Essays on auctions

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Non-Quasilinear Preferences in Premium Auctions

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4.1 Introduction

The preceding chapter has followed the existing literature on premium auctions and focused on situations where risk neutral bidders exhibit strong asymmetries prior to the auction. In particular, the attention was drawn to a situation where one strong bidder (or cartel) competed with several weak bidders. This literature remains inconclusive as to how premium auctions perform in circumstances beyond these special cases. The objective of this chapter is to provide a theory of premium auctions that broadens the scope of the existing studies. We study a class of English premium auctions (henceforth, EPA), in which risk averse or risk loving bidders compete in a standard symmetric private values setting. By comparing this type of auctions with the standard English auction (henceforth, EA), we develop new insights into the circumstances where it does, or does not, make sense for the seller to employ a premium auction.

The EPA proceeds in two stages. In the first stage, the seller raises the price until all but two bidders (finalists) have withdrawn. In the second stage, the price is raised further and stops as soon as one finalist withdraws. The remaining finalist wins the object and pays the price at which the auction stops. In addition, both finalists receive a premium determined by a pre-specified function of the difference between the ending prices of the two stages. Our model is a generalization of the "Amsterdam Second-Price Auction" model studied in Goeree and Offerman (2004). Instead of limiting to the linear premium rules and uniform value distributions, we allow for a general class of premium functions and value distribution functions, as well as for bidders to have a general utility function.

The few existing theoretical investigations of such premium tactics were mainly guided by the researchers' intuition about the circumstances where premium auctions would perform well. Goeree and Offerman (2004) consider a setting in which there is a strong bidder and a population of weak bidders. In their model, it is common knowledge that the strong bidder will have the highest value for the object for sale. Hence, the weak bidders have little incentive to participate in a standard first or second-price auction. The lack of competition could then allow the strong bidder to win the object at a very low price. In a premium auction, the same weak bidders can be attracted to the auction and bid competitively for the premium, thereby enhancing the seller's expected revenue. Milgrom (2004, p. 239-241) analyzes an example with endogenous entry that is similar in spirit. In his model, there is a small positive entry cost. In equilibrium, the weak bidders enter the premium auction with positive probability even though they know that a strong bidder will have the largest value for the good for sale. Other tactics of a similar sort, such as using bidding credits or set-asides, have also been shown to enhance competition when bidders are asymmetric (e.g., Ayres and Cramton, 1996).

Another strand of literature studies how risk aversion affects the expected utilities of the bidders, apart from that of the seller, under different auction policies. For instance, Matthews (1987) shows that in symmetric independent private values settings, the buyers who exhibit constant Arrow-Pratt absolute risk aversion has the same expected utilities in any of the standard auctions: English, Dutch, first-price, or second-price. If the buyers exhibit decreasing (increasing) risk aversion, however, then they have strictly higher (lower) expected utilities in the English or second-price auctions. Understanding the buyers' preferences over auction formats can be important because even though an individual bidder may not have much power to influence the auction design, he has nevertheless the choice to "vote with his feet." This possibility is relevant to the seller especially where the potential bidders are few and they face certain costs of participating in the auction (e.g., Smith and Levin, 1996).

In this chapter, we first characterize an EPA symmetric equilibrium and derive the basic properties of the bid function in such an equilibrium (Theorem 4). We then show that an EPA symmetric equilibrium exists and, moreover, has to be unique for a certain class of premium and utility functions (Theorem 5). In Theorem 6, we derive a "net-premium effect" of EPA that is key to the welfare conclusions of this paper. This effect predicts that a finalist's conditional expected utility for the premium from the ongoing auction, when calculated in isolation of other random payoffs, is always the same as his utility for the premium if he drops out at the current price level. An important implication of this effect is that at the start of the second stage of EPA, both finalists would have a conditional expected utility for the premium that is equal to zero. This finding is interesting on its own, as it adds a new insight into the types of auctions à la Vickrey (1961)—for the more general utility functions of the bidders. In particular, the net-premium effect implies revenue equivalence when the bidders are risk neutral.

In Theorem 7, we show that for any arbitrary premium function, the expected revenue decreases as the bidders become more risk averse. Since the expected revenue will be invariant with the premium when the bidders are risk neutral (Myerson, 1981), Theorem 7 implies that a risk neutral seller is better (worse) off to offer a premium only when the bidders are risk loving (averse). At first sight, this result may seem to be counter intuitive, especially given the result of Lemma 5 that the premium has the effect of reducing the riskiness of the payment in an English auction. But then what causes risk lovers to bid more aggressively than risk averters in an EPA? The answer is given by the net-premium effect. Risk lovers stay longer in an EPA because they derive a higher expected utility from the uncertain premium.

The result that risk seekers, or speculators, bid aggressively in a premium auction might suggest that they "love" the premiums and will therefore be more willing to participate in an EPA rather than an EA. This intuition turns out to be incorrect. From the bidders' perspective, we show in Theorem 8 that under certain conditions the bidders prefer an EPA to an EA if and only if they are risk averse. Therefore, the conventional wisdom that premium auctions tend to attract risk-seeking speculators does not apply in our symmetric auction environment. Indeed, our results suggest a conflict of interests between the revenue-maximizing seller and the bidders over the choice between EA and EPA. This conflict of interests continues to hold when the seller is risk averse but the bidders are risk loving, or vice versa (Theorem 9). However, the seller and the bidders may simultaneously prefer the EPA to the EA when the bidders are risk neutral or marginally risk averse, and the seller is sufficiently risk averse.

In general, our results suggest that there are circumstances under which a premium auction performs better than a standard English auction, and circumstances under which it performs worse. As Klemperer (2002) already emphasized, auction design is not a matter of one size fits all. Instead, it calls for "different horses for different courses." The seller should judge the field of bidders and choose the auction format accordingly. In this respect, it is interesting to note that some auction houses do switch repeatedly between using the premium auctions and the standard Ebay-like auction procedures.[1]

The rest of the chapter is organized as follows. Section 4.2 presents the EPA model and the basic assumptions. Section 4.3 characterizes, and establishes the existence and uniqueness, of an EPA equilibrium. A closed-form equilibrium solution is derived in this section for the case where the bidders exhibit constant absolute risk aversion (CARA). Section 4.4 derives the net-premium effect and analyzes its consequences for the expected revenue and the players' expected utilities. Section 4.5 concludes.

[1] For instance, Troostwijk adopted a premium auction for its recent sale of a Boeing 737-400 in November 2009, although the auction house stays with the standard procedures more often. See http://www.troostwijkauctions.com/nl/ for more examples.

4.2 The Model

A single object is to be sold to one of n (> 2) bidders via an English premium auction (EPA).[2] Each bidder has a private value (v) for the object that is independently distributed ex ante according to cumulative distribution F, which has a continuously differentiable density function $f = F'$ that is strictly positive on its support $[L, H]$.

Although the auction can be conducted incessantly until the object is sold, it is equivalent and analytically convenient to perceive it, as we do, as a two-stage auction. In the first stage, a price for the object rises continuously from a sufficiently low level and each bidder stays in the auction until he chooses to quit (e.g., by pressing an electronic button). This stage rounds up as soon as only two bidders, called *finalists*, remain and the price level (X) at which the last bidder quits, called the *bottom price*, will serve as a reserve price onwards. In the second stage, the price level rises from the bottom price X until one of the finalists quits. The last one who stays wins the object and pays the price (b) at which the other finalist quits. In addition, both finalists receive a cash premium from the seller that is equal to $\varphi(b - X)$,[3] where $\varphi : [0, H - L] \to \mathbb{R}_+$ is a twice continuously differentiable function such that $\varphi(0) = 0$, and $0 < \varphi' \leq 1/2$. We call such φ a *premium function*. As usual, ties are assumed to be resolved randomly in both stages. If two or more bidders simultaneously withdraw at price X in the first stage, with only one bidder left, then a random device will choose one of these bidders to be a finalist. If both finalists withdraw at the same price b, then both will receive a premium equal to $\varphi(b - X)$, and one of them will be randomly chosen to receive the object and pays the price b. Clearly, if $\varphi \equiv 0$ then the model reduces

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[2] We focus on the English-type of premium auctions in this study mainly because of their popularity in practice.

[3] A virtually equivalent model is to award the premium only to the highest losing bidder. This will lead to a different equilibrium bid function as derived in Theorem 4, but the qualitative conclusions of Theorems 5-9 will remain the same. See footnote 8.

to an English auction (EA)—with a possible (coffee) break when only two bidders remain.

The bidders have the same utility function, $u : \mathbb{R} \to \mathbb{R}$, that is strictly increasing and twice continuously differentiable. If a bidder quits before entering the second stage, his utility is normalized to be $u(0) = 0$. We also normalize the seller's reservation value for the object to be zero, and assume that $L \geq 0$ so that the sale will take place with certainty.

Some of our results are obtained under the following assumptions:

(A1) $\varphi'' \leq 0$.

(A2) $\forall z \in \mathbb{R}$, $\ln[u(w) - u(z)]$ is weakly concave in w on (z, ∞).

(A3) $\left(\dfrac{f(v)}{1 - F(v)}\right)' > 0$ on (L, H).

All these assumptions are quite standard. (A1) requires that the premium function be weakly concave, which includes the linear premium rule that is frequently observed in practice. (A2) assumes that the utility surplus $u(w) - u(z)$ is (weakly) log-concave in w for all $w > z$. This condition is commonly invoked for the existence of equilibria in the first-price sealed-bid auctions (e.g., Athey (2001)), which allows the bidders to be risk neutral, risk averse, and to some extent risk loving.[4] Clearly, (A2) is satisfied automatically for any degree of convexity of u that is exponential, a special case that will be studied in the Example in Section 4.3. The assumption (A3) says that the hazard rate of F is increasing, which is a well-known property that many common distributions possess.

4.3 The EPA Symmetric Equilibrium

We say that $b : [L, H]^2 \to [L, H]$ represents a bid function if $b(v, p)$ strictly increases in v, and $b(v, p) - p$ strictly decreases in p. The bid function b is an *EPA symmetric equilibrium (EPA-SE)* if the following strategy maximizes

[4] We use the terms risk loving, risk preferring, or risk seeking interchangeably—all refer to the case where the utility function is (weakly) convex.

each bidder's expected utility in each stage of the auction, conditional on updated information and the common belief that the strategy will be adopted by everyone.[5] In the first stage, as price p increases continuously, each bidder with value v remains in the auction as long as $b(v,p) > p$, and the bidder quits the auction as soon as $b(v,p) = p$. Given the bottom price X where the first stage ends, each of the two finalists adopts the bid function $b(\cdot, X)$ in the second stage such that with value v, the bidder remains in the auction until he wins the object or quits when the price level reaches $p = b(v, X)$.

Clearly, an EPA-SE implies that the low value bidders will drop out first. It can also be shown (see the proof of Theorem 4) that if $b(v,p)$ is an EPA-SE, then it is differentiable in v so that $b_1(v,p) > 0$.[6] This implies that at any price $p \geq X$ in the second stage, there is an r solving $b(r, X) = p$ such that if both finalists remain in the auction they must have values higher than r. For $v \geq r$, $[F(v) - F(r)] / [1 - F(r)]$ is thus each finalist's updated probability that the other finalist has a lower type than v. We call such r the *current screening level* (or screening level for short), which is implicitly defined through its one-to-one relation with the ongoing price p $(\geq X)$ in the second stage of EPA, given any bottom price X.

Now fix a bottom price $X \in [L, H)$ and consider the second-stage EPA that is going on at price $p \geq X$. For ease of notation, w.l.o.g. we may directly refer to the screening level r $(b(r, X) = p)$ rather than the price level p in describing an ongoing second-stage EPA. If both finalists adopt the bid function $b(\cdot, X)$, then each with value $v \geq r$ will have a conditional

[5] It is known that even in ex ante symmetric settings there may exist multiple asymmetric equilibria (e.g., Maskin and Riley, 2003). We focus on symmetric equilibria in this study.

[6] For functions with two variables, we use subscripts to denote their partial derivatives with respect to the corresponding variable.

expected utility equal to

$$\begin{aligned}
& U(v|r, b(\cdot, X)) \\
\equiv \quad & \frac{1}{1-F(r)} \int_r^v u\left(v - b(y, X) + \varphi(b(y, X) - X)\right) dF(y) \\
& + \frac{1-F(v)}{1-F(r)} u\left(\varphi(b(v, X) - X)\right), \qquad (4.1)
\end{aligned}$$

where the first term comes from the event that the bidder wins, and the last term from the event that the bidder loses.

An interesting aspect of the premium auction is that even though the bidders have private values, the premium in effect introduces a strong "affiliation" of the values for the two finalists. A higher value of the opponent can now be "good news" as it increases the expected premium for each bidder. However, there are some important (strategic) differences between EPA and the standard English auction (EA) with affiliated signals and interdependent values (e.g., Milgrom and Weber, 1982; Eso and White, 2004). For instance, when a bidder decides to drop out at a price in an EA "he would be just indifferent between winning and losing at that price." (Milgrom and Weber, 1982, p.1105). This is true for the first-stage EPA but is not true for the second-stage EPA. It will be shown that the equilibrium bid in the EPA is strictly higher than a bidder's true value. Consequently, should both finalists simultaneously drop out, they would both prefer losing rather than winning. Moreover, the EA equilibrium strategy calls for each bidder to stay in the auction until his expected utility conditional on winning in a tie is zero (or equal to the status-quo utility level that is commonly known beforehand). This property implies a straightforward solution for the symmetric bid function of the EA. In the second-stage EPA, the bidder will drop out at a price level at which his expected utility is positive due to the premium collected, but this expected utility level is private information and cannot be determined unless the bid function is already given.

Nevertheless, the difficulty of applying the standard EA analysis to the EPA can be circumvented by appealing to an argument of Milgrom and

Weber (1982, p. 1105), which implies that despite the interdependency of the utility payoffs, the second-stage EPA can also be seen as a strategically equivalent second-price sealed bid (or Vickrey) premium auction. This result derives from the fact that when only two bidders remain, the bid function cannot be made dependent on the revealed value of the opponent because the auction ends as soon as the opponent drops out—it will then be too late for the winner to adjust his bid. By the same logic, this argument can be applied to any price level p of the second-stage EPA as long as the auction continues. In other words, fix any $p \geq X$, the remaining EPA can be analyzed equivalently as though it is a Vickrey premium auction with reserve price p, in which the highest bidder wins the object and pays the second highest price b, while both finalists receive a premium $\varphi(b - X)$. We shall follow this approach from now on.

The limiting case with $X = H$ is trivial, for then both finalists would have $v = H$ and both would bid $b(H, H) = H$. Now fix $X \in [L, H)$. As long as the auction continues, given the updated screening level $r \in [r(X), H)$, where $b(r(X), X) = X$, the expected utility of a finalist who has value $v \in [r, H]$ and who bids as though his value is $z \in [r, H]$ equals

$$\begin{aligned} & \overline{U}(v, z|r, b(\cdot, X)) \\ \equiv \; & \frac{1}{1 - F(r)} \int_r^z u\left(v - b(y, X) + \varphi(b(y, X) - X)\right) dF(y) \\ & + \frac{1 - F(z)}{1 - F(r)} u\left(\varphi(b(z, X) - X)\right). \end{aligned} \tag{4.2}$$

Hence, given that each finalist adopts the bid function b, equilibrium (or incentive compatibility) holds in the second stage if and only if for all $X \in [L, H)$, $r \in [r(X), H)$, and $v, z \in [r, H]$,

$$U(v|r, b(\cdot, X)) \geq \overline{U}(v, z|r, b(\cdot, X)). \tag{4.3}$$

For some of the results we will use the following lemma. It is a variation of the "Ranking Lemma" of Milgrom (2004; p.124). See also Lemma 1 in Chapter 2, and Milgrom and Weber (1982, Lemma 2).

Lemma 3 *For $-\infty < c < d < \infty$ and $h : [c,d] \to \mathbb{R}$ continuous with $h(d) \geq 0$,*

$$(i)\ \ if \left\{ \begin{array}{c} h \text{ is differentiable on } [c,d] \text{ and} \\ [\forall t \in [c,d], h(t) = 0 \Rightarrow h'(t) < 0] \end{array} \right., \text{ then } h > 0 \text{ on } [c,d);$$

$$(ii)\ \ if \left\{ \begin{array}{c} h \text{ is differentiable on } [c,d) \text{ and} \\ [\forall t \in [c,d),\ h(t) \leq 0 \Rightarrow h'(t) \leq 0] \end{array} \right., \text{ then } h \geq 0 \text{ on } [c,d].$$

Proof. The proof of part (i) is analogous to Lemma 1(i), hence is omitted. To show part (ii), assume that h is differentiable on $[c,d)$. Suppose $h(t) < 0$ for some $t \in [c,d)$. Then the continuity of h and the assumption $h(d) \geq 0$ imply the existence of $\hat{t} \in (t,d]$ such that $h(\hat{t}) = 0$ and $h(s) < 0$ for all $s \in [t,\hat{t})$. By the mean value theorem, this implies that there exists $s \in (t,\hat{t})$ such that $h'(s) = \left[h(\hat{t}) - h(t)\right]/(\hat{t} - t) > 0$. Therefore, the hypothesis in the square brackets of part (ii) does not hold. ■

The next theorem provides the characterization of an EPA symmetric equilibrium.

Theorem 4 (Necessary and sufficient condition for EPA-SE) *For any utility function u and premium function φ, the function $b : [L,H]^2 \to [L,H]$ is an EPA-SE if and only if for all $X \in [L,H)$, $b(\cdot,X)$ is the solution of the following differential equation and boundary condition: for $v \in [L,H)$,*

$$\begin{aligned} b_1(v,X) &= \frac{u\left(\varphi(b-X)\right) - u\left(v - b + \varphi(b-X)\right)}{u'\left(\varphi(b-X)\right)\varphi'(b-X)} \frac{f(v)}{1-F(v)} && (4.4) \\ b(H,X) &= \lim_{v \uparrow H} b(v,X) = H && (4.5) \end{aligned}$$

on the domain

$$\mathcal{D}(X) = \{(v,b) \in [L,H)^2 : b(v,X) \in [X,H)\}.$$

Before proving this theorem, we first present a lemma and show that (4.4)-(4.5) imply that any solution b necessarily satisfies the required properties of an EPA-SE, i.e., $b(v,p)$ strictly increases in v and $b(v,p)-p$ strictly decreases in p. These two properties imply that as $p \uparrow H$, only the bidder(s) with $v = H$ will remain in the auction; and hence $b(H,H) = H$.

Lemma 4 *If $b : [L,H]^2 \to [L,H]$ is a solution of (4.4)-(4.5), then (i) $b_1(v,p) > 0$ on $\mathcal{D}(p)$ for all $p \in [L,H)$; and (ii) $b(v,p) - p > b(v,\hat{p}) - \hat{p}$ whenever $\hat{p} > p$, for all $v \in [L,H)$ such that $b(v,\hat{p}) \geq \hat{p}$.*

Proof. To show property (i), fix an arbitrary $p \in [L,H)$. Because the right-hand side of (4.4) is continuously differentiable in v and b (except at $v = H$), by the mean value theorem we can write, for $v < H$,

$$b_1(v,p) = \frac{u'(\xi)\,(b(v,p)-v)}{u'(\varphi(b(v,p)-p))\,\varphi'(b(v,p)-p)} \frac{f(v)}{1-F(v)},$$

where $\xi \to \varphi(H-p)$ as $v \uparrow H$ (hence $b(v,p) \to H$). Since the above expression has a 0/0 form at $v = H$, by L'Hospital's rule and taking limit as $v \uparrow H$ yields

$$\lim_{v\uparrow H} b_1(v,p) = \frac{f(H)}{\varphi'(H-p)} \lim_{v\uparrow H} \frac{b(v,p)-v}{1-F(v)} = \frac{1-\lim_{v\uparrow H} b_1(v,p)}{\varphi'(H-p)}.$$

This allows us to denote by $b_1(H,p)$ the value of the continuous extension of $b_1(v,p)$ at $v = H$:

$$b_1(H,p) = \lim_{v\uparrow H} b_1(v,p) = \frac{1}{1+\varphi'(H-p)}. \tag{4.6}$$

From (4.4) it can be seen that for all $v \in [L,H)$, $b_1(v,p) > 0$ is equivalent to $b(v,p) > v$. We now apply Lemma 3(i) to $h(v) \equiv b(v,p) - v$. Since $h(H) = 0$ and $h'(H) = b_1(H,p) - 1 < 0$ (see (4.6)), we have $h(v) > 0$ on $(\bar{v}, H)$ for some $\bar{v} < H$. If $h(v) = 0$ for some $v \leq \bar{v}$, then (4.4) implies $b_1(v,p) = 0 < 1$ so that $h'(v) < 0$. Lemma 3(i) now implies $h(v) > 0$ and therefore $b_1(v,p) > 0$ on $\mathcal{D}(p)$. Clearly, by the arbitrariness of p this property holds for all $p \in [L,H)$.

To show property (ii), fix any $\hat{p} > p$. We apply Lemma 3(i) to $h(v) \equiv b(v,p) - p - (b(v,\hat{p}) - \hat{p})$. Because $b(H,p) = b(H,\hat{p})$, by continuity $b(v,p) - b(v,\hat{p})$ is arbitrarily close to zero for v sufficiently close to H. Thus, there exists $\bar{v} < H$ such that $h > 0$ on $(\bar{v}, H]$. Now suppose $h(v) = 0$ for some $v \le \bar{v}$. Then $\varphi(b(v,p) - p) = \varphi(b(v,\hat{p}) - \hat{p})$ and $b(v,\hat{p}) > b(v,p)$. Hence,

$$\begin{aligned}
& b_1(v,\hat{p}) \\
= \; & \frac{u\left(\varphi(b(v,p)-p)\right) - u\left(v - b(v,\hat{p}) + \varphi(b(v,p)-p)\right)}{u'\left(\varphi(b(v,p)-p)\right)\varphi'(b(v,p)-p)} \frac{f(v)}{1-F(v)} \\
> \; & \frac{u\left(\varphi(b(v,p)-p)\right) - u\left(v - b(v,p) + \varphi(b(v,p)-p)\right)}{u'\left(\varphi(b(v,p)-p)\right)\varphi'(b(v,p)-p)} \frac{f(v)}{1-F(v)} \\
= \; & b_1(v,p)
\end{aligned}$$

or $h'(v) < 0$. Since $h(\bar{v}) \ge 0$, Lemma 3(i) now implies $h(v) > 0$ or $b(v,\hat{p}) - \hat{p} < b(v,p) - p$ for all $v \in [L,H)$. We conclude that $b(v,p) - p$ is a strictly decreasing function of p for all $v \in [L,H)$. ■

In addition to the stated results, from its proof we can see that Lemma 4 has also established an intuitive property that $b(v,p) > v$ for all $v < H$, i.e., the premium induces all bidders to bid higher then their true values except the one who has the highest possible value H.

Proof of Theorem 4.

We show by backward induction that b is an EPA-SE if and only if for all $X \in [L,H)$, $b(v,X)$ satisfies (4.4)-(4.5) on $\mathcal{D}(X)$.

Suppose b is an EPA-SE. We start with the second stage, assuming that the first stage ends with a bottom price $X \in [L,H)$ and that the current second-stage price level implies a screening level $r \ge r(X)$. Because $b(r,X) = p$ and $b(v,X)$ strictly increases in v, both finalists must have values higher than or equal to r if they have followed the equilibrium strategy until now. Clearly, $(v,b) \in \mathcal{D}(X)$ for $v \ge r$. Using the same arguments of Maskin and Riley (1984; p. 1485-1486)), it can be readily shown that the equilibrium condition (4.3) implies that $b(\cdot,X)$ is continuous on $[r,H]$ and differentiable on $[r,H)$.

Now for all $v, z \in [r, H)$, differentiating $\overline{U}(v, z|r, b(\cdot, X))$ (see (4.2)) with respect to z gives

$$\begin{aligned}
&\overline{U}_2(v, z|r, b(\cdot, X)) = \\
&\frac{f(z)}{1-F(r)}\left[u\left(v - b(z,X) + \varphi(b(z,X) - X)\right) - u\left(\varphi(b(z,X) - X)\right)\right] \\
&+\frac{1-F(z)}{1-F(r)}u'\left(\varphi(b(z,X) - X)\right)\varphi'(b(z,X) - X)b_1(z,X). \qquad (4.7)
\end{aligned}$$

Because the right-hand side of the above expression strictly increases in v,

$$\overline{U}_2(v, v|r, b(\cdot, X)) = 0 \text{ implies } \overline{U}_2(v, z|r, b(\cdot, X)) \begin{cases} > 0 \text{ if } z < v \\ = 0 \text{ if } z = v \\ < 0 \text{ if } z > v \end{cases}.$$

As $z \uparrow H$, the term in (4.7) vanishes (because $\overline{U}_2(v, v|r, b(\cdot, X)) = 0$ implies $\lim_{v\uparrow H} b_1(v, X) < \infty$, as shown in (4.6)). We then have

$$\begin{aligned}
&\overline{U}_2(v, H|r, b(\cdot, X)) = \frac{f(H)}{1-F(r)} \times \\
&\left[u\left(H - b(H,X) + \varphi(b(H,X) - X)\right) - u\left(\varphi(b(H,X) - X)\right)\right] \\
\lesseqqgtr\; & 0 \text{ for } b(H,X) \gtreqqless H.
\end{aligned}$$

Since $X \in [L, H)$ is arbitrary, we conclude that (4.3) holds if and only if for all $X \in [L, H)$, $b(v, X)$ satisfies (4.5) and (4.4) for all $v \in [r(X), H)$ (which is equivalent to $(v, b) \in \mathcal{D}(X)$). This completes the necessity part of the proof (and the sufficiency part of the proof for stage two).

Now consider the decision of a bidder with value v in the first stage. Suppose that b satisfies (4.4)-(4.5), and that the bidder computes his expected second-stage utility using b in (4.1). The question is: "If the bidder becomes a finalist at the current price p, does he expect a positive utility in the second stage?" Clearly, as long as $b(v, p) > p$ so that $v > r(p)$, where $r(p)$ derives from $b(r(p), p) = p$, it is a dominant strategy to stay because $U(v|r(p), b(\cdot, p)) > U(r(p)|r(p), b(\cdot, p)) = 0$. Once the price reaches the level where $b(v,p) = p$, then staying becomes a (weakly) dominated strategy, because it leads to a higher bottom price than p. It makes no difference in expected utility if the bidder can quit later in the first stage. But if the bidder quits too late and becomes a finalist, with bottom price $X > p$, he must then bid as if his value is higher than v in the second stage. This implies that his expected utility will be non-positive. (The best he might then do is to bid the bottom price X. But this is weakly dominated by quitting earlier: in case the other finalist also bids the bottom price, the random resolution of the tie could allocate the object to the bidder for too high a price, without any compensating premium.) We conclude that b constitutes an EPA-SE, having now completed the sufficiency part of the proof for both stages. ■

It is worth noting that n does not appear in (4.4)-(4.5). This suggests that, like standard private values English (or Vickrey) auctions, in an EPA the bidders in fact need not know the exact number of bidders – it suffices that each bidder only knows that there are some (more than one) other bidders competing in the first stage. Another observation is that the bid function is independent of any screening level r, which suggests that the possibility of updating the screening level will not affect bidding behavior. This observation lies at the bottom of the argument that the English (premium) auction and the Vickrey (premium) auction are strategically equivalent when only two bidders remain.

The next theorem concerns the existence and uniqueness of an EPA-SE. Because the right-hand side of differential equation (4.4) is undefined at $v = H$ (let alone Lipschitzian at this boundary point), we cannot directly apply the fundamental theorem of ordinary differential equations for a (unique) solution. The existence of a solution of (4.4)-(4.5) can be readily established, however, by employing standard techniques from ordinary differential equations theory. On the other hand, we can also look for sufficient conditions that directly guarantee the existence of an EPA-SE. As shown in Athey (2001), the set of sufficient conditions for the existence of equilibria in a large class of games of incomplete information includes a single crossing condition as proposed in Milgrom and Shannon (1994). In our context, the Milgrom-Shannon single crossing condition holds as long as $\overline{U}(v, z|r, b(\cdot, X))$ is supermodular in (v, z) (e.g., Athey (2001)).[7] We take this (shorter) approach in Theorem 5, verify that $\overline{U}(v, z|r, b(\cdot, X))$ is supermodular, and further show that the EPA-SE is unique under the assumptions (A1)-(A2).

Theorem 5 (Existence and uniqueness of EPA-SE) *For any utility function u and premium function φ, (i) there exists an EPA-SE $b : [L, H]^2 \to [L, H]$ and (ii) if the assumptions (A1)-(A2) hold, then b is unique.*

Proof. (**Existence**) By Theorem 4, it suffices to establish the existence of a second-stage equilibrium $b(\cdot, X)$ in an EPA for all $X \in [L, H]$. Theorem 4 also implies that there is no loss of generality to restrict attention to differentiable and strictly increasing bid functions in search of an EPA-SE. The case with $X = H$ is trivial, since it is then common knowledge that both finalists have values equal to H and $b(H, H) = H$. Now suppose $X \in [L, H)$ and that the opponent of a finalist adopts an increasing and differentiable bid function $b(\cdot, X)$. Then this finalist's expected utility at any screening level $r \geq r(X)$ is given by $\overline{U}(v, z|r, b(\cdot, X))$, where v is the bidder's true value and z determines his bid $b(z, X)$. Since $b(z, X)$ is continuous and strictly increasing in z, without ambiguity we can treat z as the bidder's "action." It follows from (4.2) that

$$\begin{aligned} & \overline{U}_{12}(v, z|r, b(\cdot, X)) \\ = \; & \frac{f(z)}{1 - F(r)} u'\left(v - b(z, X) + \varphi(b(z, X) - X)\right) > 0. \end{aligned}$$

This inequality implies that $\overline{U}$ is supermodular in (v, z) (Topkis (1978)). Thus, by Athey (2001; Corollary 2.1), there exists an increasing second-stage equilibrium $b(\cdot, X)$ of EPA. (It is easy to verify that other assumptions of Athey's Corollary 2.1 are satisfied in our context.) Since X is ar-

[7] See also Milgrom and Weber (1982) for the general definition of supermodularity.

bitrary, this proves that the bid function $b(\cdot, X)$ exists for all $X \in [L, H]$. By Theorem 4, $b(\cdot, X)$ is necessarily a solution of (4.4)-(4.5) whenever $b(v, X) \in [X, H)$, and, moreover, $b_1 > 0$ and $b(v, X) - X$ strictly decreases in X. These properties further imply that b meets the criteria of the first-stage equilibrium of the EPA (see the proof of Theorem 4). The existence of an EPA-SE is thus established.

(**Uniqueness**) Now assume that (A1)-(A2) hold. Let b be a solution of (4.4)-(4.5). By Theorem 4, $b(v, X) \in (v, H]$. Fix any $X \in [L, H)$ and $v \in [r(X), H)$. Then, the right-hand side of (4.4) strictly increases in b for $v < b \leq H$ as can be seen from

$$\begin{aligned}
& \frac{\partial}{\partial b} \frac{u\left(\varphi(b-X)\right) - u\left(v - b + \varphi(b-X)\right)}{u'\left(\varphi(b-X)\right) \varphi'(b-X)} \\
= \; & 1 + \frac{u'\left(v - b + \varphi\right)}{u'\left(\varphi\right)} \frac{1 - \varphi'}{\varphi'} \\
& - \frac{u\left(\varphi\right) - u\left(v - b + \varphi\right)}{u'\left(\varphi\right) \varphi'} \left(\frac{u''\left(\varphi\right) \varphi'}{u'\left(\varphi\right)} + \frac{\varphi''}{\varphi'} \right) \qquad (4.8) \\
> \; & 1 - \frac{u\left(\varphi\right) - u\left(v - b + \varphi\right)}{u'\left(\varphi\right)} \frac{u''\left(\varphi\right)}{u'\left(\varphi\right)} \quad \left(\text{by } 0 < \varphi' \leq \frac{1}{2} \text{ and (A1)}\right) \\
\geq \; & 0 \;\; (\text{by (A2)}) \qquad (4.9)
\end{aligned}$$

Suppose there exists another solution of (4.4)-(4.5) for some $X \in [L, H)$, say, $\hat{b}(v, X)$. Then we fix this X and apply Lemma 3(ii) to $h(v) \equiv b(v, X) - \hat{b}(v, X)$. We have $h(H) = 0$. From (4.4) and (4.9), it is easy to see that $h(v) \leq 0 \Rightarrow h'(v) \leq 0$ for all $v \in [L, H)$. Thus $h \geq 0$ on $[L, H]$. However, this logic applies also to $-h$, which implies $h \leq 0$ on $[L, H]$. We therefore conclude that $b(v, X) = \hat{b}(v, X)$ so that the solution of (4.4)-(4.5) is necessarily unique on $\mathcal{D}(X)$, which is equivalent to that the EPA-SE is unique.

■

Let us consider an example in which the bidders have constant absolute risk aversion (CARA). As we do not restrict attention to risk averse bidders, the Arrow-Pratt measure $-u''/u'$ can be positive (risk averse) or negative (risk loving).

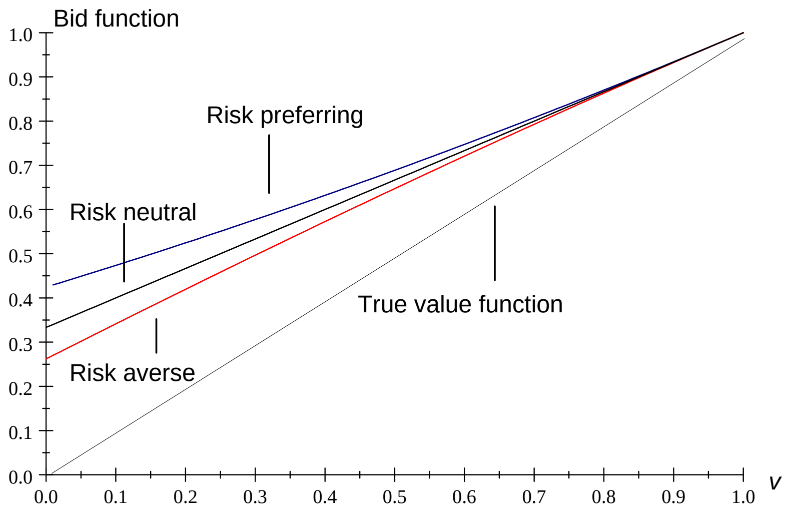

FIGURE 4.1. The premium induces the buyers to bid higher than their true values. The more risk tolerant the buyers are, the higher will be their bids. Here, F is assumed to be uniform on $[0, 1]$ and $\alpha = 0.5$. The risk averse bid function assumes $\lambda = 3$, and the risk seeking bid function assumes $\lambda = -3$.

Example. Suppose that the bidders have utility function

$$u(x) = \frac{1 - \exp(-\lambda x)}{\lambda}, \qquad \lambda \in \mathbb{R}.$$

Suppose further that φ is linear, with $\varphi(x) = \alpha x$ for some constant $\alpha \in (0, 1/2]$. The differential equation (4.4) then reduces to

$$b'_\lambda(v) = \frac{\exp(\lambda(b_\lambda(v) - v)) - 1}{\alpha\lambda} \frac{f(v)}{1 - F(v)} \tag{4.10}$$

where we observe that the bid function $b(v, X) \equiv b_\lambda(v)$ is independent of X.

The differential equation (4.10) can be solved explicitly to yield the EPA-SE. To see this, rearranging terms in (4.10), and multiplying both sides by $\exp(-\lambda b_\lambda(v))$, we obtain

$$\begin{aligned} & \alpha\lambda b_\lambda'(v)\,(1-F(v))\exp(-\lambda b_\lambda(v)) + f(v)\exp(-\lambda b_\lambda(v)) \\ = \;& f(v)\exp(-\lambda v) \end{aligned}$$

Now multiply both sides of the above equation by $(1-F(v))^{\frac{1}{\alpha}-1}$ to get

$$-\frac{\partial}{\partial v}\left(\alpha(1-F(v))^{\frac{1}{\alpha}}\exp(-\lambda b_\lambda(v))\right) = (1-F(v))^{\frac{1}{\alpha}-1} f(v)\exp(-\lambda v)$$

Then, integrating and rearranging terms yields the desired closed-form solution:

$$b_\lambda(v) = -\frac{1}{\lambda}\ln\left(\frac{1}{\alpha}\int_v^H \frac{e^{-\lambda x}}{1-F(x)}\left(\frac{1-F(x)}{1-F(v)}\right)^{\frac{1}{\alpha}} dF(x)\right)$$

Figure 4.1 depicts the bid functions of the risk averse, risk neutral, and risk preferring bidders. The figure confirms that the premium, in general, induces the bidders to bid higher than their true values. It also shows that the bids are uniformly higher (lower) if the bidders are more risk tolerant (averse).

4.4 The Premium Effects

We now investigate how the premium will affect the expected payment and the players' expected utilities. Let us start with a standard English auction (EA) without any premium, in which it is a (weakly) dominant strategy for each bidder to remain in the auction until the price reaches his true value v. When there are only two bidders left in an EA, who will also be the two finalists in an EPA, the current screening level r is equal to the current price p. The conditional expected utility at the screening level r of each remaining bidder in the EA then equals

$$\begin{aligned} W(v|r) &\equiv \frac{1}{1-F(r)}\int_r^v u(v-y)dF(y) \\ &= \frac{1}{1-F(r)}\int_r^H u\left(\max(v-y,0)\right)dF(y) \\ &= E\left[u(\max(v-y,0))|r\right], \end{aligned} \tag{4.11}$$

where y denotes the opponent's possible value, and the expectation $E(\cdot|r)$ is taken with respect to the conditional cumulative distribution of y, i.e., $\left[F(y)-F(r)\right]/\left[1-F(r)\right]$.

Consider next an EPA with a premium function φ. Suppose the first stage is over and the bottom price is X. At the start of the second-stage EPA, both finalists know that their values are higher or equal to $r(X)$ from $b(r(X),X)=X$. In order to highlight the effects of the premium, let us define a "gamble" $\Phi(\cdot|v,X)$ conditional on v and X. The payoff of Φ depends on the realization of $y\in[r(X),H]$ as follows:[8]

$$\Phi(y|v,X)=\left\{\begin{array}{cc} y-b(y,X)+\varphi(b(y,X)-X) & \text{if } y\in[r(X),v] \\ \varphi(b(v,X)-X) & \text{if } y\in(v,H] \end{array}\right. . \tag{4.12}$$

It can be the case that Φ is discontinuous at $y=v$, although it only occurs with a zero probability in the present model.

[8] If only the highest losing bidder receives the premium, then replacing the gamble Φ by Φ^0, defined as follows, will give the same predictions as in the subsequent theorems:

$$\Phi^0(y|v,X)=\left\{\begin{array}{cc} y-b(y,X) & \text{if } y\in[r,v] \\ \varphi(b(v,X)-X) & \text{if } y\in(v,H] \end{array}\right.$$

From (4.1), at any screening level $r \geq r(X)$ the conditional expected utility of a finalist can now be written as

$$\begin{aligned} & U(v|r, b(\cdot, X)) \\ = & \frac{1}{1-F(r)} \int_r^v u\left(v - y + \Phi(y|v, X)\right) dF(y) \qquad (4.13) \\ & + \frac{1-F(v)}{1-F(r)} u\left(\varphi(b(v, X) - X)\right) \\ = & \frac{1}{1-F(r)} \int_r^H u\left(\max(v-y, 0) + \Phi(y|v, X)\right) dF(y) \\ = & E\left[u(\max(v-y, 0) + \Phi(y|v, X))|r\right]. \qquad (4.14) \end{aligned}$$

Comparing the expected utility U in (4.14) to the expected utility W in (4.11) for $r = r(X)$, we find that the "gamble" Φ is entirely due to the premium offered, with the special case $\Phi \equiv 0$ corresponding to the EA.

Our next theorem establishes a *net-premium* effect in the EPA, which shows that as long as the second-stage EPA continues, the current screening level r reveals each finalist's conditional equilibrium expected utility for Φ—that is, when Φ is evaluated in *isolation* of other random payoffs. This result holds for arbitrary utility function u and premium function φ, and is independent of the bidder's private values. As can be seen from the proof of this theorem, the net-premium effect is essentially an "envelope theorem effect" as a consequence of incentive compatibility, which reduces to a revenue equivalence result in the present context when the bidders are risk neutral.

Theorem 6 (Net-premium effect) *For any utility function u and premium function φ, the second-stage EPA equilibrium implies that for all $X \in [L, H)$, $r \in [r(X), H]$, and $v \in [r, H]$,*

$$E\left[u(\Phi(y|v, X))|r\right] = u(\varphi(b(r, X) - X)). \qquad (4.15)$$

Proof. Let b be an EPA-SE. In the second stage of the EPA, for any bottom price X and screening level $r \geq r(X)$, the conditional expected utility of a finalist who has value v and who bids as though his value is z equals $\overline{U}(v,z|r,b(\cdot,X))$ (see (4.2)). Differentiating $\overline{U}$ with respect to v gives

$$\begin{aligned} & \overline{U}_1(v,z|b(\cdot,X)) \\ = \; & \frac{1}{1-F(r)}\int_r^z u'\left(v-b(y,X)+\varphi(b(y,X)-X)\right)dF(y). \end{aligned}$$

Because $\overline{U}$ is maximized at $z=v$ and $U(r|r,b(\cdot,X))=u(\varphi(b(r,X)-X))$, incentive compatibility and the envelope theorem imply

$$\begin{aligned} & U(v|r,b(\cdot,X)) \\ = \; & U(r|r,b(\cdot,X))+\int_r^v \overline{U}_1(z,z|b(\cdot,X))dz \\ = \; & U(r|r,b(\cdot,X)) \\ & +\frac{1}{1-F(r)}\int_r^v\int_r^z u'\left(z-b(y,X)+\varphi(b(y,X)-X)\right)dF(y)dz. \end{aligned}$$

Interchanging the order of integration we obtain

$$\begin{aligned} & U(v|r,b(\cdot,X)) \\ = \; & U(r|r,b(\cdot,X)) \\ & +\frac{1}{1-F(r)}\int_r^v\int_y^v u'\left(z-b(y,X)+\varphi(b(y,X)-X)\right)dzdF(y) \\ = \; & u(\varphi(b(r,X)-X)) \\ & +\frac{1}{1-F(r)}\int_r^v u\left(v-b(y,X)+\varphi(b(y,X)-X)\right)dF(y) \\ & -\frac{1}{1-F(r)}\int_r^v u\left(y-b(y,X)+\varphi(b(y,X)-X)\right)dF(y). \end{aligned} \tag{4.16}$$

On the other hand, $U(v|r, b(\cdot, X))$ has a direct expression given in (4.1). Thus, subtracting (4.1) from (4.16) yields

$$\begin{aligned}
& E\left[u(\Phi(y|v,X))|r\right] \\
= \; & \frac{1-F(v)}{1-F(r)} u\left(\varphi(b(v,X)-X)\right) \\
& + \frac{1}{1-F(r)} \int_r^v u\left(y - b(y,X) + \varphi(b(y,X)-X)\right) dF(y) \\
= \; & u(\varphi(b(r,X)-X)).
\end{aligned}$$

■

The net-premium effect is useful for gaining insight into the competitive bidding behavior in the English premium auctions. Notice that the right-hand side of (4.15) equals the bidder's utility for the premium if he quits at the current screening level r. As long as the bidder has value $v > r$, however, he will have no incentive to quit because staying in the auction gives him the same level of expected utility for the premium. In addition, from (4.14) we can see that there is an additional $\max(v - y, 0)$ to be possibly gained in case the opponent has a value $y \in [r, v)$. In equilibrium, this reasoning is common knowledge, and it therefore offers both finalists the comfort to sit back and relax, watching their premium grow up with the price. It is also common knowledge that this bidding process will continue until one of the bidders no longer expects to win.

Another useful implication of the net-premium effect is that at the time when the first stage has just ended with a bottom price X, both finalists derive a conditional expected utility for the premium that must be equal to zero. This is because $b(r(X), X) = X$ and thus at the screening level $r(X)$, $E\left[u(\Phi(y|v,X))|r(X)\right] = 0$. A special case is where the bidders are risk neutral; then, the net-premium effect reduces to an equivalent statement of the revenue equivalence principle (e.g., Myerson (1981)) that $E\left[\Phi(y|v,X)|r(X)\right] = 0$. It should be stressed, however, that the net-premium effect is an *isolated* premium effect. Only in the special case of risk neutrality does the effect imply that the bidders will be indifferent about the premiums. In general, the premium will affect the expected utilities of both risk averse and risk loving bidders in an EPA.[9]

Just as the revenue equivalence principle offers a useful tool for comparing welfare implications of various auction policies, the net-premium effect offers a handy tool for the subsequent analysis of the premium effects on the expected revenue and expected utilities of the seller and bidders in an EPA.

In what follows, unless specified otherwise we let $E(\cdot)$ denote the expectation under distribution $[1-F(v)]/[1-F(r)]$, where r is the screening level derived from the bottom price at the start of the second stage of the EPA.

Theorem 7 *For arbitrary number n (>2) of the bidders, and for arbitrary premium function φ, the expected revenue in any EPA-SE is lower (higher) when the bidders are more risk averse (loving).*

Proof. Let $\hat{u}$ be another utility function satisfying the same assumptions as u, with an absolute risk aversion measure satisfying $-\hat{u}''/\hat{u}' > -u''/u'$ at all relevant income levels. Let $\hat{b}$ and $\hat{X}$ denote the bid function and the bottom price when the bidders' preferences are represented by $\hat{u}$, and define $\hat{\Phi}$ similar to Φ as in (4.12).

Let r denote the third highest value from among the n bidders' values. It is clear that in an EPA equilibrium, r will be the screening level at the start of the second stage that is independent of the utility functional forms. (The bottom price at which the first stage ends can be different as the utility function changes.)

Denote by R and $\hat{R}$ the conditional expected payment of a finalist entering the second stage who has utility function u and $\hat{u}$, respectively. We

[9] This follows from the simple fact that for non-quasi-linear utility functions, in general

$$E\left[u\left(\max(v-y,0)+\Phi(y|v,X)\right)|r\right] \neq E\left[u\left(\max(v-y,0)\right)|r\right]+E\left[u\left(\Phi(y|v,X)\right)|r\right]$$

have

$$\begin{aligned}
& R(v|X) \\
= \; & \frac{1}{1-F(r)} \int_r^v \left[b(y,X) - \varphi(b(y,X)-X)\right] dF - \frac{1-F(v)}{1-F(r)} \varphi(b-X) \\
= \; & \frac{1}{1-F(r)} \left(\int_r^v \left[y - \Phi(y|v,X)\right] dF - (1-F(v))\, \varphi(b(v,X)-X) \right) \\
& \hat{R}(v|\hat{X}) \\
= \; & \frac{1}{1-F(r)} \left(\int_r^v \left[y - \hat{\Phi}(y|v,\hat{X})\right] dF - (1-F(v))\, \varphi(\hat{b}(v,\hat{X})-\hat{X}) \right)
\end{aligned}$$

Subtracting gives

$$\hat{R}(v|\hat{X}) - R(v|X) = E\left(\Phi(y|v,X)\right) - E\left(\hat{\Phi}(y|v,\hat{X})\right).$$

We know from Theorem 6 that

$$E\hat{u}(\hat{\Phi}(y|v,\hat{X})) = Eu(\Phi(y|v,X)) = 0.$$

Since $\hat{u}$ is more risk averse than u, the above equations imply

$$E\left(\Phi(y|v,X)\right) - E\left(\hat{\Phi}(y|v,\hat{X})\right) < 0$$

and hence $\hat{R}(v|\hat{X}) < R(v|X)$. Because the bidders are symmetric ex ante, this implies the conclusion of the theorem straightforwardly. ■

The reason why revenue decreases in the bidders' risk aversion can be seen from the expressions of R and $\hat{R}$: each finalist's conditional expected payment is the difference of the expected value of his opponent (in the event of winning) and the expected premium. Since the former has nothing to do with the utility functional forms, this difference is solely explained by the difference in the expected premiums. It follows then from the net-premium effect that the more risk averse bidders will command more expected premium in equilibrium, resulting in a lower expected payment.

The next corollary is an immediate consequence of Theorem 7.

Corollary 2 *Given any number n (> 2) of the bidders, adding a premium φ to an EA increases the expected revenue when bidders are risk loving, and decreases the expected revenue when bidders are risk averse.*

Proof. In an EA, for arbitrary utility function u, each bidder bids up to his true value v. This strategy leads to the same expected revenue in an EA when bidders are risk neutral. By the revenue equivalence theorem, the expected revenue is also the same in an EPA where the bidders are risk neutral. Therefore, by Theorem 7, the expected revenue under any premium φ is higher (lower) than that without a premium when bidders are risk loving (averse). ■

A straightforward implication of this corollary is that the problem of designing the "optimal premium function" that maximizes expected revenue, with the number of participants given, involves only a corner solution when the bidders are risk averse. In this case, the optimal premium function should be a constant zero.[10]

The results obtained so far are quite general, as they hold without much restriction regarding the shape of the distribution, premium, and utility functions (except for the uniqueness of the EPA-SE). The next two theorems require the assumptions (A1)-(A3). As shown in Theorem 5, these assumptions imply that the EPA-SE is unique. We first present a lemma that will be useful for Theorems 8-9.

Lemma 5 *Suppose that the assumptions (A1)-(A3) hold. Then, in the (unique) EPA-SE, $[1 - \varphi'(b(v,X) - X)]b_1(v,X) < 1$ for all $X \in [L,H)$ and $v \in (r(X),H)$.*

Proof. Fix $X \in [L,H)$. We apply Lemma 3(i) to

$$h(v) = 1 - [1 - \varphi'(b(v,X) - X)]\, b_1(v,X).$$

[10] If the buyers are risk preferring, the revenue maximizing premium function may not exist unless we impose some functional structure on φ. For instance, if we restrict φ to be linear such that $\varphi(b-X) = \alpha(b-X)$, where the constant α is the seller's choice variable that is restricted to be no greater than 1/2, then it can be shown that $\alpha = 1/2$ maximizes expected revenue.

Differentiating gives

$$h'(v) = -(1-\varphi')\,b_{11} + \varphi'' b_1^2.$$

Since $h(H) = 1 - \frac{1-\varphi'(H-X)}{1+\varphi'(H-X)} > 0$ and $\varphi'' \leq 0$ (by (A1)), it suffices to show that at any $v \in (r(X), H)$, $h(v) = 0$ implies $b_{11} > 0$ and hence $h'(v) < 0$. Differentiating b_1 gives

$$\begin{aligned}
b_{11}(v,X) = {} & \frac{u(\varphi) - u(v-b+\varphi)}{u'(\varphi)\varphi'(b-X)} \left(\frac{f(v)}{1-F(v)}\right)' \\
& + \frac{u'(\varphi)\varphi' b_1 - u'(v-b+\varphi)(1-(1-\varphi')b_1)}{u'(\varphi)\varphi'} \frac{f(v)}{1-F(v)} \\
& - \frac{u(\varphi) - u(v-b+\varphi)}{u'(\varphi)\varphi'} \left(\frac{u''(\varphi)}{u'(\varphi)}\varphi' + \frac{\varphi''}{\varphi'}\right) b_1 \frac{f(v)}{1-F(v)}.
\end{aligned}$$

By Lemma 4 and (A3), the first term on the right-hand side of this equation is strictly positive. If $(1-\varphi')\,b_1 = 1$ at some $v \in (L, H)$, then substituting into the above equation we have

$$\begin{aligned}
& b_{11}(v,X) \\
> {} & \left[1 - \frac{u(\varphi) - u(v-b+\varphi)}{u'(\varphi)\varphi'} \left(\frac{u''(\varphi)}{u'(\varphi)}\varphi' + \frac{\varphi''}{\varphi'}\right)\right] b_1 \frac{f(v)}{1-F(v)} \\
\geq {} & \left[1 - \frac{u(\varphi) - u(v-b+\varphi)}{u'(\varphi)} \frac{u''(\varphi)}{u'(\varphi)}\right] b_1 \frac{f(v)}{1-F(v)} \quad \text{(by (A1))} \\
\geq {} & 0 \quad \text{(by (A2))}
\end{aligned}$$

Lemma 3(i) then implies that $h > 0$, and hence for all $v \in (r(X), H]$, we have $(1 - \varphi'(b(v,X) - X))\,b_1(v,X) < 1$. ■

The role of this lemma is to establish that the "gamble" $\Phi(y|v,X)$ is an increasing function of y. An important implication of this property is that the premium reduces the riskiness of payment and therefore the riskiness of revenue.

Theorem 8 *Suppose that the assumptions (A1)-(A3) hold. Then, adding a premium φ to an EA increases the expected utility of the risk averse bidders, and decreases the expected utility of the risk loving bidders.*

Proof. Since the bidders who drop out in the first stage of an EPA have a zero expected utility regardless of any premium, we focus on the two finalists' conditional expected utilities at the start of the second stage. Fix v and X, and define $A(\theta|v, X) = Eu(\max(v - y, 0) + \theta\Phi(y|v, X))$ for $\theta \in [0, 1]$. Differentiating A with respect to θ gives

$$A'(\theta|v, X) = E\left[u'(\max(v - y, 0) + \theta\Phi(y|v, X))\Phi(y|v, X)\right].$$

Differentiate Φ with respect to $y \in (r(X), v)$ and $y \in (v, H]$, respectively, gives

$$\begin{aligned} &\Phi'(y|v, X) \\ = &\left\{ \begin{array}{ll} 1 - [1 - \varphi'(b(y, X) - X)]b_1(y, X) > 0 & \text{if } y \in (r(X), v) \\ 0 & \text{if } y \in (v, H] \end{array} \right. . \end{aligned}$$

where the inequality follows from Lemma 5. It is also easy to see that $\max(v - y, 0) + \theta\Phi(y|v, X)$ is a decreasing function of y for $\theta \in [0, 1]$. Now by Theorem 6, $Eu(\Phi) = 0$ implies that $E(\Phi) > 0$ if $u'' < 0$ and $E(\Phi) < 0$ if $u'' > 0$. Consequently,

$$\begin{aligned} &A'(\theta|v, X) \\ &\left\{ \begin{array}{l} > E\left[u'(\max(v - y, 0) + \theta\Phi(y|v, X))\right] E(\Phi) > 0 \text{ if } u'' < 0 \\ < E\left[u'(\max(v - y, 0) + \theta\Phi(y|v, X))\right] E(\Phi) < 0 \text{ if } u'' > 0 \end{array} \right. . \end{aligned}$$

where the first two inequalities follow from the established fact that u' and Φ are positively (negatively) correlated when $u'' < 0$ ($u'' > 0$). Because $A(0|v, X) = W(v|r(X))$ and $A(1|v, X) = U(v|b(\cdot, X))$, we obtain

$$U(v|b(\cdot, X)) \left\{ \begin{array}{l} > W(v|r(X)) \text{ if } u'' < 0 \\ < W(v|r(X)) \text{ if } u'' > 0 \end{array} \right. .$$

■

Intuitively, because $\Phi(y|v, X)$ and $-y$ are negatively correlated, adding a premium to an English auction reduces the risk of payment. In addition, for risk averse bidders the expected value of Φ is positive. These two effects are both favorable and hence the EPA is more attractive to risk averse bidders than the EA. The opposite holds for risk loving bidders.

Back to the Example considered in Section 4.3, we see that the bid functions depicted in Figure 4.1 confirm the results of Theorem 8. Using the risk neutral bidders' bid function as the reference, we see that the premium induces the risk loving bidders to bid "too high." Thus, the risk lovers will pay the seller a higher expected "net price" for the premium, resulting in lower expected utilities in comparison with the no-premium case (where they bid the true values). Likewise, the risk averse bidders bid "too low," and hence the seller's expected revenue is lower, and the bidders are uniformly better off with, rather than without, a premium.

Our last theorem extends the result of Theorem 8 to the case where the seller may be risk averse or risk loving.

Theorem 9 *Suppose that the assumptions (A1)-(A3) hold. Then, adding a premium φ to an English auction decreases a risk loving seller's expected utility if the bidders are (weakly) risk averse, and increases a risk averse seller's expected utility if the bidders are (weakly) risk loving.*

Proof. Let u_S denote the seller's utility function for income, and let $v^{(2)}$ and r denote the second and third highest values from among the n bidders, respectively. W.l.o.g. we normalize $u_S(0) = 0$. In either an EA or an EPA, r is revealed as soon as there are two bidders remain. Hence, we focus on the beginning of the second stage expected utilities conditional on r. The seller's utility is then uniquely determined by the realized value of $v^{(2)}$. Conditional on knowing $v^{(2)} \geq r$ in the second stage, the density function of $v^{(2)}$ is $2\left[1 - F(v^{(2)})\right] f(v^{(2)}) / \left[1 - F(r)\right]^2$. In what follows, $E(\cdot)$ denotes the expectation taken with respect to this density function.

In an EA when only two bidders remain, the seller's conditional expected utility is thus

$$E\left(u_S|\text{EA}\right) = \frac{2}{[1 - F(r)]^2} \int_r^H u_S\left(v^{(2)}\right)\left[1 - F(v^{(2)})\right] dF(v^{(2)}).$$

Likewise, in the second stage of an EPA the seller's conditional expected utility for the total net payment equals

$$\begin{aligned}
& E\left(u_S|\text{EPA}\right) \\
= \; & \frac{2}{[1-F(r)]^2} \times \\
& \int_r^H u_S\left(b(v^{(2)},X) - 2\varphi(b(v^{(2)},X) - X)\right)\left[1-F(v^{(2)})\right] dF(v^{(2)}) \\
= \; & \frac{2}{[1-F(r)]^2} \int_r^H u_S\left(v^{(2)} + \Psi(v^{(2)})\right)\left[1-F(v^{(2)})\right] dF(v^{(2)}),
\end{aligned}$$

where $\Psi(v^{(2)}) = b(v^{(2)},X) - 2\varphi(b(v^{(2)},X) - X) - v^{(2)}$. Corollary 2 implies that the expected value of Ψ satisfies

$$E\left[\Psi(v^{(2)})\right] \begin{cases} \geq 0 \text{ if } u'' \geq 0 \\ \leq 0 \text{ if } u'' \leq 0 \end{cases}.$$

By Lemma 5, $(1-\varphi')b_1 < 1$. Thus, for $\theta \in [0,1]$,

$$\begin{aligned}
\Psi'(v^{(2)}) &= b_1(v^{(2)},X)(1-2\varphi') - 1 < 0, \\
\left[v^{(2)} + \theta\Psi(v^{(2)})\right]' &= 1 + \theta\left[b_1(v^{(2)},X)(1-2\varphi') - 1\right] > 0.
\end{aligned}$$

Now define $B(\theta|v^{(2)}) = E\left[u_S\left(v^{(2)} + \theta\Psi(v^{(2)})\right)\right]$. Similar to the proof of Theorem 8, we have

$$\begin{aligned}
& B'(\theta|v^{(2)}) = E\left[u_S'\left(v^{(2)} + \theta\Psi(v^{(2)})\right)\Psi(v^{(2)})\right] \\
& \begin{cases} < E\left[u_S'\left(v^{(2)} + \theta\Psi(v^{(2)})\right)\right] E\left(\Psi\right) \leq 0 \text{ if } u'' \leq 0 \text{ and } u_S'' > 0 \\ > E\left[u_S'\left(v^{(2)} + \theta\Psi(v^{(2)})\right)\right] E\left(\Psi\right) \geq 0 \text{ if } u'' \geq 0 \text{ and } u_S'' < 0 \end{cases}.
\end{aligned}$$

Consequently, if $u'' \leq 0$ and $u_S'' > 0$, the seller prefers to choose the EA (i.e., $\theta = 0$), and if $u'' \geq 0$ and $u_S'' < 0$, the seller prefers to choose the EPA (i.e., $\theta = 1$). ■

The premium effect on the seller's expected utility is ambiguous if both the seller and bidders are either simultaneously risk averse or simultaneously risk loving. By a continuity argument, however, it can happen that when all players are risk averse, the seller also prefers the EPA to the EA as long as he is sufficiently more risk averse than the bidders. This follows from the strict preference of the seller for the EPA when the bidders are risk neutral, in which case the premium effect in reducing the revenue risk is predominant for the seller. Conversely, when all players are risk loving, it can happen that neither the seller nor the bidders would like to have the premium practice provided that the bidders are close to risk neutral.

4.5 Concluding Discussion

This chapter has studied a general English premium auction (EPA) model in a symmetric private values setting. The existence and uniqueness of the symmetric equilibrium for the class of EPA is established, along with some in-depth analyses of the effects of premium in relation to the bidders' risk preferences. When the premium is viewed as an additional "gamble" to an otherwise standard English auction, a remarkable "net-premium" effect emerges from our study. This effect implies that whatever the premium function is specified prior to the auction, and whatever is the bidders' risk preferences, the equilibrium expected utility for the premium, if calculated at the start of the second stage and in isolation of other random payoffs, must be equal to zero. This result considerably simplifies our comparative statics analysis, highlighting the reason why the premium enhances revenue if the bidders are risk loving, and the bidders prefer to have a premium if they are risk averse.

Under plausible conditions (the assumptions (A1)-(A3)), we find in Lemma 5 that the premium, in general, reduces the riskiness of the payment in the English auction. This result generalizes a similar finding in Goeree and Offerman (2004) that the premium reduces the variance of payment, calculated under a uniform distribution and linear premium rule. When the bidders exhibit constant absolute risk aversion (not necessarily risk averse), we also derive a closed-form solution for the equilibrium bid function for arbitrary distribution functions.

We conclude from this study that a seller facing ex ante symmetric bidders may consider a premium auction in two general situations. The first situation is where the seller is risk averse and where he has some good reason to believe that the bidders are approximately risk neutral or, at least, not "too" risk averse. In this situation the premium will play a (marginally) positive role in attracting entry, while at the same time reducing revenue risk. As long as the benefit of risk reduction outweighs the potential cost of a lower expected revenue, the premium auction will be preferred by both the seller and the buyers. The second situation arises where the seller is approximately risk neutral, where he is not concerned about entry and where he believes that the bidders will behave like risk seekers. Then the premium induces the bidders to overbid, resulting in a higher expected revenue. As long as the seller is not "too" risk loving himself, he will derive a higher expected utility in a premium auction.

The model presented in this chapter has assumed that the buyers have independent private values. This may be a reasonable assumption when the auctioned good is for private consumption. In other situations, allowing the bidders to have affiliated information and interdependent values will be more adequate (e.g., Milgrom and Weber, 1982). A natural extension of the present study is then to examine the potential effects of the premium tactics in the Milgrom-Weber general symmetric model with risk averse or risk loving players. Another line of extension is to endogenize the entry decision of the potential bidders when they face certain costs to participate in the auction.