Pricing long-term options with stochastic volatility and stochastic interest rates
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CHAPTER 3

The Schöbel-Zhu-Hull-White Model

*This chapter is based on:


3.1 Introduction

The derivative markets are maturing more and more. Not only are increasingly exotic structures created, the markets for plain vanilla derivatives are also growing. One of the recent advances in equity derivatives and exchange rate derivatives is the development of a market for long-maturity European options\(^1\). In this chapter we develop a stochastic volatility model aimed at pricing and risk managing long-maturity insurance contracts involving equity, interest rate and exchange rate risks.

We extend the models by Stein and Stein (1991) and Schöbel and Zhu (1999) to allow for Hull and White (1993) stochastic interest rates as well as correlation between the stock price process, its stochastic volatility and interest rates. The resulting model is dubbed as the Schöbel-Zhu Hull-White (SZHW) model. Our model enables to take into account two important factors in the pricing of long-maturity equity or exchange rate derivatives: stochastic volatility and stochastic interest rates, whilst also taking into account the correlation between those processes explicitly. It is hardly necessary to motivate the inclusion of stochastic volatility in a derivative pricing model. The addition of interest rates as a stochastic factor is important when considering long-maturity equity derivatives and has been the subject of empirical investigations most notably by Bakshi et al. (2000). These authors show that the hedging performance of delta hedging strategies of long-maturity options improves when taking stochastic interest rates into account, whilst the interest rate risk is less important for short maturity options. This result is also intuitively appealing since the interest rate risk of equity derivatives, the option’s rho,

\(^1\)The implied volatility service of MarkIT, a financial data provider, shows regular quotes on a large number of major equity indices for option maturities up to 10-15 years.
is increasing with time to maturity. The SZHW model can be used in the pricing and risk management for a range of insurance and exotic derivatives contracts. One can for example think of pension products, variable and guaranteed annuities (e.g. see Ballotta and Haberman (2003)), long-maturity PRDC FX contracts (e.g. see Piterbarg (2005)), rate of return guarantees in Unit-Linked contracts (e.g. see Schrager and Pelsser (2004)) and many other structures which have a long-term nature.

The model of this chapter can be placed in the derivative pricing literature on stochastic volatility models as it adds to or extends work by Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999) or, since our model can be placed in the affine class, in the more general context of Duffie et al. (2000), Duffie et al. (2003) and van der Ploeg (2006). The SZHW model benefits greatly from the analytical tractability typical for this class of models. Our work can also be viewed as an extension of the work by Amin and Jarrow (1992) to stochastic volatility. In a related work Ahlip (2008) considers an extension of the Schöbel-Zhu model to Gaussian stochastic interest rates for pricing of exchange rate options. Upon a closer look however the correlation structure considered in that work is limited to perfect correlation between the stochastic processes. The affine stochastic volatility models fall in the broader literature on stochastic volatility which covers both volatility modelling for the purpose of derivative pricing as well as real world volatility modelling. Previous work that covered both stochastic volatility and stochastic interest rates in derivative pricing include: Scott (1997), Bakshi et al. (1997), Amin and Ng (1993), Andreasen (2006), van der Ploeg (2007), Antonov et al. (2008), Grzelak and Oosterlee (2009) and Grzelak et al. (2009). The SZHW model distinguishes itself from these models by a closed form call pricing formula and/or explicit, rather than implicit, incorporation of the correlation between underlying and the term structure of interest rates.

Our contribution to the existing literature is fourfold:

- First, we derive the conditional characteristic function of the SZHW model in closed form and analyse pricing vanilla equity calls and puts using transform inversion.
- Second, since the practical relevance of any model is limited without a numerical implementation, we extensively consider the efficient implementation of the Fourier transform inversion (see Lord and Kahl (2008)) required to price European options. In particular we derive a theoretical result on the limiting behaviour of the conditional characteristic function of the SZHW model which allows us to calculate the inversion integral much more accurately.
- Third, we consider the pricing of forward starting options.
- Fourth, we generalize the SZHW model to be able to value FX options in a framework where both domestic and foreign interest rate processes are stochastic.

The outline for the remainder of the chapter is as follows. First, we introduce the model and focus on the analytical properties. Second, we consider the effect of stochastic interest rates and
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correlation on the implied volatility term structure. Third, we consider the numerical implementation of the transform inversion integral. Fourth, we consider the pricing of forward starting options. Fifth, we present the extension of the model for FX options involving two interest rate processes. Finally we conclude.

3.2 The Schöbel-Zhu-Hull-White model

The model we will derive here is a combination of the famous Hull and White (1993) model for the stochastic interest rates and the Schöbel and Zhu (1999) model for stochastic volatility. The model has three key variables, which we allow to be correlated with each other: the stock price \( x(t) \), the Hull-White interest rate process \( r(t) \) and the stochastic stock volatility which follows an Ornstein-Uhlenbeck process cf. Schöbel and Zhu (1999). The risk-neutral asset price dynamics of the Schöbel-Zhu-Hull-White (SZHW) read:

\[
\begin{align*}
    \text{d}S(t) &= S(t)r(t)\text{d}t + S(t)\nu(t)\text{d}W_S(t), & S(0) = S_0, \\
    \text{d}r(t) &= (\theta(t) - ar(t))\text{d}t + \sigma\text{d}W_r(t), & r(0) = r_0, \\
    \text{d}\nu(t) &= \kappa(\psi - \nu(t))\text{d}t + \tau\text{d}W_\nu(t), & \nu(0) = \nu_0,
\end{align*}
\]

where \( a, \sigma, \kappa, \psi, \tau \) are positive parameters which can be inferred from market data and correspond to the mean reversion and volatility of the short rate process, and the mean reversion, long-term volatility and volatility of the volatility process respectively. The quantity \( r_0 \) and the deterministic function \( \theta(t) \) are used to match the currently observed term structure of interest rates, e.g. see Hull and White (1993). The hidden parameter \( \nu_0 > 0 \), corresponds to the current instantaneous volatility and hence should be determined directly from market (e.g. just as the non-observable short interest rate), but is in practice often (mis-)used as extra parameter for calibration. Finally, \( \tilde{W}(t) = (W_S(t), W_r(t), W_\nu(t)) \) denotes a Brownian motion under the risk-neutral measure \( Q \) with covariance matrix:

\[
\text{Cov}(\tilde{W}(t)) = \begin{pmatrix} 1 & \rho_{Sr} & \rho_{S\nu} \\ \rho_{Sr} & 1 & \rho_{r\nu} \\ \rho_{S\nu} & \rho_{r\nu} & 1 \end{pmatrix} t 
\]

Note that as \( \nu(t) \) follows an Ornstein-Uhlenbeck process, there is a possibility that \( \nu(t) \) becomes negative; effectively this implies that the sign of instantaneous correlation between \( \ln S(t) \) and \( \nu(t) \) changes as \( \nu(t) \) goes through zero. However, the actual volatility is \( |\nu(t)| \), see Section 2.3.1, and it does not have this feature.

3.3 European option pricing

General payoffs which are a function of the stock price at maturity \( T \) can be priced using the corresponding characteristic function of the log-asset price. Therefore we evaluate the probability distribution of the \( T \)-forward stock price at time \( T \). Instead of evaluating expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying proba-
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bility measure to evaluate this expectation under the $T$-forward probability measure $Q^T$, which is equivalent to choosing the $T$-discount bond as numeraire, see Chapter 2. Hence starting from time $t$, we can evaluate the price of a European stock option ($w = 1$ for a call option, $w = -1$ for a put option) with strike $K = \exp(k)$ as

$$\mathbb{E}^{Q^T}\left[\exp\left(-\int_t^T r(u)du\right)(w(S(T) - K))^+\right|\mathcal{F}_t] = P(t, T)\mathbb{E}^{Q^T}\left[(w(F^T(T) - K))^+\right|\mathcal{F}_t], \tag{3.5}$$

where $P(t, T)$ denotes the price of a (pure) discount bond and $F^T(t) := \frac{S(t)}{p(t, T)}$ denotes the $T$-forward stock price. The above expression can be numerically evaluated by means of a Fourier inversion of the log-asset price characteristic function, see equation (2.46) of Chapter 2. This characteristic function is derived in the following subsection. Section 3.5 is concerned with the numerical implementation of the Fourier transform and presents an alternative pricing equation which transforms the integration domain to the unit interval, see e.g. Lord and Kahl (2008), and hence avoids truncation errors.

**The $T$-forward dynamics**

Recall from Chapter 2, that in the Hull-White model, one write the following for the price of a discount bond:

$$P(t, T) = \exp[A_r(t, T) - B_r(t, T)r(t)], \tag{3.6}$$

where $A_r(t, T)$ contains information on the currently observed term structure of interest rates and $B_r(t, T) := (1 - e^{-\alpha(T-t)})/\alpha$. Note that this expression is equivalent to equation (2.14) for the zero-coupon bond price, and differs only in terms of notation. The forward stock price can therefore be expressed as

$$F^T(t) = \frac{S(t)}{\exp[A_r(t, T) - B_r(t, T)r(t)]}. \tag{3.7}$$

Under the risk-neutral measure $Q$ (where we use the money market bank account as numeraire) the discount bond price follows the process $dP(t, T) = r(t)P(t, T)dt - \sigma B_r(t, T)P(t, T)dW_r(t)$. Hence, by an application of Itô’s lemma, we find the following $T$-forward stock price process:

$$dF^T(t) = \left(\sigma^2 B^2_r(t, T) + \rho_S \nu(t) \sigma B_r(t, T)\right)F^T(t)dt + \nu(t)F^T(t)dW_S(t) + \sigma B_r(t, T)F^T(t)dW_r(t) \tag{3.8}$$

By definition the forward stock price will be a martingale under the $T$-forward measure. This is achieved by defining the following transformations of the Brownian motions:

$$dW_r(t) \rightarrow dW^T_r(t) - \sigma B_r(t, T)dt, \quad dW_S(t) \rightarrow dW^T_S(t) - \rho_S \sigma B_r(t, T)dt, \quad dW_r(t) \rightarrow dW^T_v(t) - \rho_r \sigma B_r(t, T)dt. \tag{3.9}$$
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Hence under the $T$-forward measure the processes for $F^T(t)$ and $\nu(t)$ are given by

\[
\begin{align*}
    dF^T(t) &= \nu(t)F^T(t)dW_S^T(t) + \sigma B_r(t, T)F^T(t)dW_T^T(t), \\
    d\nu(t) &= \kappa(\psi - \frac{\rho_T\sigma_T}{\kappa} B_r(t, T)) - \nu(t)dt + \tau dW_T^\nu(t),
\end{align*}
\]

(3.10) (3.11)

where $W_S^T(t), W_T^T(t), W_T^\nu(t)$ are now Brownian motions under the $T$-forward $Q^T$. We can simplify (3.10) by switching to logarithmic coordinates: defining $\gamma(t) := \log(F^T(t))$ and an application of Itô’s lemma yields

\[
\begin{align*}
    dy(t) &= -\frac{1}{2}v^2_r(t)dt + \nu(t)W_S^T(t) + \sigma B_r(t, T)W_T^T(t), \\
    d\nu(t) &= \kappa(\xi(t) - \nu(t))dt + \tau dW_T^\nu(t),
\end{align*}
\]

(3.12) (3.13)

with

\[
\begin{align*}
    v^2_r(t) &:= \nu^2(t) + 2\rho_S,\nu(t)\sigma B_r(t, T) + \sigma^2 B_r^2(t, T), \\
    \xi(t) &:= \left(\psi - \frac{\rho_T\sigma_T}{\kappa} B_r(t, T)\right).
\end{align*}
\]

(3.14) (3.15)

Notice that we now have reduced the system (3.1) of the three variables $x(t), r(t)$ and $\nu(t)$ under the risk-neutral measure, to the system (3.12) of two variables $y(t)$ and $\nu(t)$ under the $T$-forward measure. What remains is to find the characteristic function of the reduced system of variables.

**Determining the characteristic function of the forward log-asset price**

We will now determine the characteristic function of the reduced system (3.12), which we will do by means of a PDE approach. That is, we apply the Feynman-Kac theorem and reduce the problem of finding the characteristic of the forward log-asset price dynamics to solving a partial differential equation; that is, the Feynman-Kac theorem implies that the characteristic function

\[
    f(t, y, \nu) = \mathbb{E}^{Q^T}[\exp(\text{i}uy(T)) | F_t],
\]

(3.16)

is given by the solution of the following partial differential equation

\[
\begin{align*}
    0 &= f_t - \frac{1}{2}v^2_r(t)f_y + \kappa(\xi(t) - \nu(t))f_\nu + \frac{1}{2}v^2_r(t)f_{yy} \\
    &\quad + (\rho_S,\nu(t) + \rho_\nu,\tau \sigma B_r(t, T))f_\nu + \frac{1}{2}\tau^2 f_{\nu\nu}.
\end{align*}
\]

(3.17)

\[
    f(T, y, \nu) = \exp\left(\text{i}uy(T)\right),
\]

(3.18)

where the subscripts denote partial derivatives and we took into account that the covariance term $dy(t)d\nu(t)$ is equal to

\[
    dy(t)d\nu(t) = (\nu(t)dW_S^T(t) + \sigma B_r(t, T)dW_T^T(t))(\tau dW_T^\nu(t)) = (\rho_S,\nu(t) + \rho_\nu,\tau \sigma B_r(t, T))dt,
\]

(3.19)
and to ease the notation we dropped the explicit \((t, y, \nu)\)-dependence for \(f\).

Solving the defining partial differential equation (3.17) subject to the boundary condition (3.18), leads to the following proposition.

**Proposition 3.3.1** The characteristic function of \(T\)-forward log-asset price of the SZHW model is given by the following closed-form solution:

\[
f(t, y, \nu) = \exp\left[A(u, t, T) + B(u, t, T)y(t) + C(u, t, T)\nu(t) + \frac{1}{2}D(u, t, T)\nu^2(t)\right],
\]

where:

\[
A(u, t, T) = \frac{1}{2}u(i + u)V(t, T)
\]

\[
B(u, t, T) = iu,
\]

\[
C(u, t, T) = -u(i + u)\frac{\gamma_3 \gamma_4 e^{-\gamma(T-t)} - (\gamma_5 e^{-\alpha(T-t)} - \gamma_6 e^{-2\alpha(T-t)}) - \gamma_7 e^{-\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}},
\]

\[
D(u, t, T) = -u(i + u)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}},
\]

with:

\[
\gamma = \sqrt{(\kappa - \rho_S \tau i u)^2 + \tau^2 u(i + u)}, \quad \gamma_1 = \gamma + (\kappa - \rho_S \tau i u),
\]

\[
\gamma_2 = \gamma - (\kappa - \rho_S \tau i u), \quad \gamma_3 = \frac{\rho_S \sigma \gamma \gamma_1 + \kappa \psi + \rho_S \sigma \tau (i u - 1)}{2},
\]

\[
\gamma_4 = \frac{\rho_S \sigma \gamma \gamma_2 - \kappa \alpha - \rho_S \sigma \tau (i u - 1)}{2}, \quad \gamma_5 = \frac{\rho_S \sigma \gamma \gamma_1 + \rho_S \alpha \tau (i u - 1)}{2},
\]

\[
\gamma_6 = \frac{\rho_S \sigma \gamma \gamma_2 - \rho_S \sigma \tau (i u - 1)}{a(\gamma + a)}, \quad \gamma_7 = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6),
\]

and:

\[
V(t, T) = \frac{\sigma^2}{a^2} ((T - t) + \frac{2}{a} e^{-\alpha(T-t)} - \frac{1}{2a} e^{-2\alpha(T-t)} - \frac{3}{2a}).
\]

**Proof** The model we are considering is not an affine model in \(y(t)\) and \(\nu(t)\), but it is if we enlarge
3.4. Impact of stochastic interest rates and correlation

the state space to include \( \nu^2(t) \):

\[
\begin{align*}
    dy(t) &= -\frac{1}{2} \nu^2_F(t) dt + \nu_F(t) dW^T_F(t) \\
    d\nu(t) &= \kappa (\xi(t) - \nu(t)) dt + \tau dW^T_\nu(t) \\
    d\nu^2(t) &= 2\nu(t) d\nu(t) + \tau^2 dt = 2\kappa \left( \frac{\tau^2}{2\kappa} + \xi(t)\nu(t) - \nu^2(t) \right) dt + 2\tau \nu(t) dW^T_\nu(t)
\end{align*}
\]

(3.27)

(3.28)

(3.29)

We can find the characteristic function of the \( T \)-forward log price by solving the partial differential equation (3.17) for joint characteristic function \( f(t, y, \nu) \) with corresponding boundary condition (3.18); substituting the partial derivatives of the functional form (3.20) into (3.17) provides us with four ordinary differential equations containing the functions \( A(t), B(t), C(t) \) and \( D(t) \). Solving this system yields the above solution, see appendix 3.9. □

We note that the strip of regularity of the SZHW characteristic function, i.e. for which values the Fourier transform of the characteristic function exists, is the same as that of the Schöbel and Zhu (1999) model, for which we refer the reader to Lord and Kahl (2008).

3.4 Impact of stochastic interest rates and correlation

To gain some insights into the impact of the correlated stochastic rates and corresponding parameter sensitivities we will look at the at-the-money implied volatility structure which we compute for different parameter settings. Besides comparing different parameter settings of the SZHW model, we also make a comparison with the classical Schöbel and Zhu (1999) model to determine the impact of stochastic rates in general. The behaviour of the ‘non-interest rate’ parameters are similar to other stochastic volatility models like Heston (1993) and Schöbel and Zhu (1999), that is the volatility of the volatility lift the wings of the volatility smile, the correlation between the stock process and the volatility process can incorporate a skew, and the short and long-term vol determine the level of the implied volatility structure. The impact of stochastic rates and the corresponding correlation are plotted in Figure 1.
Figure 1: Impact of $\rho_{Sr}$ on at-the-money implied volatilities. The graph corresponds to the (degenerate) Black-Scholes-Hull-White case with parameter values $r(t) = 0.05$, $a = 0.05$, $\sigma = 0.01$, $\nu(0) = \psi = 0.20$, $\rho_{Sv} = 0.0$ and constant volatility process.

From Figure 1, one can see that the stochastic interest rates add extra flexibility to the modelling framework; by changing the rate-asset correlation one can create an upward (or an initially downward) sloping term structure of volatility, even in case the volatility process is constant. If we compare the case with zero correlation between the equity and interest rate drivers with the ordinary process with deterministic rates, we see that the stochastic rates make the term structure upward sloping. Note that this is in correspondence with empirical data, which shows higher at-the-money volatilities the longer the maturities go. The effect becomes more apparent for maturities larger than five years; while for one years the effect of uncorrelated stochastic rates is below a basis point, the effect on a five year option is already more than ten basis points which increases to a couple of hundred basis points for a thirty year option. These model effects also correspond with a general feature of the interest rate market: the market’s view on the uncertainty of long-maturity bonds is often much higher than that of shorter bond, hence reflecting the increasing impact of stochastic interest rates for long-maturity equity options. Finally, we note that for higher positive values of linear correlation coefficient between equity and the interest rate component, the impact of stochastic rates becomes more apparent.
3.4. Impact of stochastic interest rates and correlation

From Figure 2, one can see that the effect of the correlation coefficient between the drivers of the rate and volatility process is similar, however the impact on the implied volatility structure is less severe and different in sign: a positive correlation coefficient causes a dampening effect, whereas a negative correlation increases the overall volatility, which effect can also be seen from the volatility dynamics (3.11). Note hereby that the increasing term structure for $\rho_{rv} = -$, in the Figure 2 is mainly caused by the Schöbel-Zhu stochastic volatility process in comparison to the deterministic volatility process used in Figure 1. In comparison to the Schöbel and Zhu (1999) model, we can see that the stochastic interest rates increase the slope of the term structure. More importantly, the implied volatilities do not die out, but remain upward sloping, which behaviour often corresponds with implied volatility quotes in long-maturity equity (e.g. see MarkIT) or FX (e.g. see Andreasen (2006)) options. However for strong positive correlation values this might be the other way around. In contrast to the first picture, we see somewhat smaller effects: for example the increasing effect of stochastic rates is even larger than that of the dampening effect of a positive correlation of 30% between the rate and volatility drivers. Again we see that the effects of stochastic rates become more apparent for longer maturities.

In general, we can see from Figure 1 and 2 that stochastic rates have a significant impact on the backbone of the implied volatility structure and add extra flexibility to the modelling framework. The effects become more apparent for larger maturities and for larger absolute values of the correlation coefficients. Hereby the effect of correlation coefficient between equity and interest rates seems to be the most determinant factor. One can then use these degrees of freedom in
several ways: either one jointly calibrates these parameters to implied volatility surfaces (or some other options), or one can first calibrate these and then use the other parameters to calibrate the remainder of the model. In our opinion this choice has to depend on the exotic product: if the correlations are of larger impact on a exotic product (e.g. on a hybrid equity-interest rate product) than on short-dated vanilla calls, it might then be preferable to use a historical estimate for the correlation coefficient at the cost of a slightly worse calibration result. One way or the other, the SZHW stands out by the additional freedom it offers by explicitly modelling the correlation coefficient between the underlying, the stochastic volatility and the stochastic interest rates.

3.5 Calculating the inverse Fourier transform

In Lord and Kahl (2008) the practical calculation of the inverse Fourier transform (2.46), in general and for specific models, is discussed in great detail. They recommend that

- Any truncation error is avoided by appropriately transforming the range of integration to a finite interval.
- An adaptive integration algorithm is used, hereby allowing the discretization error to be of a prescribed maximum size.
- The damping parameter $\alpha$ is chosen such that the integrand is minimized in $v = 0$, which typically leads to much more accurate prices for options which have long maturities and/or are away from the at-the-money level.

To apply these results to the SZHW model, recall from (2.46) of Chapter 2, that for general $\alpha$, the following pricing equation holds

$$C_T(k) = P(t, T) \frac{1}{\pi} \int_0^\infty \text{Re} \left( e^{-\left(\alpha + iv\right) k} \psi_T(v) \right) dv + R \left( F^T(t), K, \alpha \right), \quad (3.30)$$

with $P(t, T)$ the current time discount factor for maturing at time $T$, $F^T(t)$ the $T$-forward asset price and $k := \ln K$ the logarithm of the strike. By changing variables from $v$ to $g(v)$, which maps $[0, \infty) \mapsto [0, 1]$, the pricing equation (3.30) becomes

$$C_T(k) = P(t, T) \frac{1}{\pi} \int_0^1 \text{Re} \left( e^{-\left(\alpha + ig(v)\right) k} \psi_T(g(v)) \cdot g'(v) \right) dv + R \left( F^T(t), K, \alpha \right). \quad (3.31)$$

To avoid numerical complications, one carefully has to choose the transformation function $g$ such that the integrand remains finite over the range of integration, as it is in (3.30). To find such a transformation, we analyse the limiting behaviour of the characteristic function. In particular, suppose that the characteristic function $\phi_T(u)$ of the SZHW model for large values of $u$ behaves as

$$\exp \left( \phi_r(u) + i\phi_i(u) \right), \quad (3.32)$$
3.5. Calculating the inverse Fourier transform

with both \( \phi_r(u) \) and \( \phi(t) \) functions on the real line. The integrand in (3.30) will then have the following asymptotics

\[
\text{Re}\left(e^{-i(u-ia)k} \frac{\phi_1(u-(\alpha+1)i)}{(\alpha+iu)(\alpha+i+iu)}\right) \propto \frac{e^{-\alpha k + \phi(u-(\alpha+1)i)}}{u^2} \cdot \cos(ku - \psi (u-(\alpha+1)i)).
\]  

(3.33)

In the remainder we will determine \( \psi_r \), which will tell us which transformation function is suitable to use. Lord and Kahl (2008) already supply a number of intermediary results for the Schöbel and Zhu (1999) model, but as the notation we use here is slightly different, we will briefly restate these results. For large values of \( u \), only \( \gamma, \gamma_1 \) and \( \gamma_2 \) in (3.25) are \( O(u) \), whereas \( \gamma_3 \) to \( \gamma_6 \) tend to a constant, and \( \gamma_7 \) is actually \( O\left(\frac{1}{u}\right) \). The limits we require here are

\[
\lim_{u \to \infty} \frac{\gamma(u)}{u} = \gamma(\infty),
\]  

(3.34)

\[
\lim_{u \to \infty} \frac{\gamma_1(u)}{u} = \gamma(\infty) - i\rho_S \tau =: \gamma_1(\infty),
\]  

(3.35)

\[
\lim_{u \to \infty} \frac{\gamma_3(u)}{u} = \frac{\rho_S \gamma(\infty) + i\tau(\rho_r - \rho_S \rho_S^*)}{\alpha(u)} =: \gamma_3(\infty),
\]  

(3.36)

\[
\lim_{u \to \infty} \frac{\gamma_5(u)}{u} = \frac{\rho_S \gamma(\infty) + i\tau(\rho_r - \rho_S \rho_S^*)}{\alpha(u)} =: \gamma_5(\infty).
\]  

(3.37)

We find that the limiting behaviour for \( C(u, t, T) \) in (3.23) follows from

\[
\lim_{u \to \infty} \frac{C(u, t, T)}{u} = \frac{\gamma_3(\infty) - \gamma_5(\infty)e^{-\alpha(T-t)}}{\gamma_1(\infty)} = \frac{-i\rho_r + \rho_S \sqrt{1 - \rho_x^2}}{\sqrt{1 - \rho_x^2} + i\rho_S \sqrt{1 - \rho_x^2}} \frac{\rho_S \gamma(\infty) + i\tau(\rho_r - \rho_S \rho_S^*)}{\alpha(u)} B_r(t, T)
\]

\[
\equiv C(\infty) \frac{\sigma}{\tau} \tau B_r(t, T).
\]  

(3.38)

From the above result, the limiting behaviour of \( D(u, t, T) \) in (3.24) for large values of \( u \) follows as

\[
\lim_{u \to \infty} \frac{D(u, t, T)}{u} = -\frac{1}{\gamma_1(\infty)}.
\]  

(3.39)

Finally, we need to analyse \( A(t) = A(u, t, T) \) in (3.21). Its defining ODE (3.94) can be found in appendix 3.9, i.e.

\[
\frac{\partial A(u, t, T)}{\partial t} = \frac{-[\kappa \xi(t) + i\rho_r \tau \gamma B_r(t, T)]C(u, t, T) + 1}{2} u(i + u) \sigma^2 B_r^2(t, T)
\]

\[
-2 \tau^2 \left(C^2(u, t, T) + D(u, t, T)\right).
\]  

(3.40)

The first derivative of \( A(u, t, T) \) behaves as \( O(u^2) \) for large values of \( u \), as can be seen from

\[
\lim_{u \to \infty} \frac{1}{u^2} \frac{\partial A(u, t, T)}{\partial t} = \frac{1}{2} \left(1 - C^2(\infty) - 2i\rho_r C(\infty) \right) \sigma^2 B_r^2(t, T)
\]  

(3.41)
Finally, together with the boundary condition \( A(u, T, T) = 0 \), we have
\[
\lim_{u \to \infty} \frac{A(u, t, T)}{u^2} = -\int_t^T \lim_{u \to \infty} \frac{1}{u^2} \frac{\partial A(u, s, T)}{\partial s} \, ds = -\frac{1}{2} V(t, T) \cdot \left( 1 - C^2(\infty) - 2i\rho_r C(\infty) \right) \equiv -A(\infty),
\]
where \( V(t, T) \) denotes the integrated bond variance, i.e. as defined in (3.26). One can show that
\[
\text{Re} \left( A(\infty) \right) \geq 0 \quad \text{as} \quad V(t, T) \geq 0 \quad \text{and:}
\]
\[
\text{Re} \left( C^2(\infty) + 2i\rho_r C(\infty) \right) = \frac{\rho_{sx}^2 - 2\rho_{sr}\rho_S\rho_{sv} + \rho_{sv}^2(4\rho_{sv}^2 - 3)}{1 - \rho_{sv}^2} \leq 1.
\]
This follows by maximizing the right-hand side with respect to the constraint that the three correlations constitute a positive semi-definite correlation matrix. For example, the maximum is achieved when \( \rho_S = -\frac{1}{2} \sqrt{3} \), \( \rho_{sv} = -\frac{1}{2} \) and \( \rho_{rv} = 0 \). The above analysis determines \( \phi_r \) as
\[
\phi_r(u - (\alpha + 1)i) = -\text{Re}(A(\infty)) \cdot u^2.
\]
One can conclude that the tail behaviour of the characteristic function of the SZHW model is quite different from that of the Schöbel and Zhu (1999) model; whereas the decay in the Schöbel-Zhu model is only exponential, the decay here resembles that of a Gaussian characteristic function, caused by the addition of a Gaussian short rate process. Clearly, if \( \sigma \) (the volatility of the short rate) is zero, \( A(\infty) = 0 \) and the decay of the characteristic function becomes exponential once again. As the tail behaviour of the characteristic function is of the same form as that of the Black and Scholes (1973) characteristic function, an appropriate transformation function is \( g : [0, \infty) \mapsto [0, 1] \), as in Lord and Kahl (2008),
\[
g(u) = -\frac{\ln u}{\sqrt{A(\infty)}},
\]
which can be used in the pricing equation (3.31).

### 3.6 Forward starting options

Due to the popularity of forward starting options such as cliquets, the pricing of forward starting options recently attracted the attention of both practitioners and academics (e.g. see Lucic (2003), Hong (2004), Kruse and Nögel (2005) and Brigo and Mercurio (2006)). In this section we will show how one can price forward starting options within the SZHW framework; following Hong (2004), we consider the (forward) log return of the asset price \( S \):
\[
z(T_{i-1}, T_i) := \log \left( \frac{S(T_i)}{S(T_{i-1})} \right),
\]
(3.46)
3.6. Forward starting options

Since
\[ \log S(t) = y(t) + \log P(t, T_i), \]
we can express (3.46) also in terms of the \( T_i \)-forward log-asset price \( y(t) = \log F^T_i(t) \), i.e.
\[ z(T_{i-1}, T_i) = y(T_i) - y(T_{i-1}) - \log P(T_{i-1}, T_i), \]
with \( t \leq T_{i-1} \leq T_i \). We are interested in the following forward starting call option with strike \( K = \exp(k) \) on the return \( S(T_i) \)
\[ C_{T_{i-1}, T_i}(k) = \mathbb{E}^Q \left[ \exp \left( r(u)du \right) \left( \frac{S(T_i)}{S(T_{i-1})} - K \right) \right]_{T_{i-1}}^{T_i} \]
\[ = P(T_i) \mathbb{E}^Q \left[ (F^T_{T_i} - K) \right]_{T_{i-1}}^{T_i}, \]
where
\[ F^T_{T_i} = \exp(z(T_{i-1}, T_i)) \]
denotes the forward return between \( T_{i-1} \) and \( T_i \). Note that the above expression is nothing more than some call option under the \( T \)-forward measure. Therefore, as noted by Hong (2004), the pricing of forward starting options can be reduced to finding the characteristic function of the log forward return under the \( T \)-forward measure; by replacing the log-asset price by the forward log-return one can directly apply the pricing equation (3.30) or (3.31), i.e. by replacing the corresponding characteristic function by \( \psi_{T_{i-1}, T_i}(v) \) the characteristic function (under the \( T \)-forward measure) of the forward log-return between \( T_{i-1} \) and \( T_i \). What remains to be done for the pricing of forward starting options is the derivation of this forward characteristic function, which we will deal with in the following subsection.

3.6.1 Forward characteristic function

We will now derive the forward characteristic function of the forward log return \( z_{T_{i-1}, T_i} = y(T_i) - y(T_{i-1}) - \log P(T_{i-1}) \) in the SZHW model. In the derivation we will use the following lemma.

**Lemma 3.6.1** Let \( Z \) be a standard normal distributed random variable, furthermore let \( p \) and \( q \) be two positive constants. Then the characteristic function, of \( Y := pZ + \frac{q}{2}Z^2 \) is given by
\[ \phi_Y(u, p, q) := \mathbb{E} \exp(\int uY) = \frac{\exp\left( -\frac{p^2u^2}{2-2iuq} \right)}{\sqrt{1 - iuq}}, \]

**Proof** Either by completing the square and using properties of the non-central chi-squared distribution or by direct integration of an exponential affine form against the normal distribution, e.g. see Johnson et al. (1994) or Glasserman (2003). \( \Box \)

Before we can apply the above lemma we first need to rewrite the characteristic function of the log-return \( y(T_i) - y(T_{i-1}) \) in the form of the above lemma. To simplify the notation we write
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Let $B := iu, A(T_{i-1}) := A(u, T_{i-1}, T_i), C(T_{i-1}) := C(u, T_{i-1}, T_i)$ and $D(T_{i-1}) := D(u, T_{i-1}, T_i)$. By using the tower law for conditional expectations and the (conditional) characteristic function of the SZHW model one can then obtain

$$
\phi_{T_{i-1}, T_i}(u) = \mathbb{E}^{Q_{T_i}} \left\{ \exp \left[ iu [y(T_i) - y(T_{i-1}) - \log P(T_{i-1}, T_i)] \right] \mid \mathcal{F}_i \right\}
$$

(3.51)

$$
= \mathbb{E}^{Q_{T_i}} \left\{ \mathbb{E}^{Q_{T_i}} \left[ \exp \left[ iu [y(T_i) - y(T_{i-1}) - \log P(T_{i-1}, T_i)] \right] \mid \mathcal{F}_i \right] \right\}
$$

$$
= \mathbb{E}^{Q_{T_i}} \left\{ \exp [A(T_{i-1}) - iuA_r(T_{i-1}, T_i)] \cdot \mathbb{E}^{Q_{T_i}} \left[ \exp [iuB_r(T_{i-1}, T_i)r(T_{i-1}) + C(T_{i-1})v(T_{i-1}) + \frac{1}{2} D(T_{i-1})v^2(T_{i-1})] \mid \mathcal{F}_i \right] \right\}.
$$

Since the pair $(r(T_{i-1}), v(T_{i-1}))$, conditional on the information set $\mathcal{F}_i$ on time $t$, follows a joint Gaussian distribution with means $\mu_r, \mu_v$ (see (3.56), (3.58)) and variances $\sigma_r^2, \sigma_v^2$ (see (3.57), (3.59)), we can write the sum of dependent normal variates $r(T_{i-1}), v(T_{i-1})$ in terms of two independent standard normal distributions $Z_1$ and $Z_2$ (e.g. by a Cholesky decomposition):

$$
iu r(T_{i-1}) + cv(T_{i-1}) + \frac{1}{2} dv^2(T_{i-1}) \\
= iu \left[ \mu_r + \sigma_r \left[ \rho_{rv}(t, T_{i-1}) Z_1 + \sqrt{1 - \rho_{rv}^2(t, T_{i-1})} Z_2 \right] \right] \\
+ c \left[ \mu_v + \sigma_v Z_1 \right] + \frac{1}{2} d \left[ \mu_v + \sigma_v Z_1 \right]^2 \\
= iu \mu_r + cu \mu_v + \frac{1}{2} du_v^2 + iu \sigma_r \sqrt{1 - \rho_{rv}^2(t, T_{i-1})} Z_2 \\
+ c \sigma_v + du_v \sigma_v \left[ \rho_{rv}(t, T_{i-1}) \right] Z_1 + \frac{1}{2} d \sigma_v^2 Z_2,
$$

(3.52)

where the correlation $\rho_{rv}(t, T_{i-1})$ between $r(T_{i-1})$ and $v(T_{i-1})$ over the interval $[t, T_{i-1}]$ is given by

$$
\rho_{rv}(t, T_{i-1}) = \frac{\rho_{rv} \sigma_r}{\sigma_r \sigma_v (a + k)} \left[ 1 - e^{-[(a + k)(t - T_{i-1})]} \right].
$$

(3.53)

Hence, using the independence of $Z_1, Z_2$ and equation (3.52), one can find the following expression for the forward characteristic function

$$
\phi_{T_{i-1}, T_i}(u) = \exp \left[ A(T_{i-1}) + iu(B_r(T_{i-1}, T_i) \mu_r - A_r(T_{i-1}, T_i)) + C(T_{i-1}) \mu_v + \frac{1}{2} D(T_{i-1}) \mu_v^2 \right] \\
\mathbb{E}^{Q_{T_i}} \left\{ \exp \left[ iuB_r(T_{i-1}, T_i) \sigma_r \sqrt{1 - \rho_{rv}^2(t, T_{i-1})} Z_2 \mid \mathcal{F}_i \right] \right\} \\
\cdot \mathbb{E}^{Q_{T_i}} \left\{ \exp \left[ \left( C(T_{i-1}) \sigma_v + D(T_{i-1}) \mu_v \sigma_v + iuB_r(T_{i-1}, T_i) \rho_{rv}(t, T_{i-1}) \sigma_v \right) Z_1 \\
+ \frac{1}{2} D(T_{i-1}) \sigma_v^2 Z_1 \right] \mid \mathcal{F}_i \right\}.
$$

(3.54)

Hence we come to the following proposition

**Proposition 3.6.2** Starting from the current time $t$, the characteristic function of the forward log
3.6. Forward starting options

\[ \phi_{T_{i-1},T_{i}}(u) = \exp \left[ A(T_{i-1}) + iu \left[ B_{r}(T_{i-1}, T_{i}) \mu_{r} - A_{r}(T_{i-1}, T_{i}) \right] + C(T_{i-1}) \mu_{v} + \frac{1}{2} D(T_{i-1}) \mu_{v}^{2} \right] \]

\[ : \phi_{Z_{i}} \left( iuB_{r}(T_{i-1}, T_{i}) \sigma_{r} \sqrt{1 - \rho_{r}^{2}(t, T_{i-1})} \right) \phi_{Y}(1, P(T_{i-1}), Q(T_{i-1})) \]  \hspace{1cm} (3.55)

with

\[ P(T_{i-1}) = C(T_{i-1}) \sigma_{r} + D(T_{i-1}) \mu_{v} + i u \rho_{v}(t, T_{i-1}) B_{r}(T_{i-1}, T_{i}) \sigma_{r}, \]

\[ Q(T_{i-1}) = D(T_{i-1}) \sigma_{v}^{2}, \]

\[ \phi_{Z_{i}}(y) = \exp \left( \frac{y^{2}}{2} \right), \]

and where \( \phi_{Y}(-i, P(T_{i-1}), Q(T_{i-1})) \) is given by Lemma 3.6.1.

**Proof** The result follows directly by evaluating the expectations from expression (3.54) for the moment-generating function of the standard Gaussian distribution \( Z_{i} \) evaluated in the point

\[ iuB_{r}(T_{i-1}, T_{i}) \sigma_{r} \sqrt{1 - \rho_{r}^{2}(t, T_{i-1})}, \]

while the second expectation is the moment generating function of the random variable \( Y = P(T_{i-1}) Z_{i} + Q(T_{i-1}) Z_{1}^{2} \) evaluated in the unit point in the point \(-i\), for which an analytical expression is given by Lemma 3.6.1. \( \square \)

What yet remains, is to determine (conditional on the time-\( t \)) the \( T_{i} \)-forward mean and variance of the interest rate and volatility processes \( r(T_{i-1}) \) and \( \nu(T_{i-1}) \).

### 3.6.2 Moments of the Hull-White short interest rate

From (2.18) and Itô’s isometry, we immediately have that \( r(T_{i-1}) \), under the \( T_{i} \)-forward measure (starting from time \( t \)), is normally distributed with mean \( \mu_{r} \) and variance \( \sigma_{r}^{2} \) given by

\[ \mu_{r} = x(t)e^{-\alpha(T_{i-1}-t)} - M^{T_{i}}(t, T_{i-1}) + \beta(T_{i-1}), \]  \hspace{1cm} (3.56)

\[ \sigma_{r}^{2} = \frac{\sigma^{2}}{2a} \left( 1 - e^{-2\alpha(T_{i-1}-t)} \right), \]  \hspace{1cm} (3.57)

which can hence be used in Proposition 3.6.2.

### 3.6.3 Moments of the Schöbel-Zhu volatility process

To determine the first two moments of the Schöbel-Zhu volatility process, under the \( T_{i} \)-forward measure, for a certain time \( T_{i-1} \leq T_{i} \) and conditional on the filtration at time \( t \), one can integrate the dynamics of (3.11) to obtain

\[ \nu(T_{i-1}) = \nu(t)e^{-\kappa(T_{i-1}-t)} + \int_{t}^{T_{i-1}} \kappa \xi(u)e^{-\kappa(T_{i-1}-u)} du + \int_{t}^{T_{i-1}} \tau e^{-\kappa(T_{i-1}-u)} dW_{v}^{T}(u), \]
where $\xi(u) := \psi - \frac{\rho_{\psi\tau}}{\kappa}(1 - e^{\alpha(T_t - u)})$. Hence under the $T_t$-forward measure, from Itô’s isometry, we have the following for the mean and standard deviation of $\nu$:

$$\mu_\nu = \nu(t)e^{-\kappa(T_t - t)} + \left( \psi - \frac{\rho_{\psi\tau}}{\kappa}(1 - e^{-\kappa(T_t - t)}) \right)$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\kappa}(1 - e^{-\kappa(T_t - t)}), \tag{3.58}$$

which can hence be used in Proposition 3.6.2.

### 3.7 Schöbel-Zhu-Hull-White Foreign Exchange model

In this section we present the Schöbel-Zhu-Hull-White Foreign Exchange (SZHW-FX) model. That is, we introduce a domestic and a foreign exchange currency, which are modeled by Hull-White processes. We model the exchange rate process by geometric motion where we let the volatility follow an Ornstein-Uhlenbeck process. Moreover we allow all factors to be correlated with each other.

Notation is as follows: we let $X(t)$ denote the Foreign Exchange (FX) rate, with volatility $\nu$, between the domestic currency $r_1$ and the foreign currency $r_2$. The risk-neutral FX dynamics of the Schöbel-Zhu-Hull-White (SZHW) then read:

$$dX(t) = X(t)(r_1(t) - r_2(t))dt + X(t)\nu(t)dW_X(t), \quad X(0) = x_0, \tag{3.60}$$

$$dr_1(t) = (\theta_1(t) - a_1r_1(t))dt + \sigma_1dW_{r_1}(t), \quad r_1(0) = r_{10}, \tag{3.61}$$

$$dr_2(t) = (\theta_2(t) - a_2r_2(t) - \rho_{r_1r_2}\nu(t))dt + \sigma_2dW_{r_2}(t), \quad r_2(0) = r_{20}, \tag{3.62}$$

$$d\nu(t) = \kappa(\psi - \nu(t))dt + \tau dW_\nu(t), \quad \nu(0) = \nu_0, \tag{3.63}$$

where $a_1, \sigma_1, \kappa, \psi, \tau$ are positive parameters. Hence the domestic and the (shifted) foreign interest rate markets are modeled by Hull-White models and the exchange rate is modeled by a Schöbel-Zhu stochastic volatility model. $\tilde{W}(t) = (W_X(t), W_{r_1}(t), W_{r_2}(t), W_\nu(t))$ denotes a Brownian motion under the risk-neutral measure $Q$ with a positive covariance matrix:

$$\text{Cov}(\tilde{W}(t)) = \begin{pmatrix}
1 & \rho_{Xr_1} & \rho_{Xr_2} & \rho_{X\nu} \\
\rho_{Xr_1} & 1 & \rho_{r_1r_2} & \rho_{r_1\nu} \\
\rho_{Xr_2} & \rho_{r_1r_2} & 1 & \rho_{r_2\nu} \\
\rho_{X\nu} & \rho_{r_1\nu} & \rho_{r_2\nu} & 1
\end{pmatrix}t \quad \tag{3.64}$$

We will now show that the above model dynamics yield a closed-form expression for the price of an European FX-option with strike $K$ and maturity $T$. Hence we consider:

$$\mathbb{E}_F^Q\left[ \frac{(w(X(T) - K))^+}{N_1(T)} \middle| \mathcal{F}_t \right], \tag{3.65}$$
where $w = \pm 1$ for a call/put option and with
\[
N_1(T) = \exp\left[\int_0^T r(u)du\right]
\] (3.66)
denotes the bank-account in the domestic economy. We can also represent the expectation (3.65) in the domestic $T$-forward measure $\mathcal{Q}$ associated with a domestic zero-coupon bond option $P_1(t, T)$ which matures at time $T$, hence we obtain
\[
\mathbb{E}^\mathcal{Q}\left[\frac{(w(X(T) - K))^+}{N_1(T)}\right] = P_1(t, T)\mathbb{E}^{\mathcal{Q}}\left[\frac{(w(FFX_T(T) - K))^+}{f_T}\right],
\] (3.67)
where
\[
FFX_T(t) = \frac{X(t)P_2(t, T)}{P_1(t, T)}
\] (3.68)
denotes the forward FX-rate under the domestic $T$-forward measure and in the Hull and White (1993) model:
\[
P_i(t, T) = \exp[A_i(t, T) - B_i(t, T)x_i(t)] \quad \text{with: } B_i(t, T) := \frac{1 - e^{-a_i(T-t)}}{a_i},
\] (3.69)
where $A_i(t, T)$ is a deterministic function. We can express the forward FX-rate as
\[
FFX_T(t) = \frac{X(t)\exp[A_2(t, T) - B_2(t, T)x_2(t)]}{\exp[A_1(t, T) - B_1(t, T)x_1(t)]},
\] (3.70)
Note that under their own risk-neutral measures (where the money market bank account of their own currency is used as numeraire) the discount bond prices follow the SDEs:
\[
\frac{dP_i(t, T)}{P_i(t, T)} = r_i(t)dt - \sigma_i B_i(t, T)dW_i(t),
\] (3.71)
hence, by an application of Itô’s lemma, we find the following dynamics for the $T$-forward stock price process
\[
\frac{dFFX_T(t)}{FFX_T(t)} = \left(\sigma_1^2 B_1^2(t, T) + \rho_{\lambda_1} \nu(t) \sigma_1 B_1(t, T) - \rho_{\lambda_1} \sigma_2 B_2(t, T) \sigma_1 B_1(t, T)\right)dt \\
+ \nu(t) dW_{\lambda_1}(t) + \sigma_1 B_1(t, T)dW_{\sigma_1}(t) - \sigma_2 B_2(t, T)dW_{\sigma_2}(t).
\] (3.72)
By definition the forward FX-rate is a martingale process under the domestic $T$-forward measure. This is achieved by defining the following transformations of the Brownian motion(s):

\[
\begin{align*}
    dW_{r_1}(t) &\quad \mapsto \quad dW^T_{r_1}(t) - \sigma_1 B_1(t, T) dt, \\
    dW_{r_2}(t) &\quad \mapsto \quad dW^T_{r_2}(t) - \rho_{r_1 r_2} \sigma_1 B_1(t, T) dt, \\
    dW_{\tilde{X}}(t) &\quad \mapsto \quad dW^T_{\tilde{X}}(t) - \rho_{\tilde{X} r_1} \sigma_1 B_1(t, T) dt, \\
    dW_r(t) &\quad \mapsto \quad dW^T_r(t) - \rho_{r r_1} \sigma_1 B_1(t, T) dt.
\end{align*}
\]

Hence under the domestic $T$-forward measure the forward FX-rate and the associated volatility process are given by

\[
\begin{align*}
    \frac{d\text{FFX}_T(t)}{\text{FFX}_T(t)} &= \nu(t) dW^T_{\tilde{X}}(t) + \sigma_1 B_1(t, T) dW^T_{r_1}(t) - \sigma_2 B_2(t, T) dW^T_{r_2}(t) \\
    dv(t) &= \kappa \left( \psi - \frac{\rho_{r r_1} \sigma_1}{\kappa} B_1(t, T) - \nu(t) \right) dt + \tau dW^T_r(t). \tag{3.73}
\end{align*}
\]

We can simplify (3.73) by switching to logarithmic coordinates by defining $y(t) := \log(\text{FFX}_T(t))$; an application of Itô’s lemma yields

\[
\begin{align*}
    dy(t) &= -\frac{1}{2} \nu^2(t) dt + \nu(t) dW^T_{\tilde{X}}(t) + \sigma_1 B_1(t, T) dW^T_{r_1}(t) - \sigma_2 B_2(t, T) dW^T_{r_2}(t) \tag{3.75} \\
    dv(t) &= \kappa \left( \xi(t) - \nu(t) \right) dt + \tau dW^T_r(t), \tag{3.76}
\end{align*}
\]

with:

\[
\begin{align*}
    \nu^2(t) &:= \nu^2(t) + \sigma_1^2 B_1^2(t, T) + \sigma_2^2 B_2^2(t, T) + 2\rho_{\tilde{X} r_1} \nu(t) \sigma_1 B_1(t, T) - 2\rho_{r r_2} \sigma_1 B_1(t, T) \sigma_2 B_2(t, T) \tag{3.77} \\
    \xi(t) &:= \psi - \frac{\rho_{r r_1} \sigma_1}{\kappa} B_1(t, T). \tag{3.78}
\end{align*}
\]

Notice that we have now reduced the system (3.60) of the variables $X(t), r_1(t), r_2(t), \nu(t)$ under the domestic risk-neutral measure, to the system (3.75) of variables $y(t)$ and $\nu(t)$ under the domestic $T$-forward measure. What now remains is to determine the characteristic function of this reduced system.

**Determining the characteristic function of the forward log-FX rate**

We will now determine the characteristic function of the forward FX rate. Since this calculation goes in a similar way as the calculation of the ordinary characteristic function of the Schöbel-Zhu-Hull-White model of Section 3.2, we restrict ourselves to the most important steps. Again we apply the Feynman-Kac theorem and reduce the search for the characteristic function of the forward-FX rate dynamics to solving a partial differential equation. That is, we try to determine the Kolmogorov backward partial differential equation of the joint probability function.
3.7. Schöbel-Zhu-Hull-White Foreign Exchange model

\[ f = f(t, y, v). \]

To this end we need to take into account the following covariance term

\[
\begin{align*}
\text{dy}(t)dv(t) & = (\nu(t)dW^X(t) + \sigma_1 B_1(t, T)dW^T_1(t) - \sigma_2 B_2(t, T)dW^T_1(t))\left(\tau dW^T_\nu(t)\right) \\
& = \left(\rho_{\nu,v}\nu(t) + \rho_{\nu,y}\sigma_1 B_1(t, T) - \rho_{\nu,y}\tau\sigma_2 B_2(t, T)\right)dt.
\end{align*}
\]

(3.79)

Hence using (3.75) and (3.79), the Feynman-Kac theorem then implies that the solution of the following PDE

\[
0 = f_t - \frac{1}{2}v^2_f(t)f_y + \kappa(\xi(t) - \nu(t))f_v + \frac{1}{2}v^2_f(t)f_{yy}
+ \left(\rho_{\nu,y}\nu(t) + \rho_{\nu,v}\sigma_1 B_1(t, T) - \rho_{\nu,y}\tau\sigma_2 B_2(t, T)\right)f_{yy} + \frac{1}{2}r^2f_{vv},
\]

(3.80)

subject to the terminal boundary condition \( f(T, y, v) = \exp(iuy(T)) \), equals the characteristic function of the forward FX-rate dynamics. Solving the above system hence leads to the following proposition.

**Proposition 3.7.1** The characteristic function of domestic \( T \)-forward log SZHW-FX-rate is given by the following closed-form solution:

\[
f(t, y, v) = \exp\left[A(t) + B(t)y(t) + C(t)v(t) + \frac{1}{2}D(t)v^2(t)\right],
\]

(3.81)

where:

\[
A(u, t, T) = \frac{1}{2}(B^2 - B)V_{FX}(t, T)
\]

(3.82)

\[
B = iy,
\]

(3.83)

\[
C(u, t, T) = -u(i + u)\frac{\left((\gamma_8 - \gamma_4e^{-2\gamma(T-t)}) - (\gamma_5e^{-\gamma_1(T-t)} - \gamma_6e^{-2\gamma_1(T-t)}) - \gamma_7e^{-\gamma(T-t)}\right)}{\gamma_1 + \gamma_2e^{-2\gamma(T-t)}}
\]

(3.84)

\[
+u(i + u)\frac{\left((\gamma_8 - \gamma_2e^{-2\gamma(T-t)}) - (\gamma_10e^{-\gamma_1(T-t)} - \gamma_11e^{-2\gamma_1(T-t)}) - \gamma_12e^{-\gamma(T-t)}\right)}{\gamma_1 + \gamma_2e^{-2\gamma(T-t)}}
\]

\[
D(u, t, T) = -u(i + u)\frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2e^{-2\gamma(T-t)}}
\]

(3.85)
Our model incorporates the closed-form pricing of European options by Fourier transforming the more realistic model, which is of practical importance for the pricing and hedging of long-term changes in the interest rates and the volatility. Therefore, having the flexibility to correlate the As insurance contracts typically involve long maturities, they are are much more sensitive to ration of the correlation between the underlying asset and the term structure of interest rates.

We have introduced the SZHW model which allows for the pricing of insurance contracts under both stochastic volatility and stochastic interest rates in conjunction with an explicit incorporation of the correlation between the underlying asset and the term structure of interest rates.

3.8 Conclusion

We have introduced the SZHW model which allows for the pricing of insurance contracts under both stochastic volatility and stochastic interest rates in conjunction with an explicit incorporation of the correlation between the underlying asset and the term structure of interest rates. As insurance contracts typically involve long maturities, they are much more sensitive to changes in the interest rates and the volatility. Therefore, having the flexibility to correlate the underlying asset price with both the stochastic volatility and the stochastic interest rates yields a more realistic model, which is of practical importance for the pricing and hedging of long-term options.

Our model incorporates the closed-form pricing of European options by Fourier transforming the

with:

\[ \gamma = \sqrt{(\kappa - \rho_X \tau B)^2 - \tau^2 (B^2 - B)} \]
\[ \gamma_1 = \gamma + (\kappa - \rho_X \tau B), \]
\[ \gamma_3 = \frac{\rho_{Xt} \sigma_1 \gamma_1 + \kappa a_1 \psi + \rho_{r_2} \sigma_1 \tau (iu - 1)}{a_1 \gamma}, \]
\[ \gamma_5 = \frac{\rho_{Xt} \sigma_1 \gamma_1 + \rho_{r_2} \sigma_1 \tau (iu - 1)}{a_1 (\gamma - a_1)}, \]
\[ \gamma_8 = \frac{\rho_{Xt} \sigma_2 \gamma_1 + \rho_{r_2} \sigma_2 \tau B}{a_2 \gamma}, \]
\[ \gamma_10 = \frac{\rho_{Xt} \sigma_2 \gamma_1 + \rho_{r_2} \sigma_2 \tau B}{a_2 (\gamma - a_2)}, \]
\[ \gamma_7 = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6), \]
\[ \gamma_2 = \gamma - (\kappa - \rho_X \tau B), \]
\[ \gamma_4 = \frac{\rho_{Xt} \sigma_1 \gamma_2 - \kappa a_1 \psi - \rho_{r_2} \sigma_1 \tau (iu - 1)}{a_1 \gamma}, \]
\[ \gamma_6 = \frac{\rho_{Xt} \sigma_1 \gamma_2 - \rho_{r_2} \sigma_1 \tau (iu - 1)}{a_1 (\gamma + a_1)}, \]
\[ \gamma_9 = \frac{\rho_{Xt} \sigma_2 \gamma_2 - \rho_{r_2} \sigma_2 \tau B}{a_2 \gamma}, \]
\[ \gamma_{11} = \frac{\rho_{Xt} \sigma_2 \gamma_2 - \rho_{r_2} \sigma_2 \tau B}{a_2 (\gamma + a_2)}, \]
\[ \gamma_{12} = (\gamma_8 - \gamma_9) - (\gamma_{10} - \gamma_{11}) \]

and:

\[ V_{FX}(t, T) := \frac{\sigma^2}{a_1^2} (T - t) + \frac{2}{a_1} e^{-a_1(T-t)} - \frac{1}{2 a_1} e^{-2a_1(T-t)} - \frac{3}{2 a_1} \]
\[ + \frac{\sigma^2}{a_2^2} (T - t) + \frac{2}{a_2} e^{-a_2(T-t)} - \frac{1}{2 a_2} e^{-2a_2(T-t)} - \frac{3}{2 a_2} \]
\[ - 2\rho_{r_1} \sigma_1 \sigma_2 \frac{a_1 a_2}{a_1 a_2} (T - t) + \frac{e^{-a_1(T-t)} - \frac{1}{a_1}}{a_1} + \frac{e^{-a_2(T-t)} - \frac{1}{a_2}}{a_2} - \frac{e^{-(a_1+a_2)(T-t)} - 1}{a_1 + a_2}. \]

Proof See appendix 3.9.1. □

The strip of regularity and the decay of the characteristic function can be determined analogous to the SZHW model. The function \( C(u, t, T) \) once again determines the strip of regularity, whereas \( A(u, t, T) \) ensures the characteristic function decays like \( \exp(-C(u, t, T)u^2) \), where the exact constant follows from an analysis similar to that in Section 3.5.

3.8 Conclusion

We have introduced the SZHW model which allows for the pricing of insurance contracts under both stochastic volatility and stochastic interest rates in conjunction with an explicit incorporation of the correlation between the underlying asset and the term structure of interest rates. As insurance contracts typically involve long maturities, they are much more sensitive to changes in the interest rates and the volatility. Therefore, having the flexibility to correlate the underlying asset price with both the stochastic volatility and the stochastic interest rates yields a more realistic model, which is of practical importance for the pricing and hedging of long-term options.

Our model incorporates the closed-form pricing of European options by Fourier transforming the
conditional characteristic function of the asset price in closed-form. We extensively considered
the numerical implementation of the pricing formulas which enables a fast and accurate valuation
of European options, which is a big advantage for the calibration (and sensitivity analysis)
of the model to market prices. We have also derived a closed-form pricing formula for forward
starting options, which allows for a calibration of the model to forward smiles.
The SZHW model will be especially useful in the pricing and risk management of insurance con-
tracts and other long-maturity exotic derivatives. Examples include pension products, variable
and guaranteed annuities, rate of return guarantees, unit-linked contracts and exotic options like
PRDC FX options which have a long-term nature. For these products it is especially important
to consider the risk of the underlying in conjunction with the interest rate risk of the contract.
Given empirical data on option prices our model can be used to examine the pricing and es-
pecially hedging performance of stochastic volatility models while correcting for interest rate
risk. An empirical study on the relative performance of the SZHW model versus other stochas-
tic volatility models, as well as the relative benefit of the modelling of stochastic interest rates
(covered earlier by Bakshi et al. (1997)), is beyond the scope of this thesis, and is left for future
research.
3.9 Appendix

In this appendix we will show that the partial differential equation (3.17)

\[ f_t + \kappa(\xi(t) - \nu(t))f_y + \frac{1}{2}v_F^2(t)(f_{yy} - f_y) + (\rho_S\tau\nu(t) + \rho_{\nu\tau}\sigma B_y(t, T))f_{yy} + \frac{1}{2}\tau^2 f_{yy} = 0, \tag{3.88} \]

subject to the terminal condition

\[ f(T, y, \nu) = \psi(y, \nu) := \exp(\hat{u}y(T)). \]

has a solution given by (3.20) - (3.24).

To ease the notation, we from here on omit the explicit dependence on \( u \) and \( T \) in the \( A, B, C, D \) terms and hence write \( A(t) \) instead of \( A(u, t, T) \) for these terms. Using the ansatz

\[ f(t, y, \nu) = \exp[A(t) + B(t)y(t) + C(t)\nu(t) + \frac{1}{2}D(t)\nu^2(t)], \tag{3.89} \]

we find the following partial derivatives for \( f = f(t, y, \nu) \):

\[
\begin{align*}
    f_t &= f \cdot (A'(t) + B'(t)y(t) + C'(t)\nu(t) + \frac{1}{2}D'(t)\nu^2(t)), \quad f_y = fB(t), \\
    f_{\nu} &= f \cdot (C(t) + D(t)\nu(t)), \quad f_{\nu y} = fB'(t), \quad f_{\nu y} = fB(t)(C(t) + D(t)\nu(t)) \quad f_{\nu y} = f \cdot (C'(t) + D(t) + 2C(t)D(t)\nu(t) + D'(t)\nu^2(t)) \tag{3.90}
\end{align*}
\]

Substituting these partial derivatives into the partial differential equation (3.88) then gives

\[
\begin{align*}
    \left(A'(t) + B'(t)y(t) + C'(t)\nu(t) + \frac{1}{2}D'(t)\nu^2(t) \right) + \kappa(\xi(t) - \nu(t))(C(t) + D(t)\nu(t)) \\
    + \frac{1}{2}\left(v_F^2(t) + 2\rho_S\nu(t)\sigma B_y(t, T) + \sigma^2 B^2_y(t, T)\right)(B^2(t) - B(t)) \\
    + (\rho_S\tau\nu(t) + \rho_{\nu\tau}\sigma B_y(t, T))B(t)(C(t) + D(t)\nu(t)) \\
    + \frac{1}{2}\tau^2(C'(t) + D(t) + 2C(t)D(t)\nu(t) + D'(t)\nu^2(t)) = 0. \tag{3.90}
\end{align*}
\]

Collecting terms for \( y(t), \nu(t) \), and \( \frac{1}{2}\nu^2(t) \) then yields the following four ordinary differential equa-
3.9. Appendix

tions for the functions $A(t), \ldots, D(t)$:

$$0 = B'(t), \quad (3.91)$$

$$0 = D'(t) - 2(\kappa - \rho_{s,v} \tau B)D(t) + \tau^2 D^2(t) + (B^2 - B), \quad (3.92)$$

$$0 = C'(t) + (\rho_{s,v} \tau B - \kappa + \tau^2)C(t) + \rho_{s,v} \sigma B_v(t, T)(B^2 - B)$$
$$+ (\kappa \xi(t) + \rho_{s,v} \tau \sigma B_v(t, T)) D(t), \quad (3.93)$$

$$0 = A'(t) + (\kappa \xi(t) + \rho_{s,v} \tau \sigma B_v(t, T)) C(t)$$
$$+ \frac{1}{2} \sigma^2 B_v^2(t, T)(B^2 - B) + \frac{1}{2} \tau^2 (C^2(t) + D(t)). \quad (3.94)$$

It immediately that follows $B(t) = B$ equals a constant, subject to the boundary condition $(3.9)$ we find

$$B = i u. \quad (3.95)$$

The second equation $(3.92)$ yields a Riccati equation with constant coefficients with boundary condition $D(T) = 0$:

$$D'(t) = -(B^2 - B) + 2(\kappa - \rho_{s,v} \tau B) D(t) - \tau^2 D^2(t)$$
$$= q_0 + q_1 D(t) + q_2 D^2(t) \quad (3.96)$$

Making the substitution $D(t) \equiv \frac{v'(t)}{q_2}$ transforms the Riccati equation into the following second order linear differential equation with constant coefficients:

$$v''(t) - q_1 v'(t) + q_0 q_2 v(t) = 0,$$ 

which solution is given by

$$v(t) = \gamma_1 \exp[\lambda_+(T-t)] + \gamma_2 \exp[\lambda_-(T-t)],$$
$$\lambda_\pm = -\frac{q_1}{2} \pm \sqrt{q_1^2 - 4q_0 q_2}$$

Hence defining $\gamma = \sqrt{q_1^2 - 4q_0 q_2}$ we find:

$$D(t) = \frac{-v'(t)}{q_2 v(t)} = -\frac{1}{\tau^2} \frac{\gamma_1 \gamma_2 e^{\gamma(T-t)} - \gamma_1 \gamma_2 e^{-\gamma(T-t)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}$$
$$= (B^2 - B) \frac{e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} = -u(i + u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \quad (3.97)$$

with: $\gamma = \sqrt{(\kappa - \rho_{s,v} \tau B)^2 - \tau^2 (B^2 - B)}, \quad (3.98)$

$$\gamma_1 = \gamma + \frac{1}{2} q_1 = \gamma + (\kappa - \rho_{s,v} \tau B), \quad (3.99)$$

$$\gamma_2 = \gamma - \frac{1}{2} q_1 = \gamma - (\kappa - \rho_{s,v} \tau B). \quad (3.100)$$

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Here the constants in equation (3.97) are determined from the identity \((\gamma + \frac{1}{2}q_1)(\gamma - \frac{1}{2}q_1) = -(B^2 - B)\tau^2\) and the boundary condition \(D(T) = 0\).

The third equation (3.93) looks pretty daunting, but is merely a first order linear ordinary differential equation of the form \(C'(t) + g(t)C(t) + h(t) = 0\). Subject to the boundary condition \(C(T) = 0\) and using (3.15), we can hence represent a solution for \(C(t)\) as:

\[
C(t) = \int T_t h(s) \exp\left(\int_t^s g(w)dw\right) ds, \tag{3.101}
\]

with:
\[
g(w) = -(\kappa - \rho_{Sv}\tau B) + \tau^2 D(w), \tag{3.102}
\]
\[
h(s) = \rho S\sigma B_s(s,T)(B^2 - B) + (\kappa\xi(s) + \rho_{Sv}\tau\sigma B_1(s,T))D(s)
  = \rho S\sigma B_s(s,T)(B^2 - B) + (\kappa\xi + \rho_{Sv}(B - 1)\tau\sigma B_1(s,T))D(s). \tag{3.103}
\]

We first consider the integral over \(g\): dividing equation (3.92) by \(D(t)\), rearranging terms and integrating we find the surprisingly simple solution:

\[
\int g(w)dw = \int -(\kappa - \rho_{Sv}\tau B) + \tau^2 D(w)dw
  = \int (\kappa - \rho_{Sv}\tau B) - \frac{(B^2 - B)}{D(w)} - \frac{D'(w)}{D(w)}dw
  = \log\left(\frac{\gamma_1 e^{\gamma(T-t)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}\right) + C, \tag{3.104}
\]

where \(C\) denotes the integration constant. Hence taking the exponent and filling in the required integration boundaries yields

\[
\exp\left[\int_t^s g(w)dw\right] = \frac{\gamma_1 e^{\gamma(T-s)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}, \tag{3.105}
\]

and after a straightforward calculation we get for \(C(t)\):

\[
C(t) = \frac{1}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}} \int T_t h(s)\left(\gamma_1 e^{\gamma(T-s)} + \gamma_2 e^{-\gamma(T-s)}\right) ds
  = (B^2 - B) \frac{\left((\gamma_3 e^{\gamma(T-t)} - \gamma_4 e^{-\gamma(T-t)}) - (\gamma_3 e^{\gamma(T-t)} - \gamma_4 e(-\gamma(T-t)) - \gamma_7\right)}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}
  = -\mu(i + \mu) \frac{\left((\gamma_3 - \gamma_4 e^{-2\gamma(T-t)}) - (\gamma_3 e^{-\mu(T-t)} - \gamma_4 e^{-(2\gamma(T-t))} - \gamma_7 e^{-\gamma(T-t)}\right)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{3.106}
\]

with \(\gamma, \gamma_1, \ldots, \gamma_7\) as defined in (3.25).
Finally, by solving equation (3.94), we find the following expression for $A(t)$:

$$A(t) = \int_{t}^{T} \frac{1}{2} (B^2 - B) \sigma^2 B_s^2(s, T) ds$$

$$+ \int_{t}^{T} \left[ (\kappa \xi(t) + \rho_{s} \tau \sigma B_s(s, T) B) C(s) + \frac{1}{2} \tau^2 \left( C^2(s) + D(s) \right) \right] ds$$

$$= -\frac{1}{2} u(i + u) V(t, T)$$

$$+ \int_{t}^{T} \left[ (\kappa \psi + \rho_{s}(iu - 1) \tau \sigma B_s(s, T)) C(s) + \frac{1}{2} \tau^2 \left( C^2(s) + D(s) \right) \right] ds \quad (3.107)$$

where $V(t, T)$ can be found by simple integration and is given by

$$V(t, T) = \frac{\sigma^2}{a^2} \left( (T - t) + \frac{2}{a} e^{-a(T-t)} - \frac{1}{2a} e^{-2a(T-t)} - \frac{3}{2a} \right). \quad (3.108)$$

It is possible to write a closed-form expression for the remaining integral in (3.107). As the ordinary differential equation for $D(s)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2007) in order to avoid any (complex) discontinuities. The main problem however lies in the integrals over $C(s)$ and $C^2(s)$, which will involve the Gaussian hypergeometric function $_2F_1(a, b; c; z)$. The most efficient way to evaluate this hypergeometric function (according to Numerical Recipes, Press and Flannery (1992)) is to integrate the defining differential equation. Since all of the terms involved in $D(u)$ are also required in $C(u)$, numerical integration of the second part of (3.107) seems to be the most efficient method for evaluating $A(t)$. Hereby we conveniently avoid any computational issues regarding complex discontinuities altogether.

### 3.9.1 Deriving the log FX-rate characteristic function

In this appendix we will prove that the partial differential equation (3.80), i.e.

$$0 = f_t + \kappa (\xi(t) - \nu(t)) f_r + \frac{1}{2} v^2_r(t)(f_{yy} - f_y)$$

$$+ \left( \rho_{s} \tau \nu(t) + \rho_{s} \tau \sigma_1 B_1(t, T) - \rho_{s} \tau \sigma_2 B_2(t, T) \right) f_y + \frac{1}{2} \tau^2 f_{yy}, \quad (3.109)$$

subject to the terminal condition $f(T, y, \sigma) = \exp(i u y(T))$ has a solution given by (3.81)-(3.86); we follow the same approach as in section (3.9), that is we use the ansatz (3.81), find the corresponding partial derivatives and substitute these in the PDE (3.109).

Expanding $v^2_r(t)$ according to (3.77) and collecting the terms for $y(t), \nu(t)$ and $\frac{1}{2} v^2(t)$ yields the
following system of ordinary differential equations for the functions \(A(t), \ldots, D(t)\):

\[
\begin{align*}
0 &= B'(t), \\
0 &= D'(t) - 2(\kappa - \rho_{\lambda r}\tau B)D(t) + \tau^2 D^2(t) + (B^2 - B), \quad (3.110) \\
0 &= C'(t) + (\rho_{\lambda r}\tau B - \kappa + \tau^2 D)C(t) + (\rho_{\lambda r} \gamma_1 B_1(t, T) - \rho_{\lambda r} \gamma_2 B_2(t, T))(B^2 - B) \\
&\quad + \left(\kappa \xi(t) + (\rho_{\lambda r} \tau \gamma_1 B_1(t, T) - \rho_{\lambda r} \tau \gamma_2 B_2(t, T))\right)D(t), \quad (3.112) \\
0 &= A'(t) + (\kappa \xi(t) + \rho_{\lambda r} \tau \gamma_1 B_1(t, T)B - \rho_{\lambda r} \tau \gamma_2 B_2(t, T))C(t) \\
&\quad + \left(\frac{1}{2} \sigma_1^2 B_1(t, T) + \frac{1}{2} \sigma_2^2 B_2(t, T) - \rho_{\lambda r} \gamma_1 B_1(t, T) \gamma_2 B_2(t, T)\right)(B^2 - B) \\
&\quad + \frac{1}{2} \tau^2 (C^2(t) + D(t)) \quad (3.113)
\end{align*}
\]

Hence we end up with an analogue system of ordinary differential equations as in section (3.9): the first two differential equations (3.110) and (3.111) for \(B\) and \(D(t)\) are equivalent to (3.91) and (3.92) whose solutions are given in the equations (3.95) and (3.97)-(3.100). The third equation (3.112) for \(C(t)\) looks pretty daunting, but is again merely a first order linear differential equation of the form \(C'(t) + g(t)C(t) + h(t) = 0\), with associated boundary condition \(C(T) = 0\). Hence expanding \(\xi(t)\) according to (3.78), we can represent a solution for \(C(t)\) as:

\[
C(t) = \int_t^T h(s) \exp \left[ \int_s^T g(w)dw \right] ds, \quad (3.114)
\]

with:

\[
\begin{align*}
g(w) &= -(\kappa - \rho_{\lambda r} \tau B) + \tau^2 D(w), \\
h(s) &= \left(\rho_{\lambda r} \gamma_1 B_1(s, T) - \rho_{\lambda r} \gamma_2 B_2(s, T)\right)(B^2 - B) \\
&\quad + \left(\kappa \xi(s) + (\rho_{\lambda r} \tau \gamma_1 B_1(s, T) - \rho_{\lambda r} \tau \gamma_2 B_2(s, T))\right)D(s) \\
&= \rho_{\lambda r} \gamma_1 B_1(s, T)(B^2 - B) + (\kappa \psi + \rho_{\lambda r} \tau - 1) \gamma_2 B_1(s, T))D(s) \\
&\quad - \rho_{\lambda r} \gamma_2 B_2(s, T)(B^2 - B) - (\rho_{\lambda r} \tau \gamma_2 B_2(s, T))D(s). \quad (3.116)
\end{align*}
\]

Now notice that the integral over \(g\) is equivalent to (3.104), hence its solution is given by equation (3.105), i.e.

\[
\exp \left[ \int_t^T g(w)dw \right] = \frac{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}, \quad (3.117)
\]

with \(\gamma, \gamma_1\) and \(\gamma_2\) defined in (3.86). Substituting this expression into (3.114) we find (after a long
but straightforward calculation) for $C(t)$:

$$
C(t) = \frac{(B^2 - B)\left((\gamma_4 e^{\gamma(t-T)} - \gamma_4 e^{-\gamma(t-T)}) - (\gamma_5 e^{(\gamma-a_1)(T-t)} - \gamma_6 e^{-(\gamma+a_1)(T-t)}) - \gamma_7 \right)}{\gamma_1 e^{\gamma(t-T)} + \gamma_2 e^{-\gamma(t-T)}}
$$

$$
-(B^2 - B)\left((\gamma_8 e^{\gamma(T-t)} - \gamma_9 e^{-\gamma(t-T)} - (\gamma_{10} e^{(\gamma-a_2)(T-t)} - \gamma_{11} e^{-(\gamma+a_2)(T-t)}) - \gamma_{12} \right)
$$

$$
\frac{1}{\gamma_1 + \gamma_2 e^{2\gamma(t-T)}} \left(\gamma_3 - \gamma_4 e^{2\gamma(t-T)} - (\gamma_5 e^{-a_1(T-t)} - \gamma_6 e^{-(2\gamma+a_1)(T-t)}) - \gamma_7 e^{-\gamma(t-T)} \right)
$$

$$
\frac{1}{\gamma_1 + \gamma_2 e^{2\gamma(t-T)}} \left(\gamma_8 - \gamma_9 e^{2\gamma(t-T)} - (\gamma_{10} e^{-a_2(T-t)} - \gamma_{11} e^{-(2\gamma+a_2)(T-t)}) - \gamma_{12} e^{-\gamma(t-T)} \right)
$$

with $\gamma, \gamma_1, \ldots, \gamma_{12}$ as defined in (3.86).

Finally, by solving equation (3.113), we find the following expression for $A(t)$:

$$
A(t) = \frac{1}{2} (B^2 - B)(\sigma_1^2 B_1^2(s, T) + \sigma_2^2 B_2^2(s, T) - 2\rho_{rz} \sigma_1 B_1(s, T) \sigma_2 B_2(s, T)) ds
$$

$$
+ \int_T^T \left[k(\xi(s) + \rho_{rz} B \sigma_1 B_1(t, T) - \rho_{rz} B t \sigma_2 B_2(t, T)) C(s) + \frac{1}{2} \tau^2 (C^2(s) + D(s)) \right] ds
$$

$$
= \frac{1}{2} (B^2 - B) V_F X(t, T)
$$

$$
+ \int_T^T \left[(k \phi + \rho_{rz}(i u - 1) \tau \sigma_1 B_1(s, T) - \rho_{rz} i u \tau \sigma_2 B_2(s, T)) C(s) + \frac{1}{2} \tau^2 (C^2(s) + D(s)) \right] ds,
$$

where $V_F X(t, T)$ can be found by simple integration and is given by:

$$
V_F X(t, T) := \frac{\sigma_1^2}{a_1^2} \left((T-t) + \frac{2}{a_1} e^{-a_1(T-t)} - \frac{1}{2a_1} e^{-2a_1(T-t)} - \frac{3}{2a_1} \right)
$$

$$
+ \frac{\sigma_2^2}{a_2^2} \left((T-t) + \frac{2}{a_2} e^{-a_2(T-t)} - \frac{1}{2a_2} e^{-2a_2(T-t)} - \frac{3}{2a_2} \right)
$$

$$
-2 \rho_{rz} \sigma_1 \sigma_2 \left(T-t + \frac{e^{-a_1(T-t)} - 1}{a_1} + \frac{e^{-a_2(T-t)} - 1}{a_2} - \frac{1}{a_1 + a_2} \right).\quad (3.120)
$$

Analogous to (3.107), integrating over the $C(s)$ and $C^2(s)$ terms in (3.119) seems to be the most efficient method to evaluate $A(t)$.