V_{II}(r)$ for $r < H$.*

(ii) *If $\hat{u}_B$ satisfies the same assumptions as u_B, with $\widehat{R}_B > R_B$, and $\widehat{V}_I(r)$ is the corresponding FPA equilibrium seller payoff, then $\widehat{V}_I(r) > V_I(r)$ for $r < H$.*

Proof. Part (i) follows from Theorem 5 in Maskin and Riley (1984). To prove (ii), fix $r < H$ and $(v_1, \ldots, v_n) \in [L, H]^n$. Let $v^m = \max_i v_i$. In either case, u_B or $\hat{u}_B$, a sale occurs if and only if $v^m \geq r$. The price is then $b(v^m, r)$ or $\hat{b}(v^m, r)$, since b_1 and $\hat{b}_1$ are positive. By Proposition 3(i), $\hat{b}(v, r) > b(v, r)$ for $v > r$. Thus, for almost all value vectors resulting in

[12] Related results are in Vickrey (1961), Matthews (1980), Riley and Samuelson (1981), Waehrer et al. (1998), and Milgrom (2004, Theorem 4.10).

a sale, the sale price is higher given $\hat{u}_B$ than u_B. Since a sale occurs with positive probability because $r < H$, we have $\widehat{V}_I(r) > V_I(r)$. ■

The seller's preferences over auctions with the same fixed reserve price extend immediately to the setting in which the seller sets reserve prices. For example, if R_B and/or R_S is positive, and r_I (r_{II}) is an optimal reserve price for the seller in the FPA (SPA), then Proposition 4(i) immediately implies $V_I(r_I) > V_{II}(r_{II})$.

2.6 Optimal Reserve Prices

We now derive expressions for $V_i(r)$ and $V_i'(r)$ in order to study the seller's optimal reserve prices. The rules of the auctions and the nature of their equilibria imply

$$V_I(r) = n \int_r^H u_S(b(v,r))G(v)dF(v) + F(r)^n u_S(v_0) \tag{2.7}$$

and

$$\begin{aligned} V_{II}(r) &= nG(r)(1-F(r))u_S(r) \\ &+n \int_r^H u_S(y)(1-F(y))dG(y) + F(r)^n u_S(v_0). \end{aligned} \tag{2.8}$$

Differentiating (2.7) yields

$$\begin{aligned} V_I'(r) &= n \int_r^H u_S'(b(v,r))b_2(v,r)G(v)dF(v) \\ &\quad -nG(r)f(r)\left[u_S(r) - u_S(v_0)\right]. \end{aligned} \tag{2.9}$$

The first term in (2.9) is the seller's marginal benefit from raising the reserve price in the FPA, due to the resulting increase in the bid function on $[r, H]$. The second term is the marginal cost, due to the lost sales at price $b(r,r) = r$ caused by a marginal increase in the reserve price.

Differentiating (2.8) yields

$$\begin{aligned} V'_{II}(r) &= nG(r)\,(1-F(r))\,u'_S(r) \\ &\quad -nG(r)f(r)\,[u_S(r)-u_S(v_0)]\,. \end{aligned} \tag{2.10}$$

Again, the first and second terms are the seller's marginal benefit and marginal cost of raising the reserve price. Comparing (2.9) to (2.10) shows that the marginal cost is the same in the SPA as in the FPA. The marginal benefit of raising the reserve price in the SPA differs, as it is due to the resulting increase in the price received in the event that precisely one bidder has a value greater than r.

The next proposition establishes that in both auctions, optimal reserve prices exist, and they are all strictly between v_0 and H. Furthermore, the optimal reserve price in the SPA is unique and invariant to the number of bidders under the regularity assumption that a bidder's virtual valuation increases in his value (Myerson, 1981).

For $i = I, II$, the proposition refers to the set of reserve prices in $[L, H]$ that maximize V_i, denoted as $\mathcal{R}_i$. It also refers to a function defined by

$$\Phi(r) \equiv \frac{u_S(v_0)-u_S(r)}{u'_S(r)} + \frac{1-F(r)}{f(r)}, \tag{2.11}$$

which is of relevance because (2.10) implies

$$V'_{II}(r) = nG(r)f(r)u'_S(r)\Phi(r). \tag{2.12}$$

Proposition 5 *Both $\mathcal{R}_I$ and $\mathcal{R}_{II}$ are nonempty subsets of (v_0, H). Any $r_{II} \in \mathcal{R}_{II}$ satisfies $\Phi(r_{II}) = 0$, and $\mathcal{R}_{II}$ is a singleton and independent of the number of bidders if $v - \frac{1-F(v)}{f(v)}$ is strictly increasing on (L, H).*

Proof. Because $[L, H]$ is compact and V_i is continuous, $\mathcal{R}_i \neq \emptyset$. For any $r \in (L, v_0)$, the second term in (2.9) is nonpositive, and the first term is positive by Proposition 1. Hence, $V'_I > 0$ on (L, v_0). Expression (2.10) directly implies $V'_{II} > 0$ on (L, v_0). We thus have $\mathcal{R}_i \subseteq [v_0, H]$. From (2.9) and (2.10) we see that

$$V'_i(r) \to nf(H)\,[u_S(H) - u_S(v_0)] < 0$$

as $r \uparrow H$. Hence, $\mathcal{R}_i \subseteq [v_0, H)$.

Assume for now that $\mathcal{R}_{II} \subseteq (v_0, H)$. Then, any $r_{II} \in \mathcal{R}_{II}$ satisfies $V'_{II}(r_{II}) = 0$ and $r_{II} > L$, and hence $\Phi(r_{II}) = 0$. Differentiating (2.11) yields

$$\Phi'(r) = \left(\frac{u_S(v_0) - u_S(r)}{u'_S(r)}\right) R_S(r) - \left(r - \frac{1 - F(r)}{f(r)}\right)'.$$

The first term is nonpositive for $r \geq v_0$. Hence, if $v - \frac{1-F(v)}{f(v)}$ is strictly increasing, then $\Phi' < 0$ on $[v_0, H]$. This interval then contains a unique r_{II} satisfying $\Phi(r_{II}) = 0$, and so $\mathcal{R}_{II} = \{r_{II}\}$. Since Φ does not depend on n, neither does r_{II}.

It remains only to show $v_0 \notin \mathcal{R}_i$, and so $\mathcal{R}_i \subseteq (v_0, H)$. From (2.11) we have $\Phi(v_0) > 0$, since $v_0 \in [L, H)$ and $f(v_0) < \infty$. Hence, (2.12) implies that $V'_{II}(v_0) \geq 0$, and that $\bar{r} > v_0$ exists such that $V'_{II}(r) > 0$ for $r \in (v_0, \bar{r})$. This proves $v_0 \notin \mathcal{R}_{II}$.

To prove $v_0 \notin \mathcal{R}_I$, note first that for $v_0 > L$, we have $V'_I(v_0) > 0$ from (2.9) and Proposition 1, and hence $v_0 \notin \mathcal{R}_I$. So assume $v_0 = L$. Then, since $b_2(\cdot, L) = 0$ on (L, H) (Proposition 1), $V'_I(v_0) = V'_I(L) = 0$. Define

$$m \equiv \exp\left(-\int_L^H b_1(y, L) R_B(y - b(y, L)) dy\right).$$

The function $b_1(\cdot, L)$ is bounded on $[L, H]$, as it is continuous on $(L, H]$ and $b_1(v, L) \to \frac{n-1}{n}$ as $v \downarrow L$ (footnote. 8). The integral in the definition of m is thus finite, and so $m > 0$. Note now that from Proposition 1, for any $v \in (L, H]$ we have

$$\lim_{r \to L} b_2(v, r)\frac{G(v)}{G(r)} = \exp\left(-\int_L^v b_1(y, L) R_B(y - b(y, L)) dy\right) \geq m.$$

Consequently, since $f(L) < \infty$, there exists $\bar{r} \in (L, H)$ such that for $r \in (L, \bar{r})$,

$$\begin{aligned} V_I'(r) = nG(r) &\int_r^H u_S'(b(v,r)) \left[b_2(v,r) \frac{G(v)}{G(r)} \right] dF(v) \\ &- nG(r)f(r) \left[u_S(r) - u_S(L) \right] \\ &\geq nG(r) \left(\int_r^H u_S'(b(v,r)) \left[\tfrac{1}{2} m \right] dF(v) - f(r) \left[u_S(r) - u_S(L) \right] \right) \\ &> 0. \end{aligned}$$

This proves $v_0 \notin \mathcal{R}_I$. ■

We now show that the seller sets a lower reserve price in the FPA than in the SPA if he and/or the bidders are risk averse. The proof is based on the observation that because the seller's marginal cost of raising the reserve price is the same in both auctions, the difference in his incentives is the difference in the marginal benefits: (2.9) and (2.10) yield

$$\begin{aligned} V_I'(r) - V_{II}'(r) &= \underbrace{n \int_r^H u_S'(b(v,r)) b_2(v,r) G(v) dF(v)}_{MB_I} \\ &\quad - \underbrace{nG(r) \left(1 - F(r)\right) u_S'(r)}_{MB_{II}}. \end{aligned} \tag{2.13}$$

It is easy to see that this difference is negative if the bidders and/or the seller is risk averse. By the revenue equivalence theorem, $V_I'(r) = V_{II}'(r)$ if they are all risk neutral, and so then $MB_I = MB_{II}$. As the seller becomes risk averse, the ratio $u_S'(b(v,r))/u_S'(r)$ falls because $b(v,r) > r$, and hence MB_I falls relative to MB_{II}. As the bidders become risk averse, b_2 falls by Proposition 1, which lowers MB_I and leaves MB_{II} unchanged. The proof of our first theorem makes this logic precise.

Theorem 1 *Suppose R_B and/or R_S is positive. Then, for any $r_I \in \mathcal{R}_I$ and $r_{II} \in \mathcal{R}_{II}$, we have $r_I < r_{II}$.*

Proof. Write (2.13) as

$$\begin{aligned} & V_I'(r) - V_{II}'(r) \\ = \; & nG(r)u_S'(r) \int_r^H \left[\left(\frac{u_S'(b(v,r))}{u_S'(r)} \right) \left(\frac{G(v)}{G(r)} b_2(v,r) \right) - 1 \right] dF(v). \end{aligned}$$

Since $r > v_0 \geq L$, this expression is positive if and only if the integral is positive. Fix $v > r$. Since $b(v,r) > r$ and u_S is concave we have $u_S'(b(v,r))/u_S'(r) \leq 1$, and this inequality is strict if $R_S > 0$. From Proposition 1 we have $G(v)b_2(v,r)/G(r) \leq 1$, and this inequality is strict if $R_B > 0$. Hence, as at least one of R_S and R_B is positive, the integrand in the above expression is negative at all $v \in (r, H]$. This proves $V_I' < V_{II}'$ on (L, H). Since $r_{II} \in (v_0, H)$ by Proposition 5, Lemma 2 now implies $r_I < r_{II}$. ■

Our second theorem shows that in either auction, a more risk averse seller sets a lower reserve price. The intuition is that the more risk averse the seller is, the more he wishes to avoid the risk of not selling the object for a profitable price.

Theorem 2 *Let $\hat{u}_S$ satisfy the same assumptions as u_S, with $\widehat{R}_S > R_S$. Let $\widehat{\mathcal{R}}_i$ and $\mathcal{R}_i$ be the sets of optimal reserve prices given $\hat{u}_S$ and u_S, for $i = I, II$. Then, for any $\hat{r}_i \in \widehat{\mathcal{R}}_i$ and $r_i \in \mathcal{R}_i$, we have $\hat{r}_I < r_I$ and $\hat{r}_{II} < r_{II}$.*

Proof. We first prove $\hat{r}_{II} < r_{II}$. Let $\hat{\Phi}(r)$ be defined by replacing u_S by $\hat{u}_S$ in (2.11). By Pratt (1964, Theorem 1), $\hat{R}_S > R_S$ implies that for $r > v_0$,

$$\frac{\hat{u}_S(v_0) - \hat{u}_S(r)}{\hat{u}_S'(r)} < \frac{u_S(v_0) - u_S(r)}{u_S'(r)}, \tag{2.14}$$

and hence $\hat{\Phi}(r) < \Phi(r)$. Since $\Phi(r)$ is invariant to any linear transformation of u_S, and since $V_{II}'(r) = nG(r)f(r)u_S'(r)\Phi(r)$, a linear transformation of u_S will only lead to a positive ratio transformation of $V_{II}'(r)$ and hence will not affect the optimal reserve prices. Now let $\hat{r}_{II} = \max \widehat{\mathcal{R}}_{II}$. By

Proposition 5, $\widehat{\mathcal{R}}_{II}$ is a nonempty subset of (v_0, H) and therefore $\hat{\Phi}(\hat{r}_{II}) = 0 < \Phi(\hat{r}_{II})$. W.l.o.g., we now normalize u_S such that $u_S(v_0) = \hat{u}_S(v_0)$ and $u_S(\hat{r}_{II}) = \hat{u}_S(\hat{r}_{II})$. Since $\hat{R}_S > R_S$, by Pratt (1964, Theorem 1) $u_S(r) < \hat{u}_S(r)$ for all $r \in (v_0, \hat{r}_{II})$. Consequently, for all $r \in (v_0, \hat{r}_{II})$,

$$\begin{aligned} & V_{II}(r) - \hat{V}_{II}(r) \\ = \; & n \int_r^H \left[(u_S(r) - \hat{u}_S(r))\, G(r) + \int_r^v (u_S(y) - \hat{u}_S(y))\, dG(y) \right] dF(v) \\ < \; & n \int_{\hat{r}_{II}}^H \int_{\hat{r}_{II}}^v (u_S(y) - \hat{u}_S(y))\, dG(y) dF(v) \\ = \; & V_{II}(\hat{r}_{II}) - \hat{V}_{II}(\hat{r}_{II}), \end{aligned}$$

which implies $V_{II}(\hat{r}_{II}) - V_{II}(r) > \hat{V}_{II}(\hat{r}_{II}) - \hat{V}_{II}(r) \geq 0$ since $\hat{r}_{II}$ maximizes $\hat{V}_{II}$. This establishes $\hat{r}_{II} \leq r_{II}$. The strict inequality now follows from $V'_{II}(\hat{r}_{II}) > 0$ since $\Phi(\hat{r}_{II}) > 0$.

We now use a similar approach to prove $\hat{r}_I < r_I$. W.l.o.g., we may assume $\hat{r}_I = \max \widehat{\mathcal{R}}_I$. Define function Ψ by

$$\Psi(r) \equiv \frac{u_S(v_0) - u_S(r)}{u'_S(r)} + \int_r^H \frac{u'_S(b(v,r))}{u'_S(r)} \frac{b_2(v,r)G(v)}{G(r)f(r)} dF(v), \qquad (2.15)$$

which allows (2.9) to be written as $V'_I(r) = nG(r)f(r)u'_S(r)\Psi(r)$. The role of Ψ is analogous to that of Φ in that $\Psi(r) \geq 0$ iff $V'_I(r) \geq 0$, and that a linear transformation of u_S does not affect Ψ and only leads to a positive ratio transformation of $V'_I(r)$. Consequently, without affecting the optimal reserve prices we can now normalize u_S such that $u_S(v_0) = \hat{u}_S(v_0)$ and $u_S(\hat{r}_I) = \hat{u}_S(\hat{r}_I)$. It follows from $\hat{R}_S > R_S$ that $u_S(r) < \hat{u}_S(r)$ for all $r \in (v_0, \hat{r}_I)$, and that

$$\frac{u'_S(r)}{\hat{u}'_S(r)} > \frac{u_S(r) - u_S(v_0)}{\hat{u}_S(r) - u_S(v_0)} > 0 \quad \forall r \in (\hat{r}_I, H).$$

For all r and v such that $r \leq \hat{r}_I < v$, since $b_2 \geq 0$ and $b(v, \hat{r}_I) > \hat{r}_I$, we then have

$$\max\{u_S(b(v,r)) - \hat{u}_S(b(v,r)), 0\} \leq u_S(b(v,\hat{r}_I)) - \hat{u}_S(b(v,\hat{r}_I))$$

The above conditions imply that for $r \in (v_0, \hat{r}_I)$,

$$\begin{aligned}
V_I(r) - \hat{V}_I(r) &= n \int_r^H [u_S(b(v,r)) - \hat{u}_S(b(v,r))]\, G(v) dF(v) \\
&= n \int_r^{\hat{r}_I} [u_S(b(v,r)) - \hat{u}_S(b(v,r))]\, G(v) dF(v) \\
&\quad + n \int_{\hat{r}_I}^H [u_S(b(v,r)) - \hat{u}_S(b(v,r))]\, G(v) dF(v) \\
&< n \int_{\hat{r}_I}^H [u_S(b(v,r)) - \hat{u}_S(b(v,r))]\, G(v) dF(v) \\
&\leq n \int_{\hat{r}_I}^H \max\{u_S(b(v,r)) - \hat{u}_S(b(v,r)), 0\} G(v) dF(v) \\
&\leq n \int_{\hat{r}_I}^H [u_S(b(v,\hat{r}_I)) - \hat{u}_S(b(v,\hat{r}_I))]\, G(v) dF(v) \\
&= V_I(\hat{r}_I) - \hat{V}_I(\hat{r}_I),
\end{aligned}$$

or that $V_I(\hat{r}_I) - V_I(r) > \hat{V}_I(\hat{r}_I) - \hat{V}_I(r) \geq 0$ since $\hat{r}_I$ maximizes $\hat{V}_I$. This implies $\hat{r}_I \leq r_I$. Now define $\widehat{\Psi}(r)$ by replacing u_S by $\hat{u}_S$ in (2.15). Since $b(v,r) > r$ and $b_2 \geq 0$, the second term in (2.15) decreases in the seller's risk aversion. Thus, using (2.14) we obtain $\Psi(r) > \widehat{\Psi}(r)$ for all $r \in (v_0, H]$ and, in particular, $\Psi(\hat{r}_I) > \widehat{\Psi}(\hat{r}_I) = 0$. Therefore we must have $\hat{r}_I < r_I$. ■

Our third and final theorem establishes that under two fairly general conditions, the seller sets a lower reserve price in the FPA if the bidders are more risk averse. The logic of the result is twofold. First, under the assumed conditions the FPA bid function increases in the reserve price at a slower rate if the bidders are more risk averse. This lowers the incentive of the seller to raise the reserve price. Second, because more risk averse bidders bid more, the increase in their bids in response to an increase in the reserve price generates a lower marginal utility increase for the (weakly) risk averse seller. The proof reflects these two forces.

Theorem 3 *Let $\hat{u}_B$ satisfy the same assumptions as u_B, with $\widehat{R}_B > R_B$. Let $\mathcal{R}_I$ ($\widehat{\mathcal{R}}_I$) be the set of optimal reserve prices for the seller given u_B*

$(\hat{u}_B)$. *Then, for any* $r_I \in \mathcal{R}_I$ *and* $\hat{r}_I \in \widehat{\mathcal{R}}_I$, *we have* $\hat{r}_I < r_I$ *if either*

(a) (DRH) *holds and* R_B *and/or* $\widehat{R}_B$ *is nonincreasing, or*

(b) $\min_{t\in D} \widehat{R}_B(t) > \max_{t\in D} R_B(t)$, *where* $D = [0, H - L]$.

Proof. Let $\widehat{V}(r)$ be the seller's payoff given $\hat{u}_B$ and reserve r. We show that (a) and (b) each imply $\widehat{V}_I'(r) < V_I'(r)$ for $r > L$. This and Lemma 2 then yield the result, $\hat{r}_I < r_I$, since these reserve prices are in the interval (v_0, H).

From (2.9) we obtain

$$\begin{aligned}&\widehat{V}_I'(r) - V_I'(r)\\ = \; & n\int_r^H \Big[u_S'(\hat{b}(v,r))\hat{b}_2(v,r) - u_S'(b(v,r))b_2(v,r)\Big]\, G(v)dF(v).\end{aligned}$$

The concavity of u_S, together with $\hat{b}(v,r) > b(v,r)$ (Proposition 3(i)), yields $u_S'(\hat{b}) \leq u_S'(b)$ for $v > r$. Hence, to show that (a) and (b) each imply $\widehat{V}_I'(r) < V_I'(r)$ for $r > L$, it suffices to show that they each imply

$$\hat{b}_2(v,r) < b_2(v,r) \text{ for } L < r < v. \tag{2.16}$$

By Proposition 3(iii), (a) implies (2.16). Now assume (b) holds. Then a constant a exists such that $\widehat{R}_B > a > R_B$ on $[0, H - L]$. Fix $L < r < v$. Letting b^a be the CARA equilibrium given by (2.2), by Proposition 3(i) we have $\hat{b} > b^a > b$. Hence, by Proposition 1,

$$\begin{aligned} G(v)b_2(v,r) &= G(r)\exp\left(-\int_r^v b_1(y,r)R_B(y - b(y,r))dy\right)\\ &> G(r)\exp\left(-a\int_r^v b_1(y,r)dy\right) \quad (\text{since } R_B < a)\\ &= G(r)\exp\left(-a(b(v,r) - r)\right)\\ &> G(r)\exp\left(-a(b^a(v,r) - r)\right) \quad (\text{since } b < b^a)\\ &= G(v)b_2^a(v,r).\end{aligned}$$

Thus, $b_2^a < b_2$. Similarly, $\widehat{R}_B > a$ yields $\hat{b}_2 < b_2^a$. So (b) indeed implies (2.16). ∎

2.7 Concluding Discussion

We have shown that when the seller sets the reserve price, he sets it lower the more risk averse he is and, in a first-price auction, the more risk averse the buyers are. The seller's optimal reserve price is lower in the first-price auction than in the second-price auction, unless all parties are risk neutral. Risk aversion thus reduces the probability of not selling the object when a buyer's value for it exceeds that of the seller, especially in the first-price auction.

The buyers may agree, ex ante, with the seller's preference for the first-price auction. Indeed, if they exhibit constant (or increasing) absolute risk aversion, every type of buyer weakly (strictly) prefers at the interim stage the first-price to a second-price auction that has the same reserve price (Matthews, 1987). *Ipso facto,* in these cases the buyers prefer the first-price auction if it has the lower reserve price, as it does when the seller sets the reserve price and he or the buyers are risk averse. By continuity, the buyers must also prefer the first-price auction if their absolute risk aversion measure is approximately constant, so long as they and/or the seller are risk averse.[13] More generally, buyers with values in the interval $(r_I, r_{II}]$ strictly prefer the FPA, and hence so must the buyers with values in some interval $(r_I, \hat{v})$, where $\hat{v} > r_{II}$.

We have focused tightly in this chapter on the effects of risk aversion on optimal reserve prices in two standard auctions, holding fixed their other features. Endogenizing these other features and determining the effects of risk aversion on their levels is a topic for future research. For example, if the seller is able to charge bidders an entry fee, he may wish to do so if the bidders are risk averse (Maskin and Riley, 1983), but not if he is risk averse and can also set the reserve price (Waehrer et al., 1998). The nature of optimal combinations of entry fees and reserves when the seller

[13] Formally, if $|R_B(y) - a| < \varepsilon$ for all y, and if $a > 0$ and/or $R_S > 0$, then $\bar{\varepsilon} > 0$ exists such that when $\varepsilon < \bar{\varepsilon}$, every type of buyer interim prefers the FPA to the SPA when the seller sets the reserve prices.

or buyers are risk averse is unknown. Another example is entry: if each of a large number of potential bidders must pay a cost to learn his value, the number of bidders becomes endogenous. In this case the seller may want to lower the reserve price in order to increase entry.[14] Our results suggest that risk aversion on the part of the seller or buyers should strengthen this effect, especially in the first-price auction.[15].

Future work may also generalize the setting of our results. It may be fruitful, for example, to consider asymmetric bidders with different value distributions, which give rise to a different ex post inefficiency (sale to the wrong bidder) than the one (no sale) that we have considered. Settings with ex post risk or interdependent values are naturally of interest as well.

2.8 Appendix

Proof of Lemma 1. (i) Assume $h(t) \leq 0$ for some $t \in (c, d]$. The hypothesis and the continuity of h imply the existence of $\hat{t} \in [c, t)$ such that $h(\hat{t}) < 0$. Let $\bar{s} = \sup\{s \in [c, \hat{t}) : h(s) \geq 0\}$. As h is continuous, $\bar{s} < \hat{t}$ and $h(\bar{s}) = 0$. The hypothesis now implies the existence of $s \in (\bar{s}, \hat{t})$ such that $h(s) > 0$. This contradicts the definition of $\bar{s}$.

(ii) Assume $h(t) \leq 0$ for some $t \in (c, d]$. Let m be the largest minimizer of h on $[c, t]$. Since $h(c) \geq h(t)$, $m > c$. We thus have $h'(m) \leq 0$, as well as $h(m) \leq h(t) \leq 0$. This contradicts the hypothesis. ■

Proof of Lemma 2. Let $i \in \{1, 2\}$ be such that $t_i \in (c, d)$, and let $j \neq i$ be the other index. Then $h'_j(t_i) \neq h'_i(t_i) = 0$. This proves $h_j(t_j) > h_j(t_i)$, and hence $t_1 \neq t_2$. Defining $h = h_2 - h_1$, we now have $h(t_2) > h(t_1)$. By

[14] The effects of endogenous entry on optimal reserve prices are studied, in risk neutral settings, by McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), and Levin and Smith (1994).

[15] Endogenous entry can reverse the seller's preference for the FPA, since the SPA may induce more entry if the buyers have DARA risk preferences, as is shown in Smith and Levin (1996). This reversal should occur less often, however, when the seller sets the reserve price, since he sets it lower in the FPA.

the mean value theorem, there exists t strictly between t_1 and t_2 such that

$$(t_2 - t_1)h'(t) = h(t_2) - h(t_1) > 0.$$

This proves $t_1 < t_2$, since $h'(t) > 0$. ■