Extremes of multidimensional Gaussian processes

Dębicki, K.; Kosiński, K.M.; Mandjes, M.; Rolski, T.

DOI
10.1016/j.spa.2010.08.010

Publication date
2010

Document Version
Final published version

Published in
Stochastic Processes and their Applications

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Stochastic Processes and their Applications 120 (2010) 2289–2301
stochastic processes and their applications
www.elsevier.com/locate/spa

Extremes of multidimensional Gaussian processes

K. Dębicki[a], K.M. Kosiński[b,c,*], M. Mandjes[b,c,d], T. Rolski[a]

[a] *Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland*
[b] *Korteweg–de Vries Institute for Mathematics, University of Amsterdam, The Netherlands*
[c] *Eurandom, Eindhoven University of Technology, The Netherlands*
[d] *CWI, Amsterdam, The Netherlands*

Received 31 May 2010; received in revised form 23 August 2010; accepted 25 August 2010
Available online 8 September 2010

Abstract

This paper considers extreme values attained by a centered, multidimensional Gaussian process $X(t) = (X_1(t), \ldots, X_n(t))$ minus drift $d(t) = (d_1(t), \ldots, d_n(t))$, on an arbitrary set T. Under mild regularity conditions, we establish the asymptotics of

$$\log \mathbb{P}\left(\exists t \in T : \bigcap_{i=1}^{n} \{X_i(t) - d_i(t) > q_i u\}\right),$$

for positive thresholds $q_i > 0$, $i = 1, \ldots, n$ and $u \to \infty$. Our findings generalize and extend previously known results for the single-dimensional and two-dimensional cases. A number of examples illustrate the theory.

MSC: 60G15; 60G70

Keywords: Gaussian process; Logarithmic asymptotics; Extremes

1. Introduction

Owing to its relevance in various application domains, in the theory of stochastic processes, substantial attention has been paid to estimating the tail distribution of the maximum value

* Corresponding author at: Eurandom, Eindhoven University of Technology, The Netherlands.
E-mail addresses: Krzysztof.Debicki@math.uni.wroc.pl (K. Dębicki), K.M.Kosinski@uva.nl (K.M. Kosiński), M.R.H.Mandjes@uva.nl (M. Mandjes), Tomasz.Rolski@math.uni.wroc.pl (T. Rolski).

doi:10.1016/j.spa.2010.08.010

attained. In mathematical terms, the setting considered involves an $\mathbb{R}$-valued stochastic process $X = \{X(t) : t \in T\}$ for some arbitrary set T and a threshold level $u > 0$, where the focus is on characterizing the probability

$$\mathbb{P}\left(\sup_{t\in T} X(t) > u\right) = \mathbb{P}\left(\exists t \in T : X(t) > u\right). \tag{1}$$

More specifically, the case in which X is a Gaussian process has been studied in detail. This hardly led to any explicit results for (1), but there is quite a large body of literature on results for the asymptotic regime in which u grows large. The prototype case dealt with a centered Gaussian process with bounded trajectories for which the *logarithmic asymptotics* were found: it was shown that

$$\lim_{u\to\infty} u^{-2} \log \mathbb{P}\left(\sup_{t\in T} X(t) > u\right) = -\left(2\sigma_T^2\right)^{-1}, \tag{2}$$

where

$$\sigma_T^2 := \sup_{t\in T} \mathbb{E}X^2(t).$$

See Adler [1, p. 42] or Lifshits [9, Section 12] for this and related results. The monographs Lifshits [9] and Piterbarg [11] contain more refined results: under appropriate conditions, an explicit function $\phi(u)$ is given such that the ratio of (1) and $\phi(u)$ tends to 1 as $u \to \infty$ (so-called *exact asymptotics*). The logarithmic asymptotics (2) can easily be extended to the case of noncentered Gausssian processes if the mean function is bounded. The situation gets interesting if both trajectories and the mean function of the process are unbounded. In this respect we mention Duffield and O'Connell [6] and Dębicki [3], where the logarithmic asymptotics of $\mathbb{P}(\sup_{t\geq 0}(X(t) - d(t)) > u)$ for general centered Gaussian processes X, under some regularity assumptions on the drift function d, were derived; see also Hüsler [8], Dieker [5] and references therein.

While the above results all relate to one-dimensional suprema, considerably less attention has been paid to their multidimensional counterparts. One of few exceptions is provided by the work of Piterbarg and Stamatović [12], who considered the case of two $\mathbb{R}$-valued, possibly dependent, centered Gaussian processes $\{X_1(t_1) : t_1 \in T_1\}$ and $\{X_2(t_2) : t_2 \in T_2\}$. They found the logarithmic asymptotics of

$$\mathbb{P}(\exists (t_1, t_2) \in T : X_1(t_1) > u, X_2(t_2) > u) \tag{3}$$

for some $T \subseteq T_1 \times T_2$, under the assumption that the trajectories of X_1 and X_2 are bounded.

In this paper our objective is to obtain the logarithmic asymptotics of (following the convention that vectors are written in bold)

$$P(u) := \mathbb{P}\left(\exists \boldsymbol{t} \in T : \bigcap_{i=1}^{n} \{X_i(\boldsymbol{t}) - d_i(\boldsymbol{t}) > q_i u\}\right); \tag{4}$$

here $\{\boldsymbol{X}(\boldsymbol{t}) : \boldsymbol{t} \in T\}$, with $\boldsymbol{X}(\boldsymbol{t}) = (X_1(\boldsymbol{t}), \ldots, X_n(\boldsymbol{t}))'$, is an $\mathbb{R}^n$-valued centered Gaussian processes defined on an arbitrary set $T \subseteq \mathbb{R}^m$, for some $m, n \in \mathbb{N}$, the $d_i(\cdot)$ are drift functions and $q_i > 0$ are threshold levels, $i = 1, \ldots, n$. Our setup is rich enough to cover both of the cases in which $P(u)$ corresponds to the event in which (i) it is required that there is a *single* time epoch $t \in \mathbb{R}$ such that $X_i(t) - d_i(t) > q_i u$ for all $i = 1, \ldots, n$ and (ii) there are n epochs

$(t_1, \ldots, t_n)$ such that $X_i(t_i) - d_i(t_i) > q_i u$ for all $i = 1, \ldots, n$. We get back to this issue in detail in Remark 1, where it is also noted that the theory covers a variety of situations between these two extreme situations.

Compared to the one-dimensional setting, the multidimensional case requires various technical complications to be settled. The derivations of logarithmic asymptotics usually rely on an upper and lower bound, where the latter is based on the inequality

$$P(u) \geq \sup_{\boldsymbol{t} \in T} \mathbb{P}\left(\bigcap_{i=1}^{n} \{X_i(\boldsymbol{t}) - d_i(\boldsymbol{t}) > q_i u\}\right).$$

Strikingly, in terms of the logarithmic asymptotics, this lower bound is actually tight, which is essentially due to the common 'large deviations heuristic': the decay rate of the probability of a union of events coincides with the decay rate of the most likely event among these events. A first contribution of the present paper is that we show that this argument essentially carries over to the multidimensional setting. In order to obtain the lower bound one needs asymptotics of tail probabilities that correspond to multivariate normal distributions. In this domain a wealth of results are available (see, e.g., Hashorva [7] and references therein), but for our purposes we need estimates which are, in some specific sense, uniform. A version of such estimates, that is tailored to our needs, is presented in Lemma 4.

The upper bound is based on what we call a 'saddle point equality' presented in Lemma 1. It essentially allows us to approximate suprema of multidimensional Gaussian process $\boldsymbol{X}$ by a specific one-dimensional Gaussian process, namely a properly weighted sum of the coordinates X_i of $\boldsymbol{X}$. Formally, we identify weights $w_i = w_i(t, u) \geq 0$ such that the inequality

$$P(u) \leq \mathbb{P}\left(\exists \boldsymbol{t} \in T : \sum_{i=1}^{n} w_i X_i(\boldsymbol{t}) > \sum_{i=1}^{n} w_i (u q_i + d_i(\boldsymbol{t}))\right),$$

is logarithmically asymptotically exact, as $u \to \infty$. The reduction of the dimension of the problem allows us to use one-dimensional techniques (such as the celebrated Borell inequality). Interestingly, the optimal weights can be interpreted in terms of the solution to a convex programming problem that corresponds to an associated Legendre transform of the covariance matrix of $\boldsymbol{X}$. A different weighting technique has been developed in Piterbarg and Stamatović [12] for the case $n = 2$, but without a motivation for the weights chosen. We recover the result from Piterbarg and Stamatović [12] in Remark 5. Our analysis of (4) extends the results from Dębicki [3] and Piterbarg and Stamatović [12], in the first place because $\mathbb{R}^n$-valued Gaussian processes are covered (for arbitrary $n \in \mathbb{N}$). The other main improvement relates to the considerable generality in terms of the drift functions allowed; these were not covered in Piterbarg and Stamatović [12].

The paper is organized as follows. In Section 2 we introduce notation, describe in detail the objects of main interest to us, and state our main result; we also pay special attention to the rationale behind the assumptions that we impose. In Section 3 we illustrate the main theorem by presenting a number of examples; one of these relates to Gaussian processes with regularly varying variance functions. We also explain the potential application of our result in queueing and insurance theory. In Section 4 we describe how the multidimensional process $\boldsymbol{X}$ can be approximated by a one-dimensional process Z, obtained by appropriately weighting the coordinates X_i. We prove some preliminary results about the characteristics of the process Z. This section also contains the saddle point equality mentioned above, Lemma 1, which is the crucial element of the proof of our main result. Section 4 also contains all other lemmas needed to prove Theorem 1, as well as the proof of our main result itself.

2. Model, notation, and the main theorem

In this section we formally introduce the model, state the main theorem, and provide the intuition behind the assumptions imposed.

2.1. Model and notation

Let $T \subseteq \mathbb{R}^m$, for some $m \in \mathbb{N}$. In this paper we consider an $\mathbb{R}^n$-valued (separable) centered Gaussian process $\boldsymbol{X} \equiv \{\boldsymbol{X}(\boldsymbol{t}), \boldsymbol{t} \in T\}$ given by $\boldsymbol{X}(\boldsymbol{t}) = (X_1(\boldsymbol{t}), \ldots, X_n(\boldsymbol{t}))'$. Let the so-called drift function be denoted by $\boldsymbol{d}(\boldsymbol{t}) = (d_1(\boldsymbol{t}), \ldots, d_n(\boldsymbol{t}))'$. Now, denote the covariance matrix of $\boldsymbol{X}(\boldsymbol{t})$ by $\Sigma_{\boldsymbol{t}}$. Throughout the paper it is assumed that the matrix $\Sigma_{\boldsymbol{t}}$ is invertible for every $\boldsymbol{t} \in T$. Here and in the sequel, we use the following notation and conventions:

- We say $\boldsymbol{v} \geq \boldsymbol{w}$ if $v_i \geq w_i$ for all $i = 1, \ldots, n$.
- We write $\operatorname{diag}(\boldsymbol{v})$ for the diagonal matrix with v_i on the ith position of the diagonal.
- We define $\boldsymbol{v}\boldsymbol{w} := \operatorname{diag}(\boldsymbol{v})\boldsymbol{w}' = (v_1 w_1, \ldots, v_n w_n)'$.
- For $a \in \mathbb{R}$, we let $\boldsymbol{i}(a)$ be an n-dimensional vector $(a, \ldots, a)'$ and also let $\boldsymbol{0} = (0, \ldots, 0)'$.
- We adopt the usual definitions of norms of vectors $\|\boldsymbol{x}\| := (\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.
- We let $f(u) \sim g(u)$ denote that $\lim_{u\to\infty} f(u)/g(u) = 1$.
- We write $\mathbb{R}^n_+ := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0, \boldsymbol{x} \neq \boldsymbol{0}\}$.

Throughout the paper not all vectors are of dimension n (for instance $\boldsymbol{t}$ is of dimension m), but the above notation should be understood with obvious changes.

With each $\Sigma_{\boldsymbol{t}}$ we associate the matrix $K_{\boldsymbol{t}} = (k_{i,j}(\boldsymbol{t}))_{i,j\leq n}$, defined as

$$K_{\boldsymbol{t}} = \operatorname{diag}(\partial_{1,1}^{-1/2}(\boldsymbol{t}), \ldots, \partial_{n,n}^{-1/2}(\boldsymbol{t}))\Sigma_{\boldsymbol{t}}^{-1}\operatorname{diag}(\partial_{1,1}^{-1/2}(\boldsymbol{t}), \ldots, \partial_{n,n}^{-1/2}(\boldsymbol{t}))$$

with $\Sigma_{\boldsymbol{t}}^{-1} = (\partial_{i,j}(\boldsymbol{t}))_{i,j\leq n}$. We mention that $k_{i,j}(\boldsymbol{t}) \in [-1, 1]$ and that $-k_{i,j}(\boldsymbol{t})$ is commonly interpreted as some sort of partial correlation between $X_i(\boldsymbol{t})$ and $X_j(\boldsymbol{t})$ controlling all other variables $X_k(\boldsymbol{t}), k \neq i, j$.

2.2. Main result

Throughout the paper, we impose the following assumptions.

A1 $\sup_{\boldsymbol{t}\in T} k_{i,j}(\boldsymbol{t}) < 1$ for all $i \neq j, i, j = 1, \ldots, n$.
A2 $\sup_{\boldsymbol{t}\in T}(X_i(\boldsymbol{t}) - \varepsilon d_i(\boldsymbol{t})) < \infty$ a.s. for all $i = 1, \ldots, n$ and all $\varepsilon \in (0, 1]$.

If a process $\boldsymbol{X}$ and a drift function $\boldsymbol{d}$ comply with assumptions A1–A2, then to shorten the notation, we will write that $(\boldsymbol{X}, \boldsymbol{d})$ satisfies A1–A2.

For a point $\boldsymbol{t} \in T$ and a vector $\boldsymbol{q} > \boldsymbol{0}$, define

$$M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u, \boldsymbol{t}) := \inf_{\boldsymbol{v}\geq u\boldsymbol{q}} \left\langle \boldsymbol{v} + \boldsymbol{d}(\boldsymbol{t}), \Sigma_{\boldsymbol{t}}^{-1}(\boldsymbol{v} + \boldsymbol{d}(\boldsymbol{t}))\right\rangle,$$

$$M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T) := \frac{1}{2}\inf_{\boldsymbol{t}\in T} M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u, \boldsymbol{t}).$$

With these preliminaries we are ready to state our main result. The following theorem can be seen as an n-dimensional extension of [12, Theorem 1] and [3, Theorem 2.1].

Theorem 1. *Assume that* $(\boldsymbol{X}, \boldsymbol{d})$ *satisfies* A1–A2. *Then, for any* $\boldsymbol{q} > \boldsymbol{0}$,

$$\log \mathbb{P}\left(\exists \boldsymbol{t} \in T : \boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right) \sim -M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T) \quad \text{as } u \to \infty. \tag{5}$$

Remark 1. The result stated in Theorem 1 enables us to analyze, with $T_i \subseteq \mathbb{R}$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n}\left\{\sup_{t_i \in T_i}\left(X_i(t_i) - d_i(t_i)\right) > uq_i\right\}\right). \tag{6}$$

To see this, let $T := T_1 \times \cdots \times T_n$. Also define processes $\{Y_i(\boldsymbol{t}) : \boldsymbol{t} \in T\}$, $i = 1, \ldots, n$, such that $Y_i(\boldsymbol{t}) := X_i(t_i)$, for $i = 1, \ldots, n$. Analogously, let $m_i(\boldsymbol{t}) := d_i(t_i)$, $i = 1, \ldots, n$. Then (6) equals

$$\mathbb{P}\left(\exists \boldsymbol{t} \in T : \boldsymbol{Y}(\boldsymbol{t}) - \boldsymbol{m}(\boldsymbol{t}) > u\boldsymbol{q}\right),$$

which, under the proviso that A1–A2 are complied with by the newly constructed $(\boldsymbol{Y}, \boldsymbol{m})$, fits in the framework of Theorem 1. This example naturally extends to the situation where the sets T_i are of dimension higher than 1.

2.3. Discussion of the assumptions

In this subsection we motivate the assumptions that we imposed.

Remark 2. Assumption A1 plays a crucial role in the proof of Lemma 4. It can be geometrically interpreted as follows. For a fixed $\boldsymbol{t} \in T$, the distribution of $\boldsymbol{X}(\boldsymbol{t})$ equals that of $B_{\boldsymbol{t}}\mathcal{N}$, where $B_{\boldsymbol{t}}$ is a matrix such that $\Sigma_{\boldsymbol{t}} = B_{\boldsymbol{t}}B_{\boldsymbol{t}}'$ and $\mathcal{N}$ is an $\mathbb{R}^n$-valued standard normal random variable. For some quadrant $Q_{\boldsymbol{t}}$, we need in the proof of Lemma 4 a lower estimate of $\mathbb{P}(\boldsymbol{X}(\boldsymbol{t}) \in Q_{\boldsymbol{t}}) = \mathbb{P}(\mathcal{N} \in B_{\boldsymbol{t}}^{-1}Q_{\boldsymbol{t}})$. For $i = 1, \ldots, n$ let $\boldsymbol{e}_i$ be, as usual, the standard basis vectors of $\mathbb{R}^n$. Then the cosine of the angle $\alpha_{i,j}$ between $B_t^{-1}\boldsymbol{e}_i$ and $B_t^{-1}\boldsymbol{e}_j$ is given by

$$\cos(\alpha_{i,j}) = \frac{\left\langle B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i, B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_j\right\rangle}{\|B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i\|\,\|B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_j\|} = \frac{\left\langle \boldsymbol{e}_i, \Sigma_{\boldsymbol{t}}^{-1}\boldsymbol{e}_j\right\rangle}{\|B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i\|\,\|B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_j\|} = \frac{\partial_{i,j}(\boldsymbol{t})}{\sqrt{\partial_{i,i}(\boldsymbol{t})\partial_{j,j}(\boldsymbol{t})}} = k_{i,j}(\boldsymbol{t}).$$

We thus observe that A1 entails that, for all $\boldsymbol{t} \in T$, there is no pair of vector $B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i$ and $B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i$, with $i \neq j$, that 'essentially coincide', i.e., the angles remain bounded away from 0. Therefore, for any $\boldsymbol{x} \in B_{\boldsymbol{t}}^{-1}Q_{\boldsymbol{t}}$, one can always find a set $A_{\boldsymbol{t}}$ such that $\boldsymbol{x} \in A_{\boldsymbol{t}} \subset B_{\boldsymbol{t}}^{-1}Q_{\boldsymbol{t}}$ and $A_{\boldsymbol{t}}$ has a diameter that is bounded, and a volume that is bounded away from zero, *uniformly* in $\boldsymbol{t} \in T$.

Remark 3. For $\varepsilon = 1$, assumption A2 assures that the event

$$\bigcup_{\boldsymbol{t} \in T}\{\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\}$$

is not satisfied trivially. The following example shows that if A2 is not complied with, then it is not ensured that we remain in the realm of exponential decay. Consider a one-dimensional case in which $X \equiv \{X(t) : t \geq 0\}$ is a standard Brownian motion, and for any $\delta > 0$ let $d(t) := (1+\delta)\sqrt{2t\log\log t}$. From the law of the iterated logarithm we conclude that the process X does not satisfy A2 for every $\varepsilon \in (0, 1]$. On the other hand we have (take $t := u^4$)

$$\mathbb{P}\left(\sup_{t \geq 0}\left(X(t) - (1+\delta)\sqrt{2t\log\log t}\right) > u\right) \geq \mathbb{P}\left(\frac{u\mathcal{N}}{1 + (1+\delta)u\sqrt{2\log(4\log u)}} > 1\right),$$

where here $\mathcal{N}$ is the real-valued standard normal random variable. On the logarithmic scale the latter probability behaves roughly, for u large, as

$$-(1+\delta)^2 \log\log u.$$

For the case of $n = 1$, A2 has been required in [3, Theorem 2.1] as well.

Remark 4. The drift functions $d_i, i = 1, \ldots, n$, are not assumed to be increasing, but under assumption A2 we have $\ell_i := \inf_{\boldsymbol{t}\in T} d_i(\boldsymbol{t}) > -\infty$. Because we are interested in the asymptotic behavior of the probability in (5) as $u \to \infty$, we can assume that $u > u_0 := -\min_i(\ell_i/q_i)$, and therefore the coordinates of $u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})$ stay positive for all $\boldsymbol{t} \in T$. In what follows we shall always assume that $u > u_0$.

3. Examples

In this section we present examples that demonstrate the consequences of Theorem 1. We focus on computing the decay rate $M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T)$ in two cases: (i) the case of $\boldsymbol{X}$ having bounded sample paths a.s.; (ii) the case of the X_i having stationary increments, regularly varying variance functions, and $d_i(\cdot)$ being linear. While in the former example the drift functions do not influence the asymptotics, in the latter example the drifts *do have* an impact on the decay rate.

3.1. Bounded sample paths and drift function

We here analyze the case of $(\boldsymbol{X}, \boldsymbol{d})$ satisfying

B1 The process $\boldsymbol{X}$ has bounded sample paths a.s.
B2 There exists $D < \infty$ such that $|d_i(\boldsymbol{t})| \le D$ for all $\boldsymbol{t} \in T$ and $i = 1, \ldots, n$.

We note that under B1–B2, it trivially holds that assumption A2 is complied with as well. Assumptions B1–B2 are satisfied when T is compact, $\boldsymbol{X}$ has continuous sample paths a.s. and $\boldsymbol{d}$ is continuous for instance. Let us introduce the following notation:

$$I_{\boldsymbol{X},\boldsymbol{q}}(T) := \inf_{\boldsymbol{t}\in T} \inf_{\boldsymbol{v}\ge\boldsymbol{q}} \left\langle \boldsymbol{v}, \Sigma_{\boldsymbol{t}}^{-1}\boldsymbol{v} \right\rangle.$$

The following corollary is an immediate consequence of Theorem 1.

Proposition 1. *Assume that* $(\boldsymbol{X}, \boldsymbol{d})$ *satisfies* A1 *and* B1–B2. *Then,*

$$\log \mathbb{P}\left(\exists \boldsymbol{t} \in T : \boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right) \sim -\frac{u^2}{2} I_{\boldsymbol{X},\boldsymbol{q}}(T), \quad \textit{as } u \to \infty.$$

The above proposition states that in the 'bounded case' that we are currently considering, we encounter the same asymptotic decay as in the driftless case ($\boldsymbol{d} \equiv \boldsymbol{0}$).

Remark 5. Some special cases of Proposition 1 have been treated before in the literature. In particular, let $X_1 \equiv \{X_1(t_1) : t_1 \in T_1\}$ and $X_2 \equiv \{X_2(t_2) : t_2 \in T_2\}$ be two centered and bounded $\mathbb{R}$-valued Gaussian processes. We introduce the notation $\sigma_i(t_i) := \sqrt{\mathbb{V}\mathrm{ar}(X_i(t_i))}, r(\boldsymbol{t}) := \mathbb{C}\mathrm{orr}(X_1(t_1), X_2(t_2))$ and also

$$c_{\boldsymbol{q}}(\boldsymbol{t}) := \min\left\{\frac{q_1}{\sigma_1(t_1)}\frac{\sigma_2(t_2)}{q_2}, \frac{\sigma_1(t_1)}{q_1}\frac{q_2}{\sigma_2(t_2)}\right\}.$$

Then, upon combining Proposition 1 with Remark 1, we obtain, with $T \subseteq T_1 \times T_2$,

$$\log \mathbb{P}\left(\exists (t_1, t_2) \in T : X_1(t_1) > q_1 u, X_2(t_2) > q_2 u\right) \sim -\frac{u^2}{2} \inf_{(t_1,t_2)\in T} \frac{1}{(\min\{\sigma_1(t_1)/q_1, \sigma_2(t_2)/q_2\})^2} \left(1 + \frac{(c_{\boldsymbol{q}}(\boldsymbol{t}) - r(\boldsymbol{t}))^2}{1 - r^2(\boldsymbol{t})} 1_{\{r(\boldsymbol{t}) < c_{\boldsymbol{q}}(\boldsymbol{t})\}}\right),$$

as $u \to \infty$. Observe that the above formula is also valid for $r(\boldsymbol{t}) = \pm 1$. This recovers the result of Piterbarg and Stamatović [12].

3.2. Stationary increments, linear drift

This section focuses on the logarithmic asymptotics of $\{\boldsymbol{X}(t) - \boldsymbol{i}(t) : t \geq 0\}$, where $\boldsymbol{X}(t) = S\boldsymbol{Y}(t)$ for some invertible matrix S and, as usual, $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_n(t))'$. We assume that, for $i = 1, \ldots, n$,

C1 $\{Y_i(t) : t \geq 0\}$ are mutually independent, $\mathbb{R}$-valued, centered Gaussian processes with stationary increments.

C2 The variance functions $\sigma_i^2(t) := \mathbb{V}\mathrm{ar}(Y_i(t))$ are regularly varying at ∞ with indexes $\alpha_i \in (0, 2)$. Without loss of generality we assume that $0 < \alpha_1 \leq \cdots \leq \alpha_n < 2$. Moreover, assume that there exists $\kappa \in \{1, \ldots, n\}$ such that $\sigma_1^2 \sim \cdots \sim c_\kappa \sigma_\kappa^2$ for some $c_i > 0$ and $\lim_{t\to\infty} \sigma_\kappa(t)/\sigma_{\kappa+1}(t) = 0$ (if $\kappa = 1$, then set $c_\kappa = 1$; if the first condition is satisfied with $\kappa = n$, then the second one is redundant).

C3 $\lim_{t\to 0} \sigma_i^2(t) |\log|t||^{1+\varepsilon} < \infty$ for some $\varepsilon > 0$.

We analyze

$$\mathbb{P}\left(\exists t \geq 0 : \boldsymbol{X}(t) - \boldsymbol{i}(t) \geq u\boldsymbol{q}\right). \tag{7}$$

Probabilities of this type play an important role in risk theory, describing the probability of simultaneous ruin of multiple (dependent) companies; see Avram et al. [2] for related results. The one-dimensional counterpart of (7) was considered in Dębicki [3] in the context of Gaussian fluid models. Related examples and further references can be found in the monograph [10]. In the following proposition we derive the logarithmic asymptotics of (7).

With c_i as in C2, set

$$C := \mathrm{diag}(1, c_2, \ldots, c_\kappa, 0, \ldots, 0)$$

and

$$J(C, S, \boldsymbol{q}, \alpha) := \inf_{t\geq 0} \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \frac{\left\langle S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t)), CS^{-1}(\boldsymbol{v} + \boldsymbol{i}(t))\right\rangle}{t^\alpha}.$$

Proposition 2. *Assume that* $\boldsymbol{Y}$ *satisfies* C1–C3*, and* S *is an invertible matrix. Then, for* $\{\boldsymbol{X}(t) : t \geq 0\} := \{S\boldsymbol{Y}(t) : t \geq 0\}$,

$$\log \mathbb{P}\left(\exists t \geq 0 : \boldsymbol{X}(t) - \boldsymbol{i}(t) \geq u\boldsymbol{q}\right) \sim -\frac{u^2}{2\sigma_1^2(u)} J(C, S, \boldsymbol{q}, \alpha_1), \quad \textit{as } u \to \infty.$$

Proof. We start by checking that A1–A2 are satisfied for $(\boldsymbol{X}, \boldsymbol{i})$. Indeed, let us note that the matrix $K_t = K$ is constant. Besides, since S is invertible, then K is invertible too, which combined with the fact that K is positive-definite and $k_{i,i} = 1$, straightforwardly implies that assumption A1 is satisfied.

Since $\boldsymbol{Y}$ has stationary increments, then under C1–C3 $\lim_{t\to\infty} Y_i(t)/t = 0$ almost surely and therefore (using that $\boldsymbol{X}$ consists of linear combinations of the Y_i, $i = 1, \dots, n$) assumption A2 is complied with; see [4, Lemma 3] for details. Now following Theorem 1,

$$\begin{aligned} M_{\boldsymbol{X},\boldsymbol{i},\boldsymbol{q}}(u; [0,\infty)) &= \frac{1}{2} \inf_{t\geq 0} \inf_{\boldsymbol{v}\geq u\boldsymbol{q}} \left\langle S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t)),\, R_t^{-1} S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t))\right\rangle \\ &= \frac{1}{2} \inf_{t\geq 0} \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \left\langle S^{-1}(u\boldsymbol{v} + u\boldsymbol{i}(t)),\, R_{ut}^{-1} S^{-1}(u\boldsymbol{v} + u\boldsymbol{i}(t))\right\rangle \\ &= \frac{u^2}{2} \inf_{t\geq 0} \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \left\langle S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t)),\, R_{ut}^{-1} S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t))\right\rangle, \end{aligned}$$

where the matrix R_t^{-1} equals $\operatorname{diag}(\sigma_1^{-2}(t), \dots, \sigma_n^{-2}(t))$, which is the inverse of the covariance matrix of $\boldsymbol{Y}$. Using the regular variation of $\sigma_i^2(\cdot)$, we find that, as $u \to \infty$,

$$\sigma_1^2(u) R_{ut}^{-1} \to t^{-\alpha_1} C, \quad \text{as } u \to \infty.$$

By virtue of the uniform convergence theorem we arrive at

$$M_{\boldsymbol{X},\boldsymbol{i},\boldsymbol{q}}(u; [0,\infty)) \sim \frac{u^2}{2\sigma_1^2(u)} \inf_{t\geq 0} \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \frac{\left\langle S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t)),\, C S^{-1}(\boldsymbol{v} + \boldsymbol{i}(t))\right\rangle}{t^{\alpha_1}},$$

as $u \to \infty$. This completes the proof. □

4. The proof of the main theorem

This section is devoted to the proof of our main result – Theorem 1. We will achieve this by establishing an upper bound and a lower bound. We start by presenting the following 'saddle point equality' that plays a crucial role in the upper bound.

Lemma 1. *Let A be any positive-definite matrix. Then,*

$$\sup_{\boldsymbol{w}\in\mathbb{R}_+^n} \frac{\langle \boldsymbol{w}, \boldsymbol{q}\rangle^2}{\langle \boldsymbol{w}, A\boldsymbol{w}\rangle} = \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle,$$

for any vector $\boldsymbol{q} \in \mathbb{R}_+^n$. Moreover, if $\boldsymbol{v}^\star$ is the optimizer of the infimum problem in the right-hand side, then $\boldsymbol{w}^\star := A^{-1}\boldsymbol{v}^\star$ is an optimizer of the supremum problem in the left-hand side.

Proof. Decompose $A = BB'$ for some nondegenerate matrix B. Then,

$$\frac{\langle \boldsymbol{w}, \boldsymbol{q}\rangle^2}{\langle \boldsymbol{w}, A\boldsymbol{w}\rangle} = \frac{\langle \boldsymbol{w}, \boldsymbol{q}\rangle^2}{\|B'\boldsymbol{w}\|^2} \quad \text{and} \quad \left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle = \|B^{-1}\boldsymbol{v}\|^2.$$

Now, for $\boldsymbol{w} \in \mathbb{R}_+^n$, the Cauchy–Schwarz inequality yields

$$\langle \boldsymbol{w}, \boldsymbol{q}\rangle = \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \langle \boldsymbol{w}, \boldsymbol{v}\rangle = \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \left\langle B'\boldsymbol{w}, B^{-1}\boldsymbol{v}\right\rangle \leq \|B'\boldsymbol{w}\| \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \|B^{-1}\boldsymbol{v}\|.$$

Dividing both sides by $\|B'\boldsymbol{w}\| > 0$ and optimizing the left-hand side of the previous display, we arrive at

$$\sup_{\boldsymbol{w}\in\mathbb{R}_+^n} \frac{\langle \boldsymbol{w}, \boldsymbol{q}\rangle^2}{\langle \boldsymbol{w}, A\boldsymbol{w}\rangle} \leq \inf_{\boldsymbol{v}\geq \boldsymbol{q}} \left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle.$$

To show the opposite inequality, assume that $\boldsymbol{v}^\star$ is such that

$$\inf_{\boldsymbol{v}\geq \boldsymbol{q}}\left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle = \left\langle \boldsymbol{v}^\star, A^{-1}\boldsymbol{v}^\star\right\rangle.$$

The Lagrangian function of the above problem is given by $L(\boldsymbol{v},\boldsymbol{\lambda}) := \left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle - \langle \boldsymbol{\lambda}, \boldsymbol{v}-\boldsymbol{q}\rangle$ for $\boldsymbol{\lambda}\geq \boldsymbol{0}$, and due to complementary-slackness considerations we necessarily have that $A^{-1}\boldsymbol{v}^\star \geq \boldsymbol{0}$, and if $(A^{-1}\boldsymbol{v}^\star)_i > 0$, then $v^\star{}_i = q_i$. Thus take $\boldsymbol{w}^\star = A^{-1}\boldsymbol{v}^\star \in \mathbb{R}^n_+$, so that

$$\frac{\langle \boldsymbol{w}^\star, \boldsymbol{q}\rangle^2}{\langle \boldsymbol{w}^\star, A\boldsymbol{w}^\star\rangle} = \frac{\left\langle A^{-1}\boldsymbol{v}^\star, \boldsymbol{q}\right\rangle^2}{\left\langle A^{-1}\boldsymbol{v}^\star, \boldsymbol{v}^\star\right\rangle} = \left\langle \boldsymbol{v}^\star, A^{-1}\boldsymbol{v}^\star\right\rangle.$$

Indeed, the last equality is equivalent to

$$\left\langle A^{-1}\boldsymbol{v}^\star, \boldsymbol{q}-\boldsymbol{v}^\star\right\rangle = 0,$$

but recall that if $(A^{-1}\boldsymbol{v}^\star)_i \neq 0$, then $(\boldsymbol{q}-\boldsymbol{v}^\star)_i = 0$. Hence finally,

$$\sup_{\boldsymbol{w}\in\mathbb{R}^n_+}\frac{\langle \boldsymbol{w}, \boldsymbol{q}\rangle^2}{\langle \boldsymbol{w}, A\boldsymbol{w}\rangle} \geq \inf_{\boldsymbol{v}\geq\boldsymbol{q}}\left\langle \boldsymbol{v}, A^{-1}\boldsymbol{v}\right\rangle,$$

which proves the opposite inequality. This finishes the proof. □

The main idea behind the proof of the upper bound of Theorem 1 is that the $\mathbb{R}^n$-valued process $\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t})$ can be effectively replaced by a suitably chosen $\mathbb{R}$-*valued* Gaussian process. The asymptotics of the latter process can then be handled using the familiar techniques for real-valued Gaussian processes.

For any vector $\boldsymbol{w}\in\mathbb{R}^n_+$ define

$$Z_{u,\boldsymbol{w}}(\boldsymbol{t}) := \frac{\langle \boldsymbol{w}, \boldsymbol{X}(\boldsymbol{t})\rangle}{\langle \boldsymbol{w}, u\boldsymbol{q}+\boldsymbol{d}(\boldsymbol{t})\rangle},$$

and observe that (with $u > u_0$; cf. Remark 4)

$$\mathbb{P}\left(\exists \boldsymbol{t}\in T : \boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right) \leq \mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Z_{u,\boldsymbol{w}}(\boldsymbol{t}) > 1\right).$$

The vector $\boldsymbol{w}$ in the process $Z_{u,\boldsymbol{w}}$ can be seen as a vector of weights assigned to the coordinates of $\boldsymbol{X}$. For fixed u and $\boldsymbol{w}$ the process $Z_{u,\boldsymbol{w}}$ is a centered Gaussian process. We shall show that it also has almost surely bounded sample paths.

Lemma 2. *Under* A1–A2*, the process $Z_{u,\boldsymbol{w}}$ is a centered Gaussian process with bounded sample paths almost surely, for each $\boldsymbol{w}\in\mathbb{R}^n_+$ and $u > u_0$. Moreover,*

$$\sup_{\boldsymbol{t}\in T} Z_{u,\boldsymbol{w}}(\boldsymbol{t}) \xrightarrow{\mathbb{P}} 0 \quad \text{as } u\to\infty.$$

Proof. Without loss of generality we can assume that $\|\boldsymbol{w}\| = 1$. For any $L\geq 1$, recalling the definition of $\boldsymbol{\ell}$ from Remark 4,

$$\begin{aligned}\mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Z_{u,\boldsymbol{w}}(\boldsymbol{t}) > L\right) &= \mathbb{P}\left(\exists \boldsymbol{t}\in T : \langle \boldsymbol{w}, \boldsymbol{X}(\boldsymbol{t})\rangle > \langle \boldsymbol{w}, Lu\boldsymbol{q} + L\boldsymbol{\ell} + L(\boldsymbol{d}(\boldsymbol{t}) - \boldsymbol{\ell})\rangle\right)\\ &\leq \mathbb{P}\left(\exists \boldsymbol{t}\in T : \langle \boldsymbol{w}, \boldsymbol{X}(\boldsymbol{t})\rangle > \langle \boldsymbol{w}, L(u\boldsymbol{q} + \boldsymbol{\ell}) + (\boldsymbol{d}(\boldsymbol{t}) - \boldsymbol{\ell})\rangle\right)\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{P}\left(\exists \boldsymbol{t} \in T : \langle \boldsymbol{w}, \boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t})\rangle > \langle \boldsymbol{w}, L(u\boldsymbol{q} + \boldsymbol{\ell}) - \boldsymbol{\ell}\rangle\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} w_i \sup_{\boldsymbol{t}\in T}(X_i(\boldsymbol{t}) - d_i(\boldsymbol{t})) > L\,\langle \boldsymbol{w}, u\boldsymbol{q} + \boldsymbol{\ell}\rangle - \langle \boldsymbol{w}, \boldsymbol{\ell}\rangle\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \sup_{\boldsymbol{t}\in T}(X_i(\boldsymbol{t}) - d_i(\boldsymbol{t}))^+ > L \min_i (uq_i + \ell_i)/\sqrt{n} - \|\boldsymbol{\ell}\|\right),
\end{aligned}
$$

where the last probability tends to zero with $L \to \infty$ due to A2. This proves that $Z_{u,\boldsymbol{w}}$ has bounded sample paths almost surely.

The last probability also tends to zero with $L \ge 1$ fixed and $u \to \infty$. On the other hand, for any $L < 1$ we have

$$
\begin{aligned}
\mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Z_{u,\boldsymbol{w}}(\boldsymbol{t}) > L\right) &= \mathbb{P}\left(\exists \boldsymbol{t} \in T : \langle \boldsymbol{w}, \boldsymbol{X}(\boldsymbol{t}) - L\boldsymbol{d}(\boldsymbol{t})\rangle > L\,\langle \boldsymbol{w}, u\boldsymbol{q}\rangle\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \sup_{\boldsymbol{t}\in T}(X_i(\boldsymbol{t}) - Ld_i(\boldsymbol{t}))^+ > uL \min_i q_i/\sqrt{n}\right),
\end{aligned}
$$

where the last probability tends to zero with $u \to \infty$ by virtue of A2. We therefore have that $\sup_{\boldsymbol{t}\in T} Z_{u,\boldsymbol{w}}(\boldsymbol{t})$ converges to 0 in probability. □

The above considerations remain true even if $\boldsymbol{w}$ depends on u and $\boldsymbol{t}$. This observation allows us to optimize the variance of the process $Z_{u,\boldsymbol{w}}$, while retaining its sample path properties. Notice that

$$
\mathbb{V}\mathrm{ar}(Z_{u,\boldsymbol{w}}(\boldsymbol{t})) = \frac{\langle \boldsymbol{w}, \Sigma_{\boldsymbol{t}}\boldsymbol{w}\rangle}{\langle \boldsymbol{w}, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\rangle^2}.
$$

Therefore, take $\boldsymbol{w}^\star \equiv \boldsymbol{w}^\star(u, \boldsymbol{t})$ such that

$$
\frac{\langle \boldsymbol{w}^\star, \Sigma_{\boldsymbol{t}}\boldsymbol{w}^\star\rangle}{\langle \boldsymbol{w}^\star, u\boldsymbol{q} + \boldsymbol{d}(t)\rangle^2} = \inf_{\boldsymbol{w}\in\mathbb{R}_+^n} \frac{\langle \boldsymbol{w}, \Sigma_{\boldsymbol{t}}\boldsymbol{w}\rangle}{\langle \boldsymbol{w}, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\rangle^2} \tag{8}
$$

and denote by $Y_u(\boldsymbol{t})$ the process $Z_{u,\boldsymbol{w}^\star}(\boldsymbol{t})$ with the weights $\boldsymbol{w} = \boldsymbol{w}^\star$ chosen as above. Let $\sigma_u^2(\boldsymbol{t})$ be the variance function of the process $Y_u(\boldsymbol{t})$. Then, by Lemma 1,

$$
\sigma_u^{-2}(\boldsymbol{t}) = M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u, \boldsymbol{t}). \tag{9}
$$

To estimate the tail of the supremum of the process $Y_u(\boldsymbol{t})$ we intend to use Borell's inequality [1, Theorem 2.1]. To apply this result, we need to verify that the expectation of $\sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t})$ vanishes as $u \to \infty$. This is done in the next lemma.

Lemma 3. *Under* A1–A2*, with u_0 as in Remark* 4*,*

1. $M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T) > 0$ *for each* $u > u_0$*;*
2. $\lim_{u\to\infty} M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T) = \infty$*;*
3. $\lim_{u\to\infty} \mathbb{E} \sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t}) = 0$.

Proof. From Lemma 2 we know that for a fixed u the process Y_u has bounded sample paths almost surely. This implies that $\sup_{\boldsymbol{t}\in T} \sigma_u^2(\boldsymbol{t}) < \infty$. But

$$
\sup_{\boldsymbol{t}\in T} \sigma_u^2(\boldsymbol{t}) = \sup_{\boldsymbol{t}\in T}(M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u, \boldsymbol{t}))^{-1} = \frac{1}{2}(M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u; T))^{-1}
$$

and claim (1) follows.

The proof of (2) is a consequence of the fact that under A2

$$\mathbb{P}\left(\sup_{t\in T} Y_u(\boldsymbol{t}) > 1\right) \to 0 \quad \text{as } u \to \infty,$$

and for $\mathscr{N}$ being a standard normal random variable

$$\mathbb{P}\left(\sup_{t\in T} Y_u(\boldsymbol{t}) > 1\right) \geq \sup_{t\in T} \mathbb{P}\left(Y_u(\boldsymbol{t}) > 1\right) = \mathbb{P}\left(\mathscr{N} > \inf_{t\in T} \sqrt{M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t})}\right).$$

To prove the last claim, observe that the almost sure boundedness of sample paths of $Y_u(\boldsymbol{t})$ implies that $\mathbb{E}\sup_{t\in T} Y_u(\boldsymbol{t}) < \infty$ and it easily follows that the family $(\sup_{t\in T} Y_u(\boldsymbol{t}))_u$ is uniformly integrable. Now claim 3 follows from the second part of Lemma 2. $\square$

Before we proceed to the proof of Theorem 1 we state a technical lemma, which is a prerequisite for the proof of the lower bound.

Lemma 4. *Under* A1*, there exist constants* $C_1 < \infty$, $C_2 > 0$ *such that for any* $\boldsymbol{t} \in T$

$$\log \mathbb{P}\left(\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right) \geq -\frac{1}{2} M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}) - C_1 M^{1/2}_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}) + C_2.$$

Proof. Set

$$Q_{\boldsymbol{t}} := \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} > u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\},$$

and let $B_{\boldsymbol{t}}$ be such that $B_{\boldsymbol{t}} B_{\boldsymbol{t}}' = \Sigma_{\boldsymbol{t}}$. Then $X(\boldsymbol{t}) \stackrel{d}{=} B_{\boldsymbol{t}}\mathscr{N}$, where $\mathscr{N}$ is an $\mathbb{R}^n$-valued standard normal random variable with the density function

$$f(\boldsymbol{x}) = D_n \exp\left(-\frac{1}{2}\langle \boldsymbol{x},\boldsymbol{x}\rangle\right),$$

for some normalizing constant D_n. In this notation, we have

$$\mathbb{P}\left(\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right) = \mathbb{P}\left(\boldsymbol{X}(\boldsymbol{t}) \in Q_{\boldsymbol{t}}\right) = \mathbb{P}\left(\mathscr{N} \in B_{\boldsymbol{t}}^{-1} Q_{\boldsymbol{t}}\right).$$

Now let $\boldsymbol{x}^\star = \boldsymbol{x}^\star(u,\boldsymbol{t}) \in B_{\boldsymbol{t}}^{-1} Q_{\boldsymbol{t}}$ be such that

$$M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}) = \inf_{\boldsymbol{x}\in Q_{\boldsymbol{t}}} \left\langle \boldsymbol{x}, \Sigma_{\boldsymbol{t}}^{-1}\boldsymbol{x}\right\rangle = \inf_{\boldsymbol{x}\in B_{\boldsymbol{t}}^{-1}Q_{\boldsymbol{t}}} \langle \boldsymbol{x},\boldsymbol{x}\rangle = \left\langle \boldsymbol{x}^\star, \boldsymbol{x}^\star\right\rangle,$$

and let $A_{\boldsymbol{t}} := \mathcal{B}(x^\star, 1) \cap B_{\boldsymbol{t}}^{-1} Q_{\boldsymbol{t}}$, where $\mathcal{B}(x^\star, 1)$ is a ball in $\mathbb{R}^n$ of radius 1 and center $x^\star$. Then,

$$\mathbb{P}\left(\mathscr{N} \in B_{\boldsymbol{t}}^{-1} Q_{\boldsymbol{t}}\right) \geq \int_{A_{\boldsymbol{t}}} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Set $\Delta(\boldsymbol{x},\boldsymbol{x}^\star) := \langle \boldsymbol{x},\boldsymbol{x}\rangle - \langle \boldsymbol{x}^\star,\boldsymbol{x}^\star\rangle$. Then

$$\mathbb{P}\left(\mathscr{N} \in B_{\boldsymbol{t}}^{-1} Q_{\boldsymbol{t}}\right) \geq D_n \mathrm{Vol}(A_{\boldsymbol{t}}) \exp\left(-\frac{1}{2} M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}) - \frac{1}{2}\sup_{\boldsymbol{x}\in A_{\boldsymbol{t}}} \Delta(\boldsymbol{x},\boldsymbol{x}^\star)\right).$$

Since

$$\Delta(\boldsymbol{x},\boldsymbol{x}^\star) \leq 2\|\boldsymbol{x}-\boldsymbol{x}^\star\| \left\langle \boldsymbol{x}^\star,\boldsymbol{x}^\star\right\rangle^{1/2} + \|\boldsymbol{x}-\boldsymbol{x}^\star\|^2,$$

we have that

$$\sup_{\boldsymbol{x}\in A_{\boldsymbol{t}}} \Delta(\boldsymbol{x},\boldsymbol{x}^\star) \le 2\mathrm{diam}(A_{\boldsymbol{t}}) M^{1/2}_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}) + \mathrm{diam}^2(A_{\boldsymbol{t}}).$$

Therefore the claim follows if $\mathrm{diam}(A_{\boldsymbol{t}})$ and $\mathrm{Vol}(A_{\boldsymbol{t}})$ can be bounded uniformly in $\boldsymbol{t} \in T$ from above and below, respectively.

Observe that, by the construction of $A_{\boldsymbol{t}}$, $\mathrm{diam}(A_{\boldsymbol{t}}) \le 1$. Besides, the quadrant $Q_{\boldsymbol{t}}$ is spanned by the standard basis $(\boldsymbol{e}_i)$ in $\mathbb{R}^n$ fixed in the point $u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})$. The cosine of the angle $\alpha_{i,j}$ between $B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_i$ and $B_{\boldsymbol{t}}^{-1}\boldsymbol{e}_j$ is given by $\cos(\alpha_{i,j}) = k_{i,j}$; see Remark 2. Under A1 this angle is bounded away from zero, uniformly in $\boldsymbol{t} \in T$. Therefore $\inf_{\boldsymbol{t}\in T} \mathrm{Vol}(A_{\boldsymbol{t}}) > 0$. This completes the proof. □

Now we are ready to prove the main theorem.

Proof of Theorem 1. Put $P(u) := \mathbb{P}(\exists \boldsymbol{t} \in T : \boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q})$. We split the proof into two parts: the lower and the upper bound.

Lower bound: The lower bound follows directly from Lemma 4 and the inequality

$$\log P(u) \ge \sup_{\boldsymbol{t}\in T} \log \mathbb{P}\left(\boldsymbol{X}(\boldsymbol{t}) - \boldsymbol{d}(\boldsymbol{t}) > u\boldsymbol{q}\right).$$

Upper bound: Let $\boldsymbol{w}^\star : \mathbb{R}_+ \times T \to \mathbb{R}^n_+$ be the mapping chosen in (8). Now as in the definition of the process Y_u,

$$P(u) \le \mathbb{P}\left(\exists \boldsymbol{t} \in T : \left\langle \boldsymbol{w}^\star, \boldsymbol{X}(\boldsymbol{t})\right\rangle > \left\langle \boldsymbol{w}^\star, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\right\rangle\right) = \mathbb{P}\left(\sup_{\boldsymbol{t}\in T} \frac{\langle \boldsymbol{w}^\star, \boldsymbol{X}(\boldsymbol{t})\rangle}{\langle \boldsymbol{w}^\star, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\rangle} > 1\right)$$
$$= \mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t}) > 1\right),$$

where the passage from the n-dimensional quadrant to the tangent increases the probability. Recall that the variance $\sigma_u^2(\boldsymbol{t})$ of $Y_u(\boldsymbol{t})$ equals $(M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u,\boldsymbol{t}))^{-1}$; cf. (9). Moreover, thanks to Lemma 3, the Gaussian process Y_u has bounded sample paths almost surely. Therefore, Borell's inequality implies that

$$\mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t}) > 1\right) \le 2\exp\left(-\left(1 - \mathbb{E}\sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t})\right)^2 M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u;T)\right).$$

Now from (2) and (3) of Lemma 3 we obtain

$$\limsup_{u\to\infty} \frac{\log \mathbb{P}\left(\sup_{\boldsymbol{t}\in T} Y_u(\boldsymbol{t}) > 1\right)}{M_{\boldsymbol{X},\boldsymbol{d},\boldsymbol{q}}(u;T)} \le -1$$

and the claim follows. □

Remark 6. From the proof of the upper bound we obtain the useful inequality

$$\mathbb{P}\left(\exists \boldsymbol{t} \in T : \boldsymbol{w}^\star X(\boldsymbol{t}) > \boldsymbol{w}^\star(u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t}))\right) \le \mathbb{P}\left(\exists \boldsymbol{t} \in T : \left\langle \boldsymbol{w}^\star, \boldsymbol{X}(\boldsymbol{t})\right\rangle > \left\langle \boldsymbol{w}^\star, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\right\rangle\right),$$

which we have proven to be exact in terms of logarithmic asymptotics. Let $\boldsymbol{v}^\star \equiv \boldsymbol{v}^\star(u,\boldsymbol{t})$ be such that

$$\left\langle \boldsymbol{v}^\star + \boldsymbol{d}(\boldsymbol{t}), \Sigma_{\boldsymbol{t}}^{-1}(\boldsymbol{v}^\star + \boldsymbol{d}(\boldsymbol{t}))\right\rangle = \inf_{\boldsymbol{v}\ge u\boldsymbol{q}} \left\langle \boldsymbol{v} + \boldsymbol{d}(\boldsymbol{t}), \Sigma_{\boldsymbol{t}}^{-1}(\boldsymbol{v} + \boldsymbol{d}(\boldsymbol{t}))\right\rangle.$$

Then the optimal weights $\boldsymbol{w}^\star$ are given by $\boldsymbol{w}^\star(u, \boldsymbol{t}) = \Sigma_{\boldsymbol{t}}^{-1}\boldsymbol{v}^\star(u, \boldsymbol{t})$, or alternatively, due to Lemma 1, by

$$\boldsymbol{w}^\star(u, \boldsymbol{t}) = \arg \sup_{\boldsymbol{w}\in\mathbb{R}_+^n} \frac{\langle \boldsymbol{w}, u\boldsymbol{q} + \boldsymbol{d}(\boldsymbol{t})\rangle^2}{\langle \boldsymbol{w}, \Sigma_{\boldsymbol{t}}\boldsymbol{w}\rangle}.$$

Observe that the weights do not depend on u in the case of $\boldsymbol{d} \equiv \boldsymbol{0}$.

Acknowledgements

The first and third authors thank the Isaac Newton Institute, Cambridge, UK, for hospitality. The research of the first and fourth authors was supported by MNiSW Research Grant N N201 394137 (2009–2011). The second author thanks the Mathematical Institute, University of Wrocław, Poland, for hospitality. The research of the second author was supported by NWO grant 613.000.701.

References

[1] R. Adler, An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes, in: Lecture Notes—Monograph Series, vol. 12, IMS, 1990.
[2] F. Avram, Z. Palmowski, M. Pistorius, A two dimensional ruin problem on the positive quadrant, Insurance Math. Econom. 42 (2008) 227–234.
[3] K. Dębicki, A note on LDP for supremum of Gaussian processes over infinite horizon, Statist. Probab. Lett. 44 (1999) 211–219.
[4] T. Dieker, Conditional limit theorems for queues with Gaussian input, a weak convergence approach, Stochastic Process. Appl. 115 (2005) 849–873.
[5] T. Dieker, Extremes of Gaussian processes over an infinite horizon, Stochastic Process. Appl. 115 (2005) 207–248.
[6] N. Duffield, N. O'Connell, Large deviations and overflow probabilities for general single-server queue, with applications, Math. Proc. Cambridge Philos. Soc. 118 (1995) 363–374.
[7] E. Hashorva, Asymptotics and bounds for multivariate Gaussian tails, J. Theoret. Probab. 18 (2005) 79–97.
[8] J. Hüsler, V. Piterbarg, Extremes of a certain class of Gaussian processes, Stochastic Process. Appl. 83 (1999) 257–271.
[9] M. Lifshits, Gaussian Random Functions, in: Mathematics and its Applications, vol. 322, Kluwer Academic Publishers, Dordrecht, 1995.
[10] M. Mandjes, Large Deviations for Gaussian Queues, Wiley, Chichester, 2007.
[11] V. Piterbarg, Asymptotics Methods in the Theory of Gaussian Processes and Fields, in: Translation of Mathematical Monographs, vol. 148, AMS, Providence, RI, 1996.
[12] V. Piterbarg, B. Stamatović, Crude asymptotics of the probability of simultaneous high extrema of two Gaussian processes: the dual action function, Russian Math. Surveys 60 (2005) 167–168.