Pricing long-term options with stochastic volatility and stochastic interest rates
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Chapter 2

Stochastic Models

2.1 Affine Diffusions

In this section we discuss the class of affine models, as this class of models is frequently used throughout this thesis. The class of affine models, was introduced in Duffie and Kan (1996), in an interest rate framework, and was later generalized in great detail by Duffie et al. (2000) and Duffie et al. (2003). In the remainder of this section we discuss the subclass of affine diffusions, and its Fourier transform. This subclass is not as general as the affine jump-diffusion processes considered in Duffie et al. (2003), but is sufficient for our purposes. The popularity of the affine class of models arises from their flexibility and generic setup combined with a great analytical tractability, which facilitates the calibration and simulation of such models. Eminent members of this class are the term structure models of Hull and White (1993) and Cox et al. (1985), the Black and Scholes (1973) model, and also the stochastic volatility models of Heston (1993) and Schöbel and Zhu (1999).

The class of affine diffusions can be characterized as follows; let $X$ be a real-valued $n$-dimensional Markov process satisfying

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t), \quad (2.1)$$

where $W(t)$ is a standard Brownian motion in $\mathbb{R}^n$, $\mu(X(t)) \in \mathbb{R}^n$ and $\sigma(X(t)) \in \mathbb{R}^{n \times n}$. Then, we call the process $X$ affine if and only if the diffusion coefficients are of the following form

$$\mu(x) = K_0 + K_1 x, \quad \text{for } K = (K_0, K_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}, \quad (2.2)$$

$$(\sigma(x)\sigma^T(x))_{ij} = (H_0)_{ij} + (H_1)_{ij} \cdot x, \quad \text{for } H = (H_0, H_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}. \quad (2.3)$$

In addition, the short term interest rate $r(X(t))$ is also assumed to be affine:

$$r(x) = \rho_0 + \rho_1^T x, \quad \text{for } R = (\rho_0, \rho_1) \in \mathbb{R} \times \mathbb{R}^n. \quad (2.4)$$

In total, for a process $X(t)$ to be affine, both its instantaneous drift, variance and short rate need to be an affine combination of the factors.
The first key result in Duffie et al. (2000) is that the characteristic function (and the moment generating function, when it exists) of \( X(t) \), is known in closed-form up to the solution of a system of Ordinary Differential Equations. That is, for \( u \in \mathbb{C}^n \) the Fourier transform \( \phi(u, X(t), t, T) \) of \( X(t) \) is given by

\[
\phi(u, X(t), t, T) = \mathbb{E}\left[ e^{-\int_t^T r(X(s))ds} e^{iuX(T)} \mid \mathcal{F}_t \right] = e^{A(u, t, T) + B(u, t, T)X(t)},
\]

where \( A \) and \( B \) solve the following system of Riccati equations

\[
\begin{align*}
\frac{dA(u, t, T)}{dt} &= \rho_0 - K_0 \cdot B(u, t, T) - \frac{1}{2} B^T(u, t, T)H_0 B(u, t, T), \\
\frac{dB(u, t, T)}{dt} &= \rho_1 - K_1^T B(u, t, T) - \frac{1}{2} B^T(u, t, T)H_1 B(u, t, T).
\end{align*}
\]

subject to the terminal condition \( A(u, T, T) = 0 \) and \( B(u, T, T) = u \). In general the solutions to \( A \) and \( B \) to the system of Ordinary Differential Equations (2.6) have to be computed numerically, see e.g. Press and Flannery (1992). Of course models for which \( A \) and \( B \) can be computed in closed-form have a large advantage.

In the context of stochastic volatility models, expression (2.5) provides an expression for the characteristic function of the logarithm of the asset price. Closed-form prices of European options can be obtained by inverting this characteristic function, see Section 2.5. We will provide the characteristic functions of the considered models in this thesis when start using them. For example, the characteristic functions of the Schöbel-Zhu and Heston model, extended with stochastic interest rates, are derived in Chapter 3 and 4.

### 2.2 Stochastic Interest Rates

There is a vast literature on modelling the term structure of interest rates. Approaches for the pricing and risk management of interest rate derivatives have been described by Vasicek (1977), Hull and White (1993), Cox et al. (1985), Heath et al. (1992), Miltersen et al. (1997), Brace et al. (1997), Hunt et al. (2000). These interest rate models can be subdivided into short rate, Heath-Jarrow-Morton, market and Markov functional models. We refer to Brigo and Mercurio (2006), Pelsser (2000) for an extensive overview of interest rate modelling literature. It goes beyond the scope of this thesis to discuss all (dis)advantages of each of these framework, which for example can be judged in terms of its dimensionality, number of parameters versus calibration quality, correlation structures, or its analytical tractability. However, above all, it is important that the main price sensitivities of the considered contract are properly captured by the chosen interest rate model.

In this thesis we restrict ourselves most of the time to short rate models, as they are sufficiently able capture the desired interest rate characteristics and often allow for a tractable pricing of
2.2. Stochastic Interest Rates

long-term options. An additional advantage of short rate models is that they can be placed in the affine class and therefore benefit from the analytical properties typical for this class. In affine term structure models, one can for example express the price of a zero coupon bond maturing at time $T$ into an exponential affine form of its state variable(s), i.e.

$$P(t, T) = e^{A(u,t,T)+B(u,T)X(t)},$$

(2.8)

which can be obtained from (2.5)-(2.7). As the future term structure of interest rates, implied by the discount factors, is of crucial importance for the pricing of interest rate derivatives, this result is extremely useful. Finally, note that in many short models, for instance in the models of Vasicek (1977), Hull and White (1993) and Cox et al. (1985), closed-form solutions exists for $A$ and $B$.

2.2.1 Change of Numeraire

In an arbitrage-free and complete market the value of any contingent claim can be uniquely determined as the expectation of the payoff normalized by the money market account under a unique equivalent measure, see Harrison and Kreps (1979), Harrison and Pliska (1981). Under this measure the expected return on all assets is equal to the risk-free rate, hence this measure is dubbed as “the risk-neutral measure”, denoted here by $Q$. The normalizing asset, in these papers the money market account, is called the numeraire. In Geman et al. (1996) it is shown that not only the money market account can be used as numeraire, but every strictly positive self-financing portfolio of traded assets, can be used as numeraire. Their change of numeraire technique demonstrates how to change from one numeraire to another by switching to a different probability measure. As a byproduct every positive non-dividend paying asset divided by its numeraire, is a martingale under the measure associated with that numeraire.

For example, the measure change from the money market measure $Q$ to the $T$-forward measure $Q^T$ can be established by the following Radon-Nikodym derivative

$$\frac{dQ^T}{dQ} = \frac{\exp\left[-\int_0^T r(u)du\right]}{P(0,T)},$$

and the corresponding $T$-forward measure, see Geman et al. (1996), hence uses the time $T$ zero-coupon bond price $P(0,T)$ as numeraire. In particular for the pricing time-$T$ payoffs, it can be convenient to use the time-$T$ discount factor as numeraire, since the numeraire is then equal to one at the time of maturity.

2.2.2 Hull-White model

To be able to exactly fit the currently observed term structure of interest, Hull and White (1993) extended the Vasicek (1977) model with a time-varying mean reversion level. That is, under
the money market measure $Q$ (which uses the risk-free interest rate of the bank account as numeraire), the Hull and White (1993) model provides following dynamics

$$dr(t) = [\theta(t) - ar(t)] dt + \sigma dW_r(t),$$

(2.9)

with $a, \sigma$ two constants, $W_r(t)$ a Brownian motion and where the deterministic function $\theta(t)$ is chosen to exactly fit the currently observed term structure of interest rates. As the short rate follows an Ornstein-Uhlenbeck process, this model implies a normal distribution for the short interest rate at each time. The Gaussian distribution allows for the derivation of analytical formulas and the construction of efficient numerical methods for the pricing of various interest rate derivatives. On the other, the theoretical possibility of $r$ dropping below zero, due to the Gaussian distribution, is a clear drawback of the model. However, in practice, the probability of negative rates in this model is often negligible, e.g. see Brigo and Mercurio (2006), and even so, does not lead to any issues for most products.

Following Pelsser (2000), Brigo and Mercurio (2006), the Hull and White (1993) model can more conveniently be parameterized as

$$r(t) = x(t) + \beta(t),$$

(2.10)

which separates the interest rate process $r(t)$ into a deterministic function $\beta$ and a standard Ornstein-Uhlenbeck process $x(t)$ satisfying

$$dx(t) = -ax(t)dt + \sigma dW_r(t),$$

(2.11)

In particular, in order to exactly fit the currently observed term structure of interest rates, we have that

$$\int_0^T \beta(u)du = \ln \frac{P(0, t)}{P(0, T)} + \frac{1}{2} [V(0, T) - V(0, t)],$$

(2.12)

with $V(t, T)$ defined as

$$V(t, T) := \frac{\sigma^2}{a^2} \left[ T - t + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right].$$

(2.13)

For the Hull and White (1993) model, one can use the normality and (2.5) to obtain the following expression for the price of a zero-coupon bond $P(t, T)$ maturing at time $T$

$$P(t, T) = A(t, T) e^{-B(t, T) x(t)},$$

(2.14)

with:

$$A(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left[ \frac{1}{2} \left( V(t, T) - V(0, T) + V(0, t) \right) \right],$$

(2.15)

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

(2.16)
2.3. Stochastic Volatility

and with \( P(0, s) \) denoting the market’s time zero discount factor maturing for time \( s \).

By applying the change of numeraire technique of Section 2.2, one can change the underlying numeraire from the money market account to the time \( T \) discount bond price and evaluate the model dynamics under the \( T \)-forward measure \( Q^T \). Girsanov’s theorem implies that in the Hull and White (1993) model the process \( W^T_t(t) \) defined by

\[
dW^T_t(t) = dW_t(t) + \sigma B(t, T)dt,
\]

is a Brownian motion under the \( T \)-forward measure. An explicit solution for \( x(t) \), starting from time \( s \), under \( Q^T \) is hence given by

\[
x(t) = x(s)e^{-a(t-s)} - M^T(s, t) + \sigma \int_s^t e^{-a(t-u)}dW^T_u(u),
\]

where \( M^T(s, t) \) is a deterministic function satisfying

\[
M^T(s, t) = \frac{\sigma^2}{a^2} \left( 1 - e^{-a(t-s)} \right) - \frac{\sigma^2}{2a^2} \left( e^{-a(T-t)} - e^{-a(T+t-2s)} \right).
\]

Therefore, we obtain from Itô’s isometry that \( r(t) \), conditional on time \( s \) under the \( T \)-forward measure, is normally distributed with mean \( \mu_r(s, t) \) and variance \( \sigma^2_r(s, t) \) given by

\[
\mu_r(s, t) = x(s)e^{-a(t-s)} - M^T(s, t) + \beta(t),
\]

\[
\sigma^2_r(s, t) = \frac{\sigma^2}{2a} \left( 1 - e^{-2a(t-s)} \right).
\]

Using these dynamics, and the fact that the numeraire follows a log-normal distribution in this model, many vanilla options can be priced in closed-form using Black and Scholes (1973) style formulas, see e.g. Hull and White (1993), Pelsser (2000) or Brigo and Mercurio (2006). Also the simulation of the model can be done in exact fashion as the underlying state variables follow a (joint) normal distribution, see e.g. Glasserman (2003). In general, the Hull and White (1993) model has the advantage, and owes its popularity, from the fact that it is very tractable.

2.3 Stochastic Volatility

Since the introduction of the Black and Scholes (1973) model and in particular since the equity crash of the late eighties a battery of complex models has been proposed to relax some msspecifications of the model. Though the Black and Scholes (1973) model has theoretical and practical appealing properties, most of its assumptions, like constant volatility or constant interest rates, do not find justification in the financial markets. One class of models relaxes the constant volatility assumption and incorporates a financial phenomenon known as volatility clustering, i.e. they make volatility stochastic. Examples of models belonging to this class are the stochastic volatility models of Hull and White (1987), Stein and Stein (1991), Heston (1993) and Schöbel and
Zhu (1999). In Heston (1993) it was first shown that the price for call/put options are given in terms of numerical integrals over the characteristic function of the logarithm of the asset price, thus, provided that the characteristic function can be computed in an efficient way, this allows for efficient calibrations to market option data. Note that the option pricing formula presented in the original Heston (1993) paper requires the evaluation of two numerical integrals, whereas more recent methods only use one numerical integral, see Section 2.5.

2.3.1 Heston’s and Schöbel-Zhu’s Stochastic Volatility Model

In the stochastic volatility class, the models of Heston (1993) and Schöbel and Zhu (1999) stand out by allowing for flexibility between the correlation of the asset price and its stochastic volatility, whilst they also provide a closed-form formula for their characteristic function. As these two models play a prominent role in the rest of this thesis, we provide their model dynamics here and discuss their relationships.

The Heston (1993) stochastic volatility model assumes the following dynamics

\[
\begin{align*}
    dS(t) &= S(t)\left[\mu(t)dt + \sqrt{v(t)}dW_S(t)\right], \\
    dv(t) &= \kappa(\theta - v(t))dt + \xi \sqrt{v(t)}dW_V(t),
\end{align*}
\]

(2.22)

(2.23)

In this stochastic differential equation, \(S(t)\) represents the asset price, with a stochastic variance \(v(t)\) that follows a mean reverting, square-root/Feller/CIR process. The Brownian motions \(W_S(t)\) and \(W_V(t)\) are correlated with correlation coefficient \(\rho\). In the Schöbel and Zhu (1999) stochastic volatility model, not the variance, but the volatility is modelled via a mean reverting, Ornstein-Uhlenbeck process:

\[
\begin{align*}
    dS(t) &= S(t)\left[\mu(t)dt + \nu(t)dW_S(t)\right], \\
    d\nu(t) &= \kappa(\psi - \nu(t))dt + \tau dW_\nu(t), \\
    \nu(0) &= \nu_0,
\end{align*}
\]

(2.24)

(2.25)

and the Brownian motions are again correlated with coefficient \(\rho\). We postpone the interpretation of the above model parameters to later chapters.

At first sight, one curious property of the Schöbel and Zhu (1999) model is that the volatility process \(\nu(t)\) affects the sign of the instantaneous correlation between \(\nu(t)\) and \(\ln S(t)\). Indeed, one can show that

\[
\text{Corr}\left(d\ln S(t), d\nu(t)\right) = \frac{\rho_{S,\nu} \nu(t) \tau}{\sqrt{\nu^2(t) \tau^2}} = \rho_{S,\nu} \text{sgn}(\nu(t)),
\]

(2.26)

which will effectively cause the correlation between \(\nu(t)\) and the stock price \(S(t)\) to change sign. This effect is visualized in Figure 1, where we have plotted a sample path of \(\ln S(t)\), \(\nu(t)\) and \(|\nu(t)|\).
2.3. Stochastic Volatility

Figure 1: Sample path of $\ln S(t)$, $\nu(t)$ and $|\nu(t)|$. SZ parameters are $\kappa = \tau = 1$, $\nu(0) = \psi = 25\%$, $\rho = 1$, $x(0)=100$.

Indeed, when $\nu(t)$ is negative and decreasing, the asset price is increasing, contrary to what one would expect from the parameter configuration. The key lies therein that $\nu(t)$ should not be interpreted as the volatility of the underlying asset. It is merely a latent variable which drives the true volatility of the asset, the true volatility being defined as the square root of the instantaneous variance of the logarithm of the stock price. Using the Itô-Tanaka theorem (see Revuz and Yor (1999)), one can indeed show that the correlation between $\ln S(t)$, and $|\nu(t)|$ is equal to $\rho_{S\nu}$, as we would like it to be.

If we take the pair $(S(t), \nu(t))$ as state variables, the Heston (1993) and Schöbel and Zhu (1999) model are not affine as the instantaneous variance of the stock price is equal to $\nu(t)S^2(t)$. Nonetheless, if we consider $(\ln S(t), \nu^2(t))$ as state variables for the Heston (1993) model, and $(\ln S(t), \nu(t), \nu^2(t))$ for the Schöbel and Zhu (1999) model, we do obtain affine processes. The characteristic functions of these models will be provided when we start using these models in later chapters. For instance the characteristic function of the Schöbel and Zhu (1999) model is derived in Chapter 3.

Relationship between the Heston and Schöbel-Zhu model

It was already noted in Heston (1993), that an Ornstein-Uhlenbeck process for the volatility is closely related to a square-root process for the variance process. If the volatility $\nu(t)$ is modelled by an Ornstein-Uhlenbeck process, as in the Schöbel and Zhu (1999), then Itô’s lemma shows that the variance process $\nu^2(t)$ follow the dynamics

$$
d\nu^2(t) = 2\kappa \left( \frac{\tau^2}{2\kappa} + \psi \nu - \nu^2(t) \right) dt + 2\tau \nu(t) dW(t).$$

(2.27)
From the variance dynamics (2.23), the relationship between the Heston and Schöbel-Zhu model can directly be observed: in case the long-term mean $\psi$ of the volatility process in (2.25) is equal to zero, Schöbel-Zhu model is equivalent to the Heston model in which $\lambda = 2\kappa, \xi = 2\tau$ and $\theta = \frac{\xi^2}{2\kappa}$. The overlap of the models is restricted to this very special case. However one can use this relationship to establish a relation between the characteristic functions of the Heston and the Schöbel-Zhu(-Hull-White) model, see Lord and Kahl (2007): it turns out that the characteristic function of SZHW model can be written as the Heston characteristic multiplied by a simple extra factor.

2.4 Volatility, Correlation and Stochastic Interest Rates

It is hardly necessary to motivate the inclusion of stochastic volatility in a derivative pricing model for long-term options. Also the addition of interest rates as a stochastic factor is important when considering long-maturity derivatives and has been the subject of empirical investigations, most notably by Bakshi et al. (2000). Approaches that covered both stochastic volatility and stochastic interest rates in derivative pricing were described in Scott (1997), Bakshi et al. (1997), Amin and Ng (1993), Andreasen (2006), van der Ploeg (2007), Ahlip (2008), Antonov et al. (2008) and Grzelak and Oosterlee (2009). None of these models, however, provide both an exact closed-form call pricing formula, and also explicitly incorporate the correlation between underlying asset and the term structure of interest rates.

Scott (1997), Bakshi et al. (1997), Amin and Ng (1993) consider independent interest rates or do not derive closed-form formulas for vanilla options. Andreasen (2006) does derive closed-form Fourier expressions for vanilla options, similar to Section 2.5, but uses an indirect approach in the form of a volatility displacement parameter to correlate the independent Hull and White (1993) interest rate drivers with the underlying FX rate and which can lead to unrealistic model parameters, see e.g. Antonov et al. (2008). In van der Ploeg (2007) exact closed-form solutions are derived for various models, but in all cases only under simplifying assumptions on the correlation structure between the stochastic volatility and the corresponding stochastic interest rates model. Ahlip (2008) considers an extension of the Schöbel-Zhu model with Gaussian interest rates for the pricing of exchange rate options, but assumes a perfect correlation between all stochastic processes, considering essentially only a one factor model. Antonov et al. (2008) does consider a full correlation structure between the stochastic interest rates, the underlying FX rate and uses a Heston (1993) stochastic volatility under a full correlation structure. To price vanilla options, however, approximations are being made, which deteriorate for larger maturities or more extreme model parameters. Grzelak and Oosterlee (2009) consider the same modelling framework as Antonov et al. (2008), but follow a different approach, extending the work of Giese (2004), by using approximations for the covariance term between the volatility and the interest rates. To obtain the characteristic function of logarithm of the asset price, using the approximations, one then repeatedly has to solve for a system of ODEs using numerical methods. Analogously, this method is further generalized in Grzelak et al. (2009) in a framework where the single-factor short interest rates are replaced by their corresponding multi-factor
2.4. Volatility, Correlation and Stochastic Interest Rates

In the Black-Scholes-Hull-White (BSHW) model, in which the underlying asset has a deterministic volatility, one can derive closed-form formulas for vanilla options and allow for stochastic interest rates, which are correlated with the underlying asset. In the literature this model is used for the pricing of long-term options. For example in the context of Guaranteed Annuity Options, see also Chapter 8, the BSHW model is being used in Ballotta and Haberman (2003), Boyle and Hardy (2003) and Chu and Kwok (2007). By many authors, e.g. see Ballotta and Haberman (2003) and Piterbarg (2005) it is however noted that, given the long maturities typical in such contracts, a stochastic volatility model for the underlying asset would be more suitable. To this end, frameworks that allow for both stochastic volatility, stochastic interest rates, a full correlation structure and an exact closed-form pricing of equity, FX and inflation options are derived in Chapter 3 and 4. As the BSHW model is standard in the literature, and serves as a useful benchmark, we discuss it briefly in the next section.

2.4.1 Black-Scholes-Hull-White model

The BSHW model combines the Black and Scholes (1973) and the Hull and White (1993) model and is applicable for the pricing of equity and hybrid options, see for instance Ballotta and Haberman (2003) and Brigo and Mercurio (2006). A trivial extension of the model is used for the pricing of cross-currency derivatives, e.g. see Sippel and Ohkoshi (2002), Jarrow and Yildirim (2003) and Piterbarg (2005). Under the risk-neutral measure \( \mathcal{Q} \), which uses the bank account as numeraire, the dynamics of stock price \( S(t) \) and short interest rate \( r(t) \) in the BSHW model are given by

\[
\begin{align*}
    dr(t) &= \left[ \theta(t) - ar(t) \right] dt + \sigma dW_t, \\
    dS(t) &= S(t) \left[ r(t) dt + \eta dW_S(t) \right],
\end{align*}
\]

(2.28)

(2.29)

where \( W_S(t) \) and \( W_r(t) \) are two Brownian motions, with correlation \( \rho \). Here the short interest rate \( r(t) = x(t) + \beta(t) \) follows a Hull and White (1993) process, see Section 2.2.2. Using Geman et al. (1996), analogously to Section 2.2, we can switch from the money market measure \( \mathcal{Q} \) to the \( T \)-forward measure \( \mathcal{Q}^T \), which implies the following model dynamics:

\[
\begin{align*}
    dx(t) &= -ax(t)dt - \sigma^2 B(t, T) dt + \sigma dW^T_t, \\
    dS(t) &= S(t) \left[ r(t) dt + \eta \rho \sigma B(t, T) dt + \eta dW^T_S(t) \right].
\end{align*}
\]

(2.30)

(2.31)

So, conditional on time \( s \), using Itô’s lemma and direct integration, this gives us following explicit solutions under \( \mathcal{Q}^T \):

\[
\begin{align*}
    x(t) &= x(s) e^{-at} - \int_s^t e^{-a(t-u)} \sigma^2 B(u, T) du + \int_s^t \sigma e^{-a(t-u)} dW^T_r(t), \\
    S(t) &= S(s) \exp \left[ \int_s^t (r(u) - \rho \eta \sigma B(u, T) - \frac{1}{2} \eta^2) du \right] + \int_s^t \eta dW^T_S(u).
\end{align*}
\]

(2.32)

(2.33)
Chapter 2. Stochastic Models

For every derivatives pricing model it is important that efficient, preferably closed-form, formulas are available for the calibration and pricing of vanilla options. We therefore demonstrate how vanilla call and put options can be priced in closed-form in the BSHW model.

starting from time $t$, we can evaluate the price of a European stock option ($\omega = 1$ for a call option, $\omega = -1$ for a put option) with strike $K$ as

$$ C(S(t), K, T, \omega) = \mathbb{E}^Q \left[ \exp \left[ - \int_t^T r(u)du (\omega(S(T) - K))^+ \right] \mid \mathcal{F}_t \right]. \quad (2.34) $$

Instead of evaluating expected discounted payoff under the risk-neutral bank account measure, we have shown in Section 2.2 how to change the underlying probability measure and evaluate this expectation under the $T$-forward probability measure $Q^T$, which is equivalent to choosing the $T$-discount bond as numeraire.

$$ C(S(t), K, T, \omega) = P(t, T) \mathbb{E}^{Q^T} \left[ (\omega(S(T) - K))^+ \mid \mathcal{F}_t \right]. \quad (2.35) $$

As $S(T)$ follows a log-normal distribution under $Q^T$, we can evaluate the latter expectation by means of the Black and Scholes (1973) formula, for which we need to determine the mean of $S(T)$ and variance of $\ln S(T)$ under the $T$-forward measure $Q^T$.

To this end, first note that from (2.10), (2.30), (2.12), Fubini’s theorem and some algebra, one can obtain that

$$ \int_t^T r(u)du = \int_t^T \beta(u)du + \int_t^T x(u)du $$

$$ = \ln \frac{P(0,t)}{P(0,T)} + \frac{1}{2} \left[ V(0,T) - V(0,t) \right] $$

$$ + B(t,T).x(t) - V(t,T) + \int_t^T \sigma B(u,T)dW^T_r(u), $$

so that under the $T$-forward measure $Q^T$ we have

$$ \frac{S(T)}{S(t)} = e^{\int_t^T \left( r(u) - \rho \sigma r B(u,T) - \frac{1}{2} \eta^2 \right)du + \int_t^T \eta dW^T_r(u)} $$

$$ = \frac{P(0,t)}{P(0,T)} e^{B(t,T)x(t) + \frac{1}{2} \left[ V(0,T) - V(0,t) - V(t,T) \right] - \frac{1}{2} \sigma^2 \int_t^T B(u,T)dW^T_r(u) + \eta \int_t^T B(u,T)dW^T_r(u)} $$

$$ = \frac{1}{P(t,T)} e^{-\frac{1}{2} \sigma^2 \int_t^T B(u,T)dW^T_r(u) + \eta \int_t^T B(u,T)dW^T_r(u)}, \quad (2.36) $$

which implies that $S(T)$ follows a log-normal distribution with mean and “log variance” given
by

\[ F(t, T) := \mathbb{E}^{Q_T}[S(T)\mathcal{F}_t] = \frac{S(t)}{P(t, T)}, \quad (2.37) \]

\[ v_S^2(t, T) := \text{Var}\left[\ln S(T)\mathcal{F}_t\right] = V(t, T) + 2\rho\eta \int_t^T \sigma B(u, T) + \eta^2(T-t). \quad (2.38) \]

Omitting the dependence on \( t, T \) in \( F \) and \( v_S \), we can express the option price (2.35) for a European stock option with strike \( K \) as in terms of the Black and Scholes (1973) formula as

\[ C(S(t), K, T, \omega) = P(t, T) \left[ F \omega N(\omega d_1(K, F, v_S)) - K \omega N(\omega d_2(K, F, v_S)) \right], \quad (2.39) \]

\[ d_1(K, F, v_S) = \frac{\ln(F/K) + v_S^2/2}{v_S}, \quad (2.40) \]

\[ d_2(K, F, v_S) = d_1(K, F, v_S) - v_S, \quad (2.41) \]

where \( \omega = 1 \) for a call and \( \omega = -1 \) for a put option.

Note that the difference \( v_S^2(t, T) - \eta(T-t) \), typically is increasing function of the time to maturity. Consequently, the larger the maturity, the larger the impact of the stochastic interest rates on the option price. Therefore, as expected, long-term options are more sensitive to the behaviour of the stochastic interest rates than shorter dated derivatives.

### 2.5 Characteristic Function and Option Pricing

The use of transform analysis and inversions of characteristic functions for option pricing, was pioneered by Heston (1993). His stochastic volatility option pricing model provided closed-form expressions for the price of vanilla options in terms by inverting the characteristic function of the logarithm of underlying asset. Prior to Heston, Stein and Stein (1991) used the characteristic function to calculate the stock price distribution in their stochastic volatility model. However, whereas the approach of Stein and Stein (1991) heavily relies on the independence between the asset and its stochastic volatility, the approach of Heston (1993) is applicable to any model in which the characteristic function of the logarithm of the asset price is known in closed form.

Under the \( T \)-forward measure \( Q_T \), starting from time \( t \), the price of a European option on the underlying asset with strike \( K \) is given by

\[ C(S(t), K, T) = P(t, T)\mathbb{E}^{Q_T}\left[ (\omega [S(T) - K])^+ \mathcal{F}_t \right], \quad (2.42) \]

with \( P(t, T) \) the price of a zero coupon bond, \( \mathcal{F}_t \) the time-\( t \) filtration and with \( \omega = 1 \) for a call and \( \omega = -1 \) for a put option. The price of the above option can be expressed in closed-form by inverting the characteristic function of the logarithm of the underlying asset. For instance, Heston (1993) expresses the option price in terms of a generalized version of the Black and
Scholes (1973) formula, in which cumulative probabilities are expressed as integrals over the forward characteristic function $\phi(u)$,

$$\phi(u) = \mathbb{E}^Q \left[ e^{iu \ln S(T)} | F_t \right].$$  \hspace{1cm} (2.43)

Though the approach of Heston (1993) is general and holds for every option pricing model in which the forward characteristic function of the logarithm of the asset price is known, it suffers from large numerical complications, see e.g. Lewis (2001), Lord and Kahl (2008).

To allow for more flexibility and greater numerical efficiency, Carr and Madan (1999) found an alternative representation for the European call price (2.42). These authors take the Fourier transform of the dampened call option price $c_T(k) := \exp(\alpha k) C_T(k)$ with respect the logarithm of the strike price $K$ as:

$$\psi(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk,$$ \hspace{1cm} (2.44)

the Fourier transform of which, after using Fubini’s theorem and some algebra, can be explicitly expressed in terms of the forward characteristic function $\phi(u)$, i.e.

$$\psi(v) := \frac{\phi(v - (\alpha + 1) i)}{(\alpha + iv)(\alpha + 1 + iv)}.$$ \hspace{1cm} (2.45)

By inverting the Fourier transform and undampening the call option price the following result is obtained:

$$C(S(t), K, T) = P(t, T) \frac{1}{\pi} \int_0^\infty \text{Re} \left( e^{-(\alpha+iv)k} \psi_T(v) \right) dv + R(F(t), K, \alpha),$$ \hspace{1cm} (2.46)

with forward price $F(t) := \frac{S(t)}{P(t, T)}$ and residue term $R$, see Lewis (2001) and Lord and Kahl (2008), equal to

$$R(F(t), K, \alpha) := F(t) \cdot 1_{\{\alpha \leq 0\}} - K \cdot 1_{\{\alpha \leq -1\}} - \frac{1}{2} \left( F(t) \cdot 1_{\{\alpha = 0\}} - K \cdot 1_{\{\alpha = -1\}} \right).$$ \hspace{1cm} (2.47)

The dampening parameter $\alpha$ is used to ensure that the dampened call price $c_T(k)$ is $L^1$ integrable, which is a sufficient condition for the Fourier transform to exist.

To make the existence more explicit, a sufficient condition for $c_T(k)$ to be integrable is provided by $\psi(0) = \int_{-\infty}^{\infty} c_T(k) dk$ being finite, see Carr and Madan (1999). From (2.45) one can observe that $\psi(0)$ is finite provided that

$$\phi(-(\alpha + 1)i) = \mathbb{E}^Q \left[ S(T)^{\alpha+1} \right] < \infty.$$ \hspace{1cm} (2.48)

For the Fourier transform (2.46) to exist, it therefore suffices that the $(\alpha + 1)$-th moment of the asset price is finite.
2.5. Characteristic Function and Option Pricing

The option pricing result (2.46) is analyzed in great detail by Lee (2004), Lewis (2001) and Lord and Kahl (2008). These papers conclude that an optimal choice of $\alpha$, minimizing sampling and truncation error, depends both on the chosen model and its parameters, as well as the strike of the underlying option.