Pricing long-term options with stochastic volatility and stochastic interest rates

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Chapter 1

Introduction and Outline

The markets for long-term options have witnessed an explosive growth over the last decade. One of the recent developments is the expansion of markets for long-term European equity, exchange rate and inflation options. Currently, liquid prices for maturities up to thirty years and beyond are shown for these products. Also the markets for hybrid options, which depend on multiple underlying assets, are starting to take off. Pension contracts, for instance, incorporate options which are exposed to both equity and interest rates risks. The correlation, a measure for the dependency between the underlying assets, has a large impact on the pricing and risk management of such contracts. The simultaneous decreases in asset prices and interest rates around 2003 and during the recent credit crunch are “perfect” examples hereof and caused the funding ratios of many pension funds to drop to historically low levels. Appropriate methods for the pricing and risk management of long-term contracts, should therefore at least be able to deal with such (joint) market risks.

Though the history of option contracts is dated back to ancient Greek and Babylonian times, the trading of financial securities on exchanges arose from the 16th to 18th centuries. In Antwerp, Amsterdam and London, well organized exchanges were established dealing a range of commodities and financial options. These activities included trading for future delivery, “time bargains”, as well as options. At the bourse of Amsterdam contract prices for tulip bulbs reached extraordinarily high levels around 1637 and then suddenly collapsed, which is generally considered as the first ever recorded financial bubble, referred to as “tulip mania”. In the U.S. the first formal futures and options exchange, the Chicago Board of Trade, was established in 1848 and initially served to reduce the seasonality effects of grain harvesting. At the time, the distribution of grain took place in Chicago due to its central location, however its storage facilities were unable to accommodate the enormous increase in supply after the harvest, whilst the same facilities were underutilized during other periods of the year. As a consequence spot prices for grain fell and rose severely. Futures contracts, on the other hand, allowed farmers to store the grain at other places and deliver it to Chicago at a later time. In this way, farmers were able transfer the price risk associated with the grain as it could always be sold and delivered.
anywhere else at any time. However, it was not until 1973, marked by both the creation of the Chicago Board Options Exchange and the publication of the Black and Scholes (1973) and Merton (1973) papers, that option trading really took off.

Prior to the breakthrough of Black, Scholes and Merton, market participants would have had to rely on heuristics methods and their own views of the future to determine the prices of options. Starting with Bachelier (1900), who suggested a fair game approach using a normal distribution for the underlying asset, attempts were made to develop option pricing formulas. Nevertheless, all these approaches lacked the crucial insight of Black and Scholes (1973) and Merton (1973) that, under certain assumptions, an option can be turned into a risk-free instrument using a technique called “dynamic hedging”. Under the assumption that no arbitrage opportunities exist in the financial markets, the price of the option price should equal the price of its replicating portfolio, irrespective of investor’s risk attitudes and expectations. This finding together with an increasing computational power, formed the basis for an explosive revolution in the use of derivatives, making this industry as large as it is today.

Using the replication argument, more exotic structures could also be priced. In such a procedure, the prices of actively traded contracts, like futures or European options, are used to determine the hedge and hence the involved costs of the exotic option. Therefore, an option pricing model can essentially be interpreted as an extrapolation method of the prices of these simpler instrument. A necessary requirement for such an extrapolation to make sense, is that within the model the prices of simpler contracts coincide with their market price. It became clear that this was not the case within the Black and Scholes (1973) model and that some of its assumptions, like constant volatility and constant interest rates were inappropriate. A lot of research within mathematical finance, has therefore concentrated on the development of alternative models and asset price dynamics, such that the prices of liquid vanilla options and the stochastic nature of the underlying asset are matched in a more suitable way.

Generally, to price an exotic option, one first chooses a financial model for the underlying asset, which is calibrated to match the prices of traded vanilla contracts as closely as possible. Using appropriate numerical techniques, the calibrated model is then applied to price a specific exotic option. Many mathematical models may fit such a description and additional criteria have to be considered to assess the fitness of a model to price financial derivatives. First, as a pricing model can essentially be interpreted as an extrapolation tool between liquid simple contracts to complex derivatives, it should be economically plausible and parsimonious. Secondly, practitioners demand fast and accurate prices and sensitivities of financial contracts, therefore the model should be analytically tractable. Thirdly, because financial models and option contracts are becoming increasingly complex, efficient methods have to be developed to cope with these evolutions. For a realistic pricing and risk management of long-term options, it is furthermore strongly advised to incorporate empirical phenomena as heavy-tailed returns, stochastic interest rates and general correlation structures into a derivative pricing model.

All three parts of this thesis, are devoted to the valuation of long-dated derivatives. Part I is ded-
icated to the incorporation of long-term maturities into the development of new models and the derivation of closed-form pricing formulas herein. Part II considers the pricing of exotic options using Monte Carlo simulation, in particular the development of efficient discretization schemes. Finally, Part III develops pricing formulas for embedded insurance options and performs a quantitative analysis of their valuations. For each part, the remainder of this chapter discusses its motivation, scope and contribution to the literature. Furthermore, it will elaborate how the parts are interconnected with each other.

**Part I: Stochastic Interest Rates and Stochastic Volatility**

Many of the assumptions of the Black and Scholes (1973) model, like constant volatility and constant interest rates, do not find justification in the financial markets. In particular since the equity crash of the late eighties a battery of complex models has been proposed to improve upon these misspecifications. One class of models relaxes the deterministic volatility assumption and incorporates an empirical financial phenomenon known as volatility clustering, providing more realistic, heavy-tailed asset returns. Other popular approaches involve the use of stochastic interest rates, local volatility or jumps, depending on the specific application. For instance in the case of embedded insurance derivatives, characterized by long maturities, it would be suitable to incorporate stochastic volatility and stochastic interest rates into an option pricing model.

An overview of the literature related to the pricing of long-term options under stochastic volatility and stochastic interest rates is given in Chapter 2, which also provides a short introduction to the methods and techniques used throughout this thesis.

Chapter 3 is concerned with the pricing of long-dated exotic contracts with stochastic volatility and stochastic interest rates. To incorporate the need for stochastic interest rates, the stochastic volatility models of Stein and Stein (1991) and Schöbel and Zhu (1999) are expanded to allow for Hull and White (1993) interest rates and a general correlation structure between the stock price process, its stochastic volatility and interest rates. The resulting Schöbel-Zhu-Hull-White (SZHW) model, can be placed in the general framework for affine models of Duffie et al. (2000) and Duffie et al. (2003), and benefits greatly from the analytical tractability that is typical for this class of models. Our contribution to the existing literature is threefold. First, we derive the characteristic functions of the log-asset price, which enables an efficient closed-form pricing of European options by Fourier inversion. Secondly, since the practical relevance of any model is limited without a numerical implementation, we extensively consider the efficient implementation of the transform inversion required to price European options. In particular we derive the limiting behaviour of the characteristic function of the SZHW model which allows us to calculate the inversion integral much more accurately. Thirdly, we generalize the SZHW model to be able to value foreign exchange options in a framework where both domestic and foreign interest rate processes are stochastic.

To price cross-currency derivatives, such as foreign exchange and inflation options, most
investment banks have standardized on a three-factor modelling framework. Hereby the index has a deterministic volatility, and the interest rates of the two currencies are driven by one-factor Gaussian models. This deterministic volatility assumption, though technically very convenient, does not find justification in the financial markets for equity, FX or inflation options. In fact, the markets for these products exhibit a strong volatility skew or smile, implying log index returns that deviate from normality and suggest the use of skewed and heavy tailed distributions. Moreover, many multi-currency structures are directly exposed to the shape of the volatility structure as they often incorporate multiple strikes as well as callable and knockout components. Examples hereof include Limited Price Index (Inflation) options, Power Reverse Dual Contract (Foreign Exchange) swaps and Pension Funding Ratio (Equity-Interest Rates) options. Suitable derivative pricing models, aimed to quantify this volatility exposure and corresponding hedge costs, should therefore at least be able to incorporate the volatility shapes of the vanilla option markets. Though in the current literature various methods exist to incorporate volatility smiles, to price multi-currency options also a term-structure involving various time points of the forward index is required. The incorporation of stochastic interest rates makes the connection between the two particularly non-trivial and is by Piterbarg (2005) even dubbed as “perhaps even the most important current outstanding problems for quantitative research departments worldwide”.

Chapter 4 of this thesis develops a generic multi-currency framework incorporating the need for stochastic interest rates, stochastic volatility, whilst allowing for closed-form calibration formulas for vanilla options. The modelling framework considers the pricing of inflation, foreign exchange and stock options under multi-factor Gaussian interest rates, Schöbel and Zhu (1999) and Heston (1993) stochastic volatility, hereby using a full correlation structure between all driving quantities. Relying on Fourier transform methods, we show that vanilla call and put options, forward starting options, year-on-year inflation-indexed swaps and inflation-indexed caps and floors can be valued in closed-form. We suggest a new calibration algorithm, based on a control variate technique, for the generic Heston model. This new method is compared to the Markovian projection technique of Antonov et al. (2008) and turns out to provide certain advantages over the Markovian projection approximation. Furthermore, we demonstrate that the frameworks are well able to incorporate the markets implied volatility shapes. Finally, due to the generic setup, the multi-currency framework has the additional advantages that it can be used for multi-asset purposes and is fast enough for the real life risk management of big portfolios of inflation, FX, equity, interest rates, commodities and hybrid option contracts.

Part II: Efficient Simulation Methods for Valuing Exotic Derivatives

Once a suitable financial model is selected and calibrated, the next step is to apply it in practice. Though certain models yield closed-form solutions for some contracts, the fast majority of products cannot be priced in closed-form. Monte Carlo methods provide an extremely popular and flexible pricing alternative to value such complex derivatives. Due to technical advances
such as multi-processor programming, increasing computational power and modern day variance
reduction techniques, the Monte Carlo technique is expected to become even more widely ap-
licable in the near future. By its nature, however, these methods are relatively time consuming
as they are based on repetitive simulations. Many attention of both academics as practitioners is
devoted to the efficient simulation schemes aiming to minimize the computational efforts whilst
retaining a high degree of accuracy. In Part II of this thesis we deal with efficient discretization
methods for stochastic volatility models.

Though an exact simulation method for the Heston (1993) model was developed in Broadie and
Kaya (2006), its practical use is limited due to its complexity and lack of speed. For instance, to
simulate the Non-central Chi-squared distributed variance process an acceptance and rejection
technique is suggested, which hinders the sensitivity analysis and which cannot be used in
conjunction with low-discrepancy numbers. Euler discretizations give rise to a completely
different category of problems. For example, while the continuous-time variance process of the
Heston (1993) model is guaranteed to be non-negative, its Euler discretization is not.

Chapter 5 deals with efficient discretizations schemes for the Heston (1993) stochastic volatility
model. The considered schemes improve upon disadvantages of the exact method considered in
Broadie and Kaya (2006). To overcome the acceptance and rejection sampling method of the
exact scheme, one can create a large three-dimensional cache of the inverse from the non-central
chi-squared distribution function for all conceivable values of the number of degrees of freedom,
the non-centrality parameters and its function values. Nonetheless, as the parameter-space is
potentially very large, such a brute-force caching method is not realizable for practical purposes.
Using a conditioning argument, we will however show that the three-dimensional inverse of the
Non-central Chi-squared distribution can effectively be reduced to a one dimensional search
space for the case of the Heston (1993) model. We develop three new efficient simulation
schemes, using this insight. Finally, we perform an extensive numerical comparison between
the new methods with other recent schemes. Approximations of the exact scheme based on drift
interpolations of the integrated variance process, are found to be several times more efficient
than the recent Euler, Kahl and Jäckel (2006) and exact schemes.

A major problem signaled with Euler schemes in the simulation of stochastic volatility models
is their inability to generate the proper correlation between the increments of the asset and
the stochastic volatility processes. As the correlation parameter in the stochastic volatility
models is an important determinant of the skew in implied volatilities, not being able to match
this parameter, leads to a significant mispricing of options with strikes far away from the
at-the-money level. In the Heston (1993) model, this so-called “leaking correlation” problem,
is partially caused by the fact that an Euler discretization tries to approximate a square root
process, by a Gaussian process. However even when the stochastic volatility itself is Gaussian,
such as in the Schöbel and Zhu (1999) model, the problem of “leaking correlation” is still an
issue.

In Chapter 6 discretization schemes are presented for the Schöbel and Zhu (1999) stochastic
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volatility model, tailored to match correlation between the increments of the asset and the variance processes of the continuous-time process. Particular attention is given to the 'leaking correlation' issue in the simulation of Heston (1993) and Schöbel and Zhu (1999) models. Furthermore the simulation in the Schöbel-Zhu-Hull-White extension, which incorporates the need for stochastic interest rates, is considered. This is closely related to recent advances in the development of markets for long-term derivatives, described in Part I of this thesis, for which maturities the inclusion of stochastic interest rates in a derivatives pricing model is more suitable. Though the continuous time processes of the considered stochastic volatility models are guaranteed to be a martingale, and hence have finite first moments, this does not necessarily hold for their discretizations. To this end, we derive conditions for the regularity of the developed discretization schemes and investigate how to ensure the exact martingale property. Finally, we numerically compare the new simulation schemes to other recent schemes in the literature. For a special case of the Schöbel and Zhu (1999) model which coincides with the Heston (1993) model, our proposed scheme has a similar performance to the QE-M scheme of Andersen (2008), whilst being slightly more efficient in terms of computational time required. For the Schöbel-Zhu cases not coinciding with the Heston model, it is found that our scheme consistently outperforms the Euler scheme. These results affirm that Andersen’s result is more widely applicable than to the Heston model alone; for the simulation of stochastic volatility models, it is of great importance to match the correlation between the asset price and its stochastic volatility process.

Part III: Applications to Insurance Markets

The third and last part of this thesis is concerned with the pricing of two popular type of contracts appearing in insurance markets. Using the methods of Part I and II of this thesis, we investigate the impact of stochastic volatility, stochastic interest rates and a general correlation structure on the valuation of insurance contracts. By developing closed-form solutions for the prices of forward starting and guaranteed annuity options, we are able to carry out a quantitative analysis on the pricing of these embedded options. The analysis performed in Chapter 7 and 8 stands out, compared to the existing literature, by taking both stochastic volatility and stochastic interest rates explicitly into account.

Forward starting options form the basis for many Unit-Linked guarantees, cliquet and ratchet options. Due to their popularity, these products recently attracted a lot of attention from both academics and practitioners. Forward starting contracts belong to the class of path-dependent European-style contracts in the sense that they not only depend on the terminal value of the underlying asset, but also on the asset price at an intermediate point. For instance, a forward starting option may provide the holder a call option with a strike equal to a fixed proportion of the underlying asset price at some intermediate date. These options are frequently used by insurance companies to hedge year on year Unit-Linked guarantees, but also many structured products, tailored for investors seeking for upside potential whilst preserving protection against downside movements, involve such forward starting optionalities.
Chapter 7 is concerned with a quantitative analysis on the valuation of forward starting options, where we explicitly account for stochastic volatility, stochastic interest rates and a general correlation structure between all underlying processes. This analysis is facilitated by the development of closed-form formulas, based on Fourier inversions, for these contracts. Compared to vanilla options, forward starting structures are much more sensitive to future interest rate movements, volatility smiles as well as their correlation structure with the underlying asset. It is found that it is important to take stochastic interest rates, volatility and a general correlation structure into account for a proper valuation and hedging of these securities: ignoring one of these aspect can lead to serious mispricings and hedge errors.

In Chapter 8 the pricing of guaranteed annuity options (GAOs) is investigated using a stochastic volatility model for equity prices. GAOs are options providing the right to convert a policyholder’s accumulated funds to a life annuity at a fixed rate when the policy matures. These options were a common feature in UK retirement savings contracts issued in the 1970’s and 1980’s when interest rates were high, but they caused problems for insurers as the interest rates began to fall in the 1990’s. Currently, these options are frequently sold in the U.S. and Japan as part of variable annuity products. Until now, for the pricing of these options generally a geometric Brownian motion for equity prices is assumed. However, given the long maturities of the insurance contracts a stochastic volatility model for equity prices, providing more realistic equity returns, would be more suitable.

The contribution of Chapter 8 is threefold. First, closed-form expressions are derived for prices of GAOs assuming stochastic volatility for equity prices and either a one-factor or two-factor Gaussian interest rate model. Secondly, we come up with a more efficient GAO pricing formula, than considered in Chu and Kwok (2007), for an equity model with constant volatility. Under two-factor Gaussian rates, these authors argue that no analytical pricing formula exists and hence propose several approximation methods for its valuation. In this chapter we do derive an exact closed-form pricing formula in terms of a single numerical integral. This method is preferable compared to these latter approaches, as it gives exact GAO prices over all strike levels whilst being computational very efficient to compute. Finally, using U.S. and EU market option data, we investigate the effects of a stochastic volatility model on the pricing of GAOs. For both markets, the results indicate that the impact of ignoring a stochastic volatility model can be significant.