Chapter 4

Generic pricing of FX, Inflation and Stock Options under Stochastic Interest Rates and Stochastic Volatility

*This chapter is based on:


4.1 Introduction

The markets for long maturity and hybrid derivatives are developing more and more. Not only are increasingly exotic structures created, also the markets for plain vanilla derivatives are growing. One of the recent advances is the development of long maturity option markets across various asset classes; during the last years long maturity securities, such as Target Auto Redemption Notes (TARN) equity-interest rate options (e.g. see Caps (2007)), Power-Reverse Dual-Currency (PRDC) Foreign Exchange (FX) swaps (e.g. see Piterbarg (2005)) and inflation-indexed Limited Price Indices (LPI) structures (e.g see Mercurio (2005) or Mercurio and Moreni (2006a)) have become increasingly popular. For FX, inflation and hybrid structures, which explicitly depend on future interest rates evolutions, it is apparent that the use of stochastic interest rates is crucial in any derivative pricing model. However, the addition of stochastic rates is also important for the pricing and in particularly the hedging of long maturity equity derivatives (e.g. see Bakshi et al. (2000)). First, the option’s rho, which measures/hedges the interest rate risk of the derivative, is increasing with time to maturity. Secondly, the stochastic interest rates are important for exotic option pricing since the numeraire is the discount bond associated with the maturity of the option: because the long term interest rates are to a reasonable degree correlated with FX/inflation/equity indices, the rates directly influence the pricing kernel used in exotic option pricing. Most investment banks have now standardized a three-factor modelling framework to price cross-currency (i.e. FX and inflation) options (see Piterbarg (2005), Sippel and Ohkoshi (2002) or Jar-
row and Yildirim (2003)). Here the index follows a log-normal process, and the interest rates of the two currencies are modelled with Gaussian Hull and White (1993) frameworks. The choice for Gaussian models for the interest rates and log-normality for the index has allowed for a very efficient, essentially closed-form, calibration to at-the-money options on the index, i.e. on the FX-rate or stock price. The assumption of log-normality for an index, though technically very convenient, does not find its justification in the financial equity markets (e.g. see Bakshi et al. (1997)), the FX markets (e.g. see Piterbarg (2005), Caps (2007)) nor in the inflation markets (e.g. see Mercurio and Moreni (2006a), Mercurio and Moreni (2009) or Kenyon (2008)). In fact, the markets for these products exhibit a strong volatility skew or smile, implying log index returns deviating from normality and suggesting the use of skewed and heavier tailed distributions. Moreover many multi-currency structures (like LPIs or PRDCs) are particularly sensitive to volatility skews/smiles as they often incorporate multiple strikes as well as callable/knockout components. Hence appropriate exotic option pricing models, which need to quantify the volatility exposure in such structures, should at least be able to incorporate the smiles/skews in the vanilla markets. While various methods exist to incorporate volatility smiles (i.e. local volatility, stochastic volatility and/or jumps), the calibration of such models is by no means trivial. A skew-mechanism is normally applied to the forward index price (i.e. the FX-rate, CPI/Equity index), however to price multi-currency options also a term-structure involving various time points of the forward index is required. The incorporation of stochastic interest rates makes the connection between the two particularly non-trivial (e.g. see Piterbarg (2005) or Antonov et al. (2008)). Though the issue is important, Piterbarg (2005) even dubs it as ‘perhaps even the most important current outstanding problems for quantitative research departments worldwide’, there is remarkably little literature available on the subject even though the problem attracted both the attention of practitioners as well as from academia (e.g. see van der Ploeg (2007)).

Only very recently a few approaches have been suggested. A local volatility approach is used in Piterbarg (2005) who derives approximating formulas for calibration. Andreasen (2006) combines Heston (1993) stochastic volatility with independent stochastic interest rates drivers and derives closed-form Fourier expressions for vanilla options. To correlate the independent rate drivers with the FX-rate, Andreasen (2006) uses an indirect approach in the form of a volatility displacement parameter, which has some disadvantages as that it can lead to extreme model parameters (e.g. see Antonov et al. (2008)). This latter framework is generalized by Kainth and Saravanamuttu (2007), which authors consider the pricing of double no-touch options in a model with stochastic correlation and double Heston dynamics for the stochastic volatility. The calibration of FX options stochastic interest rates with Heston (1993) stochastic volatility under a full correlation structure is undertaken in Antonov et al. (2008) who use a so-called “Markovian” projection to derive approximation formulas. Though their projection technique is elegant, the quality of their approximation deteriorates for larger maturities or more extreme model parameters. The exact pricing of FX options under Schöbel and Zhu (1999) stochastic volatility, single-factor Gaussian rates and a full correlation structure was recently has been considered in Chapter 3. The modelling of inflation smile, has been considered by several authors, e.g. see Belgrade et al. (2004), Kenyon (2008) and Mercurio and Moreni (2009).

In this chapter, building on the results of Chapter 3, Antonov et al. (2008), Andreasen (2006) and Piterbarg (2005), we consider the pricing of foreign exchange, inflation and stock options under
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Schöbel and Zhu (1999) and Heston (1993) stochastic volatility and under multi-factor Gaussian interest rates with a full correlation structure. Since stock and FX options are special (nested) cases of inflation-indexed caps/floors\(^1\) we will mainly focus on the pricing of inflation index derivatives. The stock and FX model option pricing formulas hence follow directly from our generalization of the foreign exchange inflation framework of Jarrow and Yildirim (2003). The setup of the chapter is as follows: in Section 4.2 we introduce our new model, Section 4.3 considers the basic vanilla derivatives and the pricing methodology used for the model. In Section 4.4 we derive the characteristic functions required for the Fourier-based pricing methods: under Schöbel and Zhu (1999) stochastic volatility we can derive the characteristic function of the general model in closed-form, under Heston (1993) stochastic volatility it is challenging to derive the characteristic function of the general model in closed-form, nonetheless we demonstrate how the characteristic function of the special (uncorrelated) case can be used as a simple and efficient control variate for the general model. Finally, Section 4.6 concludes.

4.2 The model

Before introducing the general model, we first consider the Jarrow and Yildirim (2003) model which can be seen as a special (degenerate) case of our model. The Jarrow and Yildirim (2003) framework for modelling inflation and real rates is based on a foreign-exchange analogy between the real and nominal economy. That is, the real rates are seen as interest rates in the real (foreign) economy, whereas the nominal rates represent the interest rates in the nominal (domestic) economy. The inflation index then represents the exchange rate between the nominal (domestic) and real (foreign) currency. There are several assumptions that can be made with respect to the evolution of these dynamics: we first discuss the classical Jarrow and Yildirim (2003) model, before turning to generalized model setups. For clarity of exposition we will use constant model parameters in both frameworks, it is however obvious to extend this to time-dependent model parameters.

4.2.1 Special case: Jarrow-Yildirim (2003) model

Jarrow and Yildirim (2003) assume that the real-world evolution of the nominal and real instantaneous forward rates is given by HJM-dynamics, whereas the inflation index is log-normally distributed. Though several choices can be made with respect to the volatility structure within an HJM-model, Jarrow and Yildirim (2003) assume that the forward rate volatilities are given by \(\sigma e^{\delta(T-t)}\). Using the equivalent formulation of the HJM-model in terms of instantaneous short rates then results in the following dynamics under the risk-neutral measure \(Q^\rho\), see Jarrow and Yildirim (2003).

\(^1\)In our framework an inflation option can be seen as forward-starting FX-option, hence the pricing of FX-option follows from the pricing of an inflation option by setting the forward starting date equal to the current date. A stock option can be seen as an FX-option in which (possibly deterministic) foreign interest rates represent the continuous dividend yield.
Proposition 4.2.1 The $Q_n$ dynamics of the instantaneous nominal rate $n(t)$, real rate $r(t)$ and the inflation index $I(t)$ are given by

\[
\begin{align*}
\frac{dn}{dt} &= \left[\theta_n(t) - \alpha_n n(t)\right]dt + \sigma_n dW_n(t), \\
\frac{dr}{dt} &= \left[\theta_r(t) - \rho_{n,r}\sigma_n \rho_{r,I}\sigma_I - \alpha_r r(t)\right]dt + \sigma_r dW_r(t), \\
\frac{dI}{dt} &= I(t)\left[n(t) - r(t)\right]dt + \sigma_I dW_I(t),
\end{align*}
\]

with $\alpha_n, \alpha_r, \sigma_n, \sigma_r, \sigma_I$ positive parameters (possibly time-dependent) and where $(W_n, W_r, W_I)$ is a Brownian motion under $Q^0$ (i.e. with the nominal bank-account as numeraire) with correlations $\rho_{n,r}, \rho_{n,I}$ and $\rho_{r,I}$, and with $\theta_n(T)$ and $\theta_r(t)$ deterministic functions which are used to exactly fit the term structure the nominal and real interest rates.

Note that the covariance in (4.2) between the inflation and real rate term $\rho_{r,I}\sigma_I \sigma_r$, arises due to a change of the real to the nominal risk-neutral measure, e.g. see Geman et al. (1996). With this particular volatility structure, Jarrow and Yildirim (2003) thus assumed that both the nominal as real (instantaneous) rates followed Hull and White (1993) processes under their own risk-neutral measure. Moreover they showed that the real rate still follows an Ornstein-Uhlenbeck process under the nominal risk-neutral measure $Q_n$ and that the inflation index $I(T)$ for each $t < T$ is log-normal distributed under $Q_n$, in particular one can write:

\[
I(T) = I(t) \exp\left(\int_t^T \left[n(u) - r(u) - \frac{1}{2}\sigma_I^2\right]du + \int_t^T \sigma_I dW_I(u)\right).
\] (4.4)

The main advantage of the Jarrow and Yildirim (2003) model is its tractability; one for example has analytical formulas for the prices of year on years inflation-indexed swaps (see Brigo and Mercurio (2006) pp.653, formula 16.15) and closed-form Black-like formulas for the prices of inflation-indexed caplets (see Brigo and Mercurio (2006) pp.663, formula 17.4). Though one can challenge the one-factor rate models, the biggest disadvantage of the Jarrow and Yildirim (2003) model for the pricing of inflation derivatives is most often the log-normal assumption of the inflation index, which does not find its justification in the markets, e.g. see Mercurio and Moreni (2006b), Kenyon (2008) or Kruse (2007).

4.2.2 General model

In this section we will present a general model, which can be seen as an extension of the models of Jarrow and Yildirim (2003) and the SZHW model from Chapter 3. The first extension is that instead of one-factor Hull and White (1993) models for the instantaneous nominal and real rates, we let the short rate be driven by multiple (correlated) factors. We use an equivalent additive formulation for Hull-White interest rates in terms of a sum of correlated Gaussian factors plus a deterministic function, i.e. we write the model into an affine factors formulation, e.g. Duffie et al. (2000) and Duffie et al. (2003). The deterministic factor can be chosen as to exactly fit the term structure of the nominal or real interest rates, e.g. see Brigo and Mercurio (2006) or Pelsser (2000). The nominal short interest rate be driven by $K$ correlated Gaussian factors and the real
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short rate by $M$ factors, the multi-factor Gaussian interest can hence be represented as:

$$n(t) = \varphi_n(t) + \mathbf{1} \cdot X_n(t), \quad r(t) = \varphi_r(t) + \mathbf{1} \cdot X_r(t),$$ (4.5)

with $\mathbf{1}$ a vector of ones and where $\varphi_n(t), \varphi_r(t)$ are the deterministic functions to fit the nominal and real term structure (in particular $\varphi_n(0) = n(0)$ and $\varphi_r(0) = r(0)$) and with $X_n(t), X_r$ Gaussian rate vectors which drive respectively the nominal and real rates, i.e. with typical elements the Gaussian factors $x_n^i(t), x_r^i(t)$.

The second extension in our model is that we make the volatility $\sigma_f$ stochastic. Moreover we let this stochastic volatility factor, which we from now on denote by $\nu(t)$, be correlated with the instantaneous interest rates and the inflation index. Two popular choices within the stochastic volatility literature are the models of Heston (1993) and Schöbel and Zhu (1999). In the latter the volatility is modeled as an Ornstein-Uhlenbeck process

$$d\nu(t) = \kappa [\psi - \nu(t)]dt + \tau dW_{\nu}(t), \quad \nu(0) = \nu_0$$ (4.6)

with $\kappa, \psi, \sigma_\nu$ positive parameters and where $W_\nu(t)$ is a Brownian motion that is correlated with the other driving factors, especially the asset price. Note that we have a positive probability that $\nu(t)$ in (4.6) can become negative, which will cause the correlation between $\nu(t)$ and the other driving factors to (temporarily) change sign.

The most popular stochastic volatility model, however, is the Heston (1993) model which mainly owns its popularity due to its analytical tractability. In the Heston model, the variance is modeled by the following Feller/CIR/square-root process

$$d\nu^2(t) = \kappa [\theta - \nu^2(t)]dt + \xi \nu(t)dW_r(t), \quad \nu^2(0) = \nu_0^2$$ (4.7)

with $\kappa, \theta, \xi$ positive parameters and where $W_r$ represents again a Brownian that is correlated with the other model factors.

With the multi-factor Gaussian rates and with stochastic volatility a la Schöbel-Zhu or Heston, we come to the following the dynamics of our model. The $Q^n$ dynamics of the $K$-factor instantaneous nominal rate $n(t)$, $M$-factor real rate $r(t)$ and the inflation index $I(t)$ are given by

$$dx_n^i(t) = -a_n^i x_n^i(t)dt + \sigma_n^i dW_{n_i}(t) \quad i = 1, \ldots, K,$$ (4.8)

$$dx_r^j(t) = [-a_r^j x_r^j(t) - \rho_{j,1} \nu(t)\sigma_r^j]dt + \sigma_r^j dW_{r_j}(t) \quad j = 1, \ldots, M,$$ (4.9)

$$dI(t) = I(t)[n(t) - r(t)]dt + \nu(t)I(t)dW_r(t)$$ (4.10)

with $a_n^i(t), a_r^j, \sigma_n^i(t), \sigma_r^j$ positive parameters, $\nu(t)$ the stochastic volatility factor with dynamics given by (4.6) or (4.7), and where $W_r(t) := (W_{n_1}, \ldots, W_{n_K}, W_{r_1}, \ldots, W_{r_M}, W_r)$ is a Brownian motion under $Q^n$ with (possibly) a full correlation structure.

The multi-factor Gaussian model is still very tractable; one, for example, has the following
analytical formulas for zero-coupon bond prices:

\[
P_n(t, T) = \mathbb{E}_n\left\{e^{-\int_t^T n(u)du}\right\} = A_n(t, T)e^{-B_n(t,T)X_n(t)}, \quad (4.11) \\
P_r(t, T) = \mathbb{E}_r\left\{e^{-\int_t^T r(u)du}\right\} = A_r(t, T)e^{-B_r(t,T)X_r(t)}. \quad (4.12)
\]

with \(B_n(t, T), B_r(t, T)\) vectors with typical elements \(B_n^k(t, T), B_r^j(t, T)\), and where \(A_n(t, T), A_r(t, T), B_n^k(t, T), B_r^j(t, T)\) are affine functions, e.g. see Appendix 4.7.2. A useful quantity for the pricing of inflation-indexed options will turn out to be the forward inflation index \(I_F(t)\) under the nominal \(T\)-forward measure for a general maturity \(T\), i.e.

\[
I_F(t) = I(t) \frac{P_r(t, T)}{P_n(t, T)}. \quad (4.13)
\]

Hence since \(I_F(T) = I(T)\), we can directly substitute the forward inflation index dynamics for the inflation index, to price European time-\(T\) options. In the following subsection we will derive the dynamics of \(I_F(t)\) under the nominal \(T\)-forward measure.

**Dynamics under the \(T\)-forward measure**

Using the change of numeraire technique of Geman et al. (1996), we will now derive the dynamics of our model under the \(T\)-forward measure for a general maturity \(T\). Note that under their risk-neutral measures the nominal and real discount bond prices follows the processes

\[
\frac{dP_n(t, T)}{P_n(t, T)} = n(t)dt + \Sigma_n(t, T)dW_n(t), \quad \frac{dP_r(t, T)}{P_r(t, T)} = r(t)dt + \Sigma_r(t, T)dW_r(t), \quad (4.14)
\]

where \(\Sigma_i(t, T), \, i \in \{n, r\}\) denotes the vector of zero bond volatilities, with typical element \(\sigma_n^k B_n^k(t, T)\), and with \(W_i\) a vector Brownian Motion. Hence, by an application of Itô’s lemma, we come to the following proposition.

**Proposition 4.2.2** The \(Q_n^T\) dynamics for the \(T\)-forward asset price, under the \(T\)-forward Brownian Motion \(W^T(i)\), has the following SDEs

\[
dx_n^k(t) = \left[-a_n^k x_n^k - R_{n,i,X_n}^k \Sigma_n(t, T)\right]dt + \sigma_n^k dW_n^T(t), \quad k = 1, \ldots, K, \quad (4.15) \\
dx_r^j(t) = \left[-a_r^j x_r^j - \sigma_r^j R_{r,i,X_r}^j \Sigma_r(t, T)\right]dt + \sigma_r^j dW_r^T(t), \quad j = 1, \ldots, M, \quad (4.16) \\
dI_F(t) \quad (4.17)
\]

where \(R_{n,i,X_n}^k, R_{r,i,X_r}^j\) denote the correlation vectors, between respectively \(x_n^k(t), x_r^j(t)\) and the vector of nominal interest rate drivers \(X_n(t)\). The stochastic volatility SDE in the Schöbel-Zhu case is
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given by

\[
d\nu(t) = k\left[\xi(t) - \nu(t)\right]dt + \tau dW_T^\nu(t),
\]

(4.18)

\[
\xi(t) = \psi - \tau \frac{R_{\nu}}{\kappa} \sum_n(t)
\]

(4.19)

while the Heston dynamics become

\[
d\nu^2(t) = k\left[\zeta(t) - \nu^2(t)\right]dt + \xi\nu(t)dW_T^{\nu^2}(t)
\]

(4.20)

\[
\zeta(t) = \theta - \nu(t)\xi \frac{R_{\nu}}{\kappa} \sum_n(t).
\]

(4.21)

Correlations are introduced via vector volatilities in the above models. Note that this the Heston parametrization (4.20) is consistent with the dynamics as provided in (2.23) of Chapter 2. However, to make the notation between the Schöbel-Zhu and Heston model more uniform, we adopt a slightly different notation here.

We can simplify (4.17) further, by switching to logarithmic coordinates: defining

\[
z(t) := \log I_T(t) = \log \left( \frac{I(t) P_n(t, T)}{P_n(t, T)} \right)
\]

(4.22)

and an application of Itô’s lemma yields

\[
dz(t) = -\frac{1}{2} \nu_F(t) dt + \nu_F(t) dW^\nu(t),
\]

(4.23)

with \(\nu_F(t) := \left[\nu(t) + \sum_n(t, T) - \sum_n(t, T)\right]\) the instantaneous variance of the forward inflation index (explicitly defined in (4.49)). Notice that we now have transformed the system of (4.2.1) of the variables \(x_n^1(t), \ldots, x_n^K(t), x_r^1(t), \ldots, x_r^M(t), I(t), \nu(t)\), under the nominal risk-neutral measure, to the system (4.23)-(4.18) of variables \(z(t), \nu_F(t)\), under the \(T\)-forward measure. This latter system will be used to determine characteristic function of log inflation rate in our model, see Section 4.4.

4.3 Pricing and Applications

In this section we will briefly discuss the main vanilla inflation, FX and equity derivatives and discuss how these securities can be priced in closed-form by our model. Before turning to the market-specific structures, we first consider the general pricing methodology.

4.3.1 Pricing

Recall from Section 2.5 from Chapter 2 that under the \(T\)-forward measure \(Q^T\), we can write the following for the price \(C_T(k)\) of an European option (\(\omega = 1\) for a call, \(\omega = -1\) for a put) maturing
at time $T$, with strike $K = \exp(k)$, on an asset $I$:

$$C_T(k, \omega) = P_n(t, T) \mathbb{E}^{Q^n}_n \left[ \left[ \omega \left( I_T^T(T) - K \right) \right] \bigg| \mathcal{F}_T \right]$$

where $P_n(t, T)$ denotes the price of a (pure) discount bond and $I_T^T(t) := \frac{I_T^T(t)}{P_n(t, T)}$ denotes the $T$-forward index price. This expression can be numerically evaluated by means of a Fourier inversion of the characteristic function $\phi_T$ of the $T$-forward log index price $z(T) := \log I_T^T(t)$, see Chapter 2. That is, provided that the regularity conditions for the Fourier Transformations are satisfied, i.e. $\alpha > 0$ for a call ($\omega = 1$) and $\alpha > 1$ for a put ($\omega = -1$), one can write the following for the corresponding European option price:

$$C_T(k, \omega, \alpha) = \frac{P_n(t, T)}{\pi} \int_0^\infty \text{Re} \left( e^{-(\omega \alpha + iv)k} \psi_T(v, \omega, \alpha) \right) dv,$$

with

$$\psi_T(v, \omega, \alpha) := \frac{\phi_T(v - (\omega \alpha + 1)i)}{(\omega \alpha + iv)(\omega \alpha + 1 + iv)},$$

and where $\phi_T(u) := \mathbb{E}^{Q^n}\left[ \exp(iuz(T)) \big| \mathcal{F}_T \right]$ denotes the $T$-forward conditional characteristic function of the log index price. The characteristic and forward characteristic function under Schöbel and Zhu (1999) volatility can respectively be found in proposition 4.4.1 and 4.4.2, where under Heston (1993) volatility these can be found in proposition 4.4.5 and 4.4.8.

### 4.3.2 Inflation derivatives

Before dealing with the pricing of inflation-index derivatives within the general model (4.8), we first discuss the main (vanilla) inflation-indexed securities. Hereby we adopt the notation that is used in Brigo and Mercurio (2006) and Mercurio (2005), to which authors we also refer for an excellent overview of interest rate and inflation-indexed derivatives and models.

#### Inflation-indexed swaps

Given a set of payment dates $T_1, \ldots, T_M$, an inflation-indexed swap (IIS) is a swap where, on each date, party A pays party B the inflation rate over a predefined period, while party B pays party A a fixed rate. This inflation rate is calculated as the percentage return of the inflation index (e.g. HICP ex Tobacco in the Eurozone) over the time interval it applies to. The two main IIS contracts that are traded in the markets are the zero-coupon inflation-indexed swap (ZCIIIS) and the year-on-year inflation-indexed swap (YYIIS).

In the ZCIIIS, the payoff at time $T_M$, assuming $T_M = M$ years, party B pays party A the fixed amount

$$N[(1 + K)^M - 1],$$

where $N$ is the notional amount.
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where $K$ is the strike (e.g. the break-even inflation rate) and $N$ the nominal value of the contract. In exchange, party A pays party B, at the time final time $T_M$, the floating amount of

$$N\left[\frac{I(T_M)}{I_0} - 1\right], \quad (4.28)$$

with $I(T_M), I_0$ the inflation/CPI index respectively at time $T_M$ and $T_0$. In the YYIIS, at each time $T_i$, party B pays party A the fixed amount

$$N\phi_i K, \quad (4.29)$$

where $\phi_i$ denotes the fixed-leg year fraction for the interval $[T_{i-1}, T_i]$, and $N$ the nominal value of the YYIIS. In exchange, at each time $T_i$, party A pays party B the floating amount

$$N\psi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right], \quad (4.30)$$

where $\psi_i$ denotes the fixed leg year fraction for the interval $[T_{i-1}, T_i]$ ($T_0 := 0$).

Let $P_n$ and $P_r$ respectively denote the (zero-coupon) discount bond prices of the real and nominal economy, then standard no-arbitrage theory and some straightforward rewriting show that the price of an ZCIIS (zero-coupon inflation-indexed swap) can be expressed as

$$\text{ZCIIS}(t, T_M, I_0, N, K) = N\left[\frac{I(t)}{I_0} P_r(t, T_M) - (1 + K)^M\right], \quad (4.31)$$

which quantity is model-independent. That is, the above price is not based on any specific assumptions on the evolution of the (real and nominal) interest rates, but simply follows from the absence of arbitrage. This is an important fact, since it allows us to calibrate our model appropriately; more specifically, it allows us to strip, without ambiguity, real zero-coupon bond prices, from the quotes prices of ZCIIS. More specifically, given a set of market quotes of $K = K(T_M)$ at time $t = 0$, we can use equation (4.31) together with the net present value (4.27) to determine discount bonds of the real economy, i.e.

$$P_r(0, T_M) = P_n(0, T_M)(1 + K(T_M))^M. \quad (4.32)$$

A completely different story applies to the valuation of a YYIIS (year-on-year inflation-indexed swap), which in fact depends on the evolution of the underlying quantities and hence its price is model dependent; note that the value at time $t < T_{i-1}$ of the payoff (4.30) at time $T_i$ is

$$\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i \mathbb{E}_n\left\{ e^{-\int_{T_i}^{T_{i-1}} n(u) du} \left[ \frac{I(T_i)}{I(T_{i-1})} - 1 \right] \bigg| \mathcal{F}_i \right\}$$

$$= N\psi_i \mathbb{E}_n\left\{ e^{-\int_{T_i}^{T_{i-1}} n(u) du} \frac{I(T_i)}{I(T_{i-1})} \bigg| \mathcal{F}_i \right\} - N\psi_i P_n(t, T_i), \quad (4.33)$$

where $\mathbb{E}_n$ denotes the expectation under the nominal risk-neutral measure. We briefly comment
on why the latter expectation is model dependent, first notice that
\[
\mathbb{E}_n\left\{ e^{-\int_{T_i}^{T_{i-1}} n(u)du} \frac{I(T_i)}{I(T_{i-1})} | \mathcal{F}_t \right\} = \mathbb{E}_n\left\{ e^{-\int_{T_{i-1}}^{T_{i-1}} n(u)du} P_r(T_{i-1}, T_i) | \mathcal{F}_t \right\},
\] (4.34)

hence we can interpret the expectation from (4.33) as the nominal price of a derivative that pay-offs off (in nominal units), the real zero-coupon bond price \( P_r(T_{i-1}, T_i) \) at time \( T_i \). Alternatively we can also evaluate the latter expectation under a different measure, e.g. see Geman et al. (1996). Denote with \( Q^T_n \) as the nominal \( T \)-forward measure for some maturity \( T \) and let \( \mathbb{E}_T^T_n \) represent the expectation taken under this measure, then we can write (4.34) as:
\[
\mathbb{E}_n\left\{ e^{-\int_{T_{i-1}}^{T_{i-1}} n(u)du} P_r(T_{i-1}, T_i) | \mathcal{F}_t \right\} = P_n(t, T_{i-1}) \mathbb{E}_T^T_n\left\{ P_r(T_{i-1}, T_i) | \mathcal{F}_t \right\}.
\] (4.35)

If the nominal or real rates are deterministic, then this expectation would reduce to the present value (in nominal units) of the forward price of the real zero-coupon bond, i.e. we would then have
\[
P_n(t, T_{i-1}) \mathbb{E}_T^T_n\left\{ P_r(T_{i-1}, T_i) | \mathcal{F}_t \right\} = P_r(T_{i-1}, T_i) P_n(t, T_{i-1}).
\] (4.36)

However for inflation-linked derivative pricing purposes it is usually desirable (if not necessary) that real rates are stochastic, and the expectation of (4.33) is model dependent. In fact, if the nominal and real rates are correlated (and hence stochastic), the change of measure will change the drift of the real rate \( r(t) \) and hence also the expectation of (4.35). In interest rate terms, this effect is known under the term convexity adjustment, e.g. see Pelsser (2000) or Brigo and Mercurio (2006). For example if one assumes one-factor Gaussian rates (as in the JY model), one will see this convexity effect for any non-zero correlation coefficient between the nominal and real rates.

Finally note that we can also evaluate the expectation of (4.33) under the \( T_i \)-forward measure, i.e.
\[
\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N \psi_i P(t, T_i) \mathbb{E}_T^T_n\left\{ \frac{I(T_i)}{I(T_{i-1})} | \mathcal{F}_t \right\} - N \psi_i P_n(t, T_i).
\] (4.37)

This latter interpretation, which expresses the YYIIS (year-on-year inflation-indexed swap) as the \( T_i \)-forward expectation of the return on the inflation index, is very useful for our pricing methodology (see Section 4.3.1), because it expresses the price of a YYIIS in terms of the distribution of \( \frac{I(T_i)}{I(T_{i-1})} \) under the \( T_{i-1} \)-forward measure.

**Inflation-indexed caplets/floorlets**

An inflation-indexed caplet can be seen as a call option on the inflation rate implied by the inflation (e.g. CPI) index. Analogously, an inflation-indexed floorlet can be seen as put option on the same inflation rate. In formulas we can write the following for the payoff of an IICplt (inflation-indexed caplet/floorlet) at time \( T_i \)
\[
N \psi_i \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - 1 - \kappa \right) \right]^+,
\] (4.38)
4.3. Pricing and Applications

where \( N \) denotes the nominal value of the contract, \( \kappa \) the strike, \( \psi_i \) the year fraction for the interval \([T_{i-1}, T_i]\) and \( \omega = 1 \) for a caplet and \( \omega = -1 \) for a floorlet. Setting \( K := 1 + \kappa \), standard no-arbitrage theory implies that the value of the payoff (4.38) at time \( t \leq T_{i-1} \) is

\[
\mathcal{IIC}plt(t, T_{i-1}, T_i, \psi_i, K, N, \omega) = N \psi_i \left( e^{-\int_{T_{i-1}}^{T_i} n(u) du} \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \right) \mid \mathcal{F}_t \right) = N \psi_i \left( \frac{I(T_i)}{I(T_{i-1})} \left( \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right)^+ \right) \mid \mathcal{F}_t \right).
\]

Since (4.39) is equivalent to a call option on the forward return of the inflation index, the pricing of an inflation-indexed caplet/floorlet is thus very similar to that of a forward starting (cliquet) option.

**Pricing**

The crucial quantity for the pricing of the inflation-indexed derivatives in our model (4.8) is the log-return \( y(T_{i-1}, T_i) \) of the inflation index over the interval \([T_{i-1}, T_i]\) under the \( T_i \)-forward measure \( Q_{T_i}^T \), i.e.

\[
y(T_{i-1}, T_i) = \log \left( \frac{I(T_i)}{I(T_{i-1})} \right),
\]

and henceforth we assume that we explicitly know the characteristic function \( \phi_{T_{i-1}, T_i}(u) \) of the log-return under the \( T_i \)-forward measure, i.e.

\[
\phi_{T_{i-1}, T_i}(u) := \mathbb{E}^T_{n} \left[ \exp(i u y(T_{i-1}, T_i)) \mid \mathcal{F}_t \right].
\]

The derivations and explicit formulas of the characteristic function(s) are discussed in Section 4.4.

**Pricing of inflation-indexed swaps**

The main two inflation-indexed swaps are the ZCIIS and the YYIIS. Recall that the zero-coupon swap is model independent and is simply given by no-arbitrage arguments, i.e. by (4.31). Given the characteristic function \( \phi_{T_{i-1}, T_i}(u) \) from (4.41) of the log-inflation return under the \( T_i \)-forward measure, the pricing of a YYIIS is extremely simple. In fact recall from (4.37) that we have the following expression for the price of a YYIIS:

\[
\text{YYIIS}(t, T_{i-1}, T_i, \psi_i, N) = N \psi_i \mathbb{P}(t, T_i) \mathbb{E}_{n}^T \left( \frac{I(T_i)}{I(T_{i-1})} \right) \mid \mathcal{F}_t \right) - N \psi_i \mathbb{P}(t, T_i) \mid \mathcal{F}_t \right),
\]

and then note that the expectation in the above expression is nothing more than the characteristic function of the log-return evaluated in the complex-valued point \(-i\),

\[
\mathbb{E}_{n}^T \left( \frac{I(T_i)}{I(T_{i-1})} \right) \mid \mathcal{F}_t \right) = \mathbb{E}_{n}^T \left( \exp \left[ i(-i) \log \left( \frac{I(T_i)}{I(T_{i-1})} \right) \right] \right) \mid \mathcal{F}_t \right) = \phi_{T_{i-1}, T_i}(-i).
\]

63
Hence the price of a YYIIS is just given by the following simple expression:

\[ YYIIS(t, T_{i-1}, T_i, \psi_i, N) = N\psi_i P(t, T_i)\phi_{T_{i-1}, T_i}(-i) - N\psi_i P_n(t, T_i). \]  

\[ (4.44) \]

**Pricing of inflation-indexed caplets/floorlets**

The pricing of forward starting options like cliquets, attracted recent attention of both practitioners as well as from academia (e.g. see Lucic (2003), Hong (2004) and Brigo and Mercurio (2006)). In this section we will show how one can price inflation call options in the framework of Carr and Madan (1999). Working under the \( T_i \)-forward measure, we are in particular interested in the \( T_i \)-forward log return \( y(T_{i-1}, T_i) \) on the inflation index between the times \( T_{i-1} \) and \( T_i \), i.e. as defined in 4.40. From (4.39) we know that we can express an inflation caplet as a call option on the forward return of the index. We can then place this directly in the Carr and Madan (1999) methodology of Section 4.3.1. Provided that \( \alpha > 0 \) for a caplet and \( \alpha > 1 \) for a floorlet, we can write the following for the price of an IICplt (inflation-indexed caplet):

\[
\text{IICplt}(t, T_{i-1}, T_i, \psi_i, K, N, \omega) = N\psi_i P_n(t, T_i) \mathbb{E}_n^{F_t}
\left\{ \left[ \omega \left( \frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \right\}
\]

\[ = N\psi_i P_n(t, T_i) \frac{1}{\pi} \int_0^\infty \text{Re}\left[ e^{-i(\omega + iv)\log K} \Psi_{T_{i-1}, T_i}(v, \omega, \alpha) \right] dv \]  

\[ (4.45) \]

with \( \Psi_{T_{i-1}, T_i}(v, \omega, \alpha) \) in (4.26) a function of the characteristic function \( \phi_{T_{i-1}, T_i}(u) \) of (4.41). Alternatively the price of a floorlet can be expressed in terms of the corresponding caplet price (and vice versa) by means of a put-call parity, e.g. see Mercurio (2005). Given that we know the characteristic function, formula (4.45) provides an efficient and accurate way for determining the prices of inflation-indexed caps/floors. What remains is the derivation of this forward characteristic function, which we will discuss in Section 4.4. The corresponding characteristic functions can be found in Propositions 4.4.2 (for Schöbel and Zhu (1999) volatility) and 4.4.8 (for Heston (1993) volatility).

**4.3.3 FX and stock derivatives**

The pricing of FX and stock derivatives within the general model (4.8) can be done using similar techniques as in the previous section with inflation-indexed derivatives. The main difference is that inflation-indexed derivatives are usually forward-starting options, whereas the vanilla FX and stock options do not share this feature. In a way, one can therefore treat FX and stock options within the FX setup of our (4.8) as nested (degenerate) cases of inflation derivatives by choosing the forward-starting date equal to the current date and normalising the stock/index price by \( I(0) \), i.e. in accordance with (4.40). In a similar spirit, one can see a stock option as a FX option in which the foreign instantaneous interest rate represents the stochastic (or deterministic) continuous dividend rate of the stock.
4.4 Characteristic function of the model

For clarity we provide the pricing formulas for FX and stock options: working under the $T$-forward measure, the pricing formulas require the conditional characteristic function

$$
\phi_T(u) := \mathbb{E}^{Q_T}[\exp(iuz(T))|F_T]
$$

(4.46)

of the log index/FX-rate/stock price $z(T) := \log I(T)$. Equipped with this characteristic function, the time-$T$ forward FX-rate $\text{FFX}(T)$ (i.e. with convexity adjustment when the foreign interest rates are stochastic) is given by

$$
\text{FFX}(T) = \mathbb{E}^{Q_T}[I(T)] = \phi_T(-i).
$$

(4.47)

Provided with the log-asset price characteristic function, one can immediately price a call/put option on the stock or FX-rate within ‘Fourier-inversion’ framework of Section 4.3.1. More specifically, one can directly substitute the characteristic function for $\phi_T$ into the pricing formulas (4.25)-(4.26). Completely analogously to inflation-indexed options, one can price forward-starting (cliquet) options on the forward return of the FX-rate/stock index by substituting the characteristic function $\phi_{T_{i-1},T_i}(u)$ of the forward log return (4.40) into the pricing equations (4.25)-(4.26). We will discuss the derivation of both these characteristic functions in the next section.

4.4 Characteristic function of the model

In this section we will turn to the derivation of the characteristic function of the log inflation return under the nominal $T$-forward measure $Q^T_n$. For an inflation index which is driven by a Schöbel-Zhu stochastic volatility process, we are able to derive a closed-form expression, whereas for the Heston stochastic volatility case we are able to approximate this characteristic function. Before turning to these derivations, we first turn to a volatility aspect of the inflation index and to the Gaussian interest rates, which treatment is common for both volatility choices.

Volatility driver and multi-factor Gaussian rates

To ease notation we introduce some matrix notation: let $\Sigma(t, T)$ denote the $1 \times (1 + K + M)$ column vector of ‘volatilities’ driving the Brownian motion of the $T$-forward inflation index, with corresponding $(1 + K + M) \times (1 + K + M)$ correlation matrix $R$, i.e.

$$
\Sigma(t, T) = \begin{bmatrix}
    \nu(t) \\
    \sigma^1_n(t)B^1_n(t, T) \\
    \vdots \\
    \sigma^K_n(t)B^K_n(t, T) \\
    -\sigma^1_r(t)B^1_r(t, T) \\
    \vdots \\
    -\sigma^M_r(t)B^M_r(t, T)
\end{bmatrix},
\quad
R = \begin{pmatrix}
    1 & \rho_{\nu,\nu} & \cdots & \rho_{\nu,s} & \rho_{\nu,s'} & \rho_{\nu,s''} \\
    \rho_{\nu,s} & 1 & \cdots & \rho_{s,s} & \rho_{s,s'} & \rho_{s,s''} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    \rho_{\nu,s'} & \rho_{s,s'} & \cdots & 1 & \rho_{s',s'} & \rho_{s,s''} \\
    \rho_{\nu,s''} & \rho_{s,s''} & \cdots & \rho_{s,s''} & 1 & \rho_{s'',s''} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
    \rho_{v,s''} & \rho_{s'',s'} & \cdots & \rho_{s'',s''} & \rho_{s',s'} & 1
\end{pmatrix},
$$

(4.48)
Hence we can write the following for the instantaneous variance \( \nu_f(t) \) of the inflation index under the \( T \)-forward measure:

\[
\nu_f^2(t) = \Sigma'(t, T)R \Sigma(t, T),
\]

with \( \Sigma' \) the transpose of \( \Sigma \). Another useful expression is the integrated variance of the multi-factor Gaussian rate process; we can write the following for the instantaneous variance \( \nu_{K, M}(t) \) of the sum of the rate processes:

\[
\nu_{K, M}^2(t, T) = \sum_{i=2}^{K+M+1} (\Sigma^{(i)}(t, T))^2 + 2 \sum_{i=2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i, j)} \Sigma^{(i)}(t, T) \Sigma^{(j)}(t, T)
\]

(4.50)

with \( \Sigma^{(i)} \) is the \( i \)-th element of the vector \( \Sigma(t, T) \) and where \( R^{(i, j)} \) denotes the element at row \( i \) and column \( j \) of the matrix \( R \). Note the shift in index, which is due the presence of the volatility driver \( \nu(t) \) in (4.48). For the integrated rate variance \( V_{K, M}(t, T) \) one has the following expression

\[
V_{K, M}(t, T) := \int_t^T \nu_{K, M}^2(u, T) du = \sum_{i=2}^{K+M+1} C^{(i, i)} + 2 \sum_{i=2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i, j)} C^{(i, j)},
\]

(4.51)

where \( C^{(i, j)} \) denotes the integrated covariance between the \( i \)-th and the the \( j \)-th element of the vector of rate volatilities \( \Sigma(t) \).

For the covariance \( C^{(2, K+M+1)} \) between the first and the \( K + M \)-th element, one for example has\(^2\)

\[
C^{(2, K+M+1)} := -\frac{\sigma_n^1 \sigma_M^1}{a_n^1 a_M^1} \left( (T - t) + \frac{e^{-\sigma_n^1(T-t)} - 1}{a_n^1} + \frac{e^{-\sigma_M^1(T-t)} - 1}{a_M^1} - \frac{e^{-(\sigma_n^1 + \sigma_M^1)(T-t)} - 1}{a_n^1 + a_M^1} \right).
\]

(4.53)

### 4.4.1 Schöbel-Zhu stochastic volatility

In this section we will determine the characteristic function (under the \( T \)-forward measure) of the forward log-inflation return \( y(T_{i-1}, T_i) \) between times \( T_{i-1} \) and \( T_i \). For this we first need to determine the characteristic function of the \( T \)-forward log-inflation rate \( z(T) \) for a general maturity \( T \). Building forth on the results of Chapter 3, in which the characteristic function for the one-factor Schöbel-Zhu-Hull-White model is derived, we will derive the characteristic function of the multi-factor model in the following subsection.

\(^2\)It is indeed possible to consider time-dependent parameters, in which case the covariance \( C^{(2, K+M+1)} \) is given by the following time-dependent integral expression

\[
C^{(2, K+M+1)} := \int_t^T (\sigma_n^1(u) B_n^1(u, T)) (-\sigma_M^1(u) B_M^1(u, T)) du.
\]

(4.52)

We can do this for all formulas in this chapter. However as the resulting integral expressions become obscure, whilst the generalization is obvious, we use constant parameters for clarity of exposition.
4.4. Characteristic function of the model

Characteristic function of the log-index price

We will now determine the characteristic function of the reduced system (4.23), for which we shall use a partial differential equation approach. Recall from (4.22) that $z(t) := \log I_F(t)$ is defined as the $T$-forward log-asset price; subject to the terminal condition

$$f(T, z, \nu) = \exp\left(iuz(T)\right). \quad (4.54)$$

The Feynman-Kac theorem implies that the expected value of $\exp(iuz(T))$ equals the solution of the Kolmogorov backward partial differential equation for the joint probability distribution function $f(t, z, \nu)$, i.e.

$$f := f(t, z, \nu) = \mathbb{E}^Q\left[\exp(iuz(T))|\mathcal{F}_t\right]. \quad (4.55)$$

Thus the solution for $f$ equals the characteristic function of the forward asset price dynamics (starting from $z$ at time $t$). To obtain the Kolmogorov backward partial differential equation for the joint probability distribution function $f = f(t, y, \nu)$, we need to take into account the covariance term $dz(t)dv(t)$ which equals

$$dz(t)dv(t) = \left(\nu(t) + \Sigma_n(t, T) - \Sigma_b(t, T)\right)dw^*(t)\left(\tau dw^T(t)\right)$$

$$= \left(\rho_1 t\nu(t) + \tau R_{\nu, B}(t) - \tau \rho_{\nu, B}(t)\right)dt. \quad (4.56)$$

The model we are considering is not an affine model in $z(t)$ and $\nu(t)$, but it is if we enlarge the state space to include $\nu^2(t)$:

$$dz(t) = -\frac{1}{2}\nu^2(t)dt + \nu(t)dw^*(t) \quad (4.57)$$

$$dv(t) = \kappa \left[\xi(t) - \nu(t)\right]dt + \tau dw^T(t) \quad (4.58)$$

$$dv^2(t) = 2\nu(t)dv(t) + \tau^2 dt = 2\kappa \left(\frac{\tau^2}{2\kappa} + \xi(t)\nu(t) - \nu^2(t)\right)dt + 2\tau\nu(t)dw_i(t). \quad (4.59)$$

Using (4.57) and (4.56), we obtain the following partial differential equation for $f(t, z, \nu)$:

$$0 = f_t - \frac{1}{2}\nu^2(t)f_{zz} + \kappa \left(\xi(t) - \nu(t)\right)f_{z} + \frac{1}{2}\nu^2(t)f_{zz}$$

$$+ \left(\rho_1 t\nu(t) + \tau R_{\nu, B}(t) - \tau \rho_{\nu, B}(t)\right)f_{zz} + \frac{1}{2}\tau^2 f_{vv}. \quad (4.60)$$

Solving this partial differential equation, subject to the terminal boundary condition (4.54), provides us with the characteristic function of the forward asset price dynamics (starting from $z$ at time $t$). Due to the affine structure of our model, we come to the following proposition.

**Proposition 4.4.1** The characteristic function of the domestic $T$-forward log inflation-rate of the model with Schöbel and Zhu (1999) stochastic volatility is given by the following closed-form
solution:

\[ f(t,z,\nu) = \exp\left[A(u,t,T) + B(u,t,T)z(t) + C(u,t,T)\nu(t) + \frac{1}{2}D(u,t,T)\nu^2(t)\right], \]  

(4.61)

where:

\[ A(u,t,T) = -\frac{1}{2}u(i+u)V_{K,M}(t,T) \]

\[ + \int_0^T \left[ \left( \kappa\psi + (iu - 1) \sum_{i=1}^K \rho_{\psi,i}\tau\sigma_iB_i(t,T) - iu \sum_{j=1}^M \rho_{\psi,j}\tau\sigma_jB_j(t,T) \right)C(u,s,T) \right. \]

\[ \left. + \frac{1}{2}\tau^2\left(C^2(u,s,T) + D(u,s,T)\right) \right] ds, \]

(4.62)

\[ B(u,t,T) = B := iu, \]

(4.63)

\[ C(u,t,T) = \frac{u(i+u)}{\gamma_1} \left( \gamma_0 + e^{-2\gamma(T-t)} \right) \]

\[ + \sum_{i=1}^K \left[ \left( \gamma_3 - \gamma_4 e^{-2\gamma(T-t)} \right) - \left( \gamma_5 - \gamma_6 e^{-2\gamma+i\nu(T-t)} \right) \right] \]

\[ - \sum_{j=1}^M \left[ \left( \gamma_8 - \gamma_9 e^{-2\gamma(T-t)} \right) - \left( \gamma_{10} - \gamma_{11} e^{-2\gamma+i\nu(T-t)} \right) \right] \]

(4.64)

\[ D(u,t,T) = -u(i+u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1} \]

(4.65)

with \(V_{K,M}(t,T)\), as defined in (4.51), the integrated variance of the Gaussian rate processes. The constants are:

\[ \gamma = \sqrt{(\kappa - \rho_{\psi}\tau B)^2 - \tau^2(B^2 - B)}, \]

\[ \gamma_1 = \gamma + (\kappa - \rho_{\psi}\tau B), \]

\[ \gamma_3 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i\gamma}, \]

\[ \gamma_5 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i(\gamma - a_0^i)}, \]

\[ \gamma_8 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i\gamma}, \]

\[ \gamma_{10} = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i(\gamma - a_0^i)}; \]

\[ \gamma_4 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i\gamma}, \]

\[ \gamma_6 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i(\gamma - a_0^i)}, \]

\[ \gamma_9 = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i(\gamma + a_0^i)}, \]

\[ \gamma_{11} = \frac{\rho_{\psi,0}\tau a_0^i\gamma_1 + \rho_{\psi,i}\tau a_0^i\gamma_1}{a_0^i(\gamma + a_0^i)}; \]

\[ \gamma_{12} = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6), \]

\[ = \frac{\kappa\psi}{\gamma} \]

(4.66)
4.4. Characteristic function of the model

**Proof** See appendix 4.7.1.

In the following section we are able to derive the forward starting characteristic of the log-inflation index return, using the above characteristic function of the log-inflation index under the $T$-forward measure.

**Characteristic function of the log index return**

Recently the pricing of forward starting options attracted the attention of both practitioners as well as from academia e.g. see Lucic (2003), Hong (2004), Chapter 3 and in an inflation context Mercurio and Moreni (2006a) and Kruse (2007). In this section we will consider the pricing of forward starting options like inflation caplets within the general model setup combined with Schöbel-Zhu volatility. In particular, using the framework of Carr and Madan (1999), as described in Section 4.3.1, it suffices to know the characteristic function of the following log-inflation index return under the $T_i$-forward measure:

$$y(T_{i-1}, T_i) := \log \left( \frac{I(T_i)}{I(T_{i-1})} \right) = \log I(T_i) - \log I(T_{i-1}).$$  \hfill (4.67)

Since $I(t) := I_F(t) F_{t,T_i}^{P_i(t),P_{0,T_i}}$, we can also express this return in terms of the $T_i$-forward log inflation rate $z(t) := \log(I_F(t))$, i.e.

$$y(T_{i-1}, T_i) = z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_i(T_{i-1}, T_i).$$ \hfill (4.68)

We are interested in the characteristic function $\phi_{T_{i-1}, T_i}(u)$ of the log-inflation index return $y(T_{i-1}, T_i)$ under the $T_i$ forward measure, i.e.

$$\phi_{T_{i-1}, T_i}(u) := \mathbb{E}^Q \left[ \exp \left( iu(y(T_{i-1}, T_i)) \right) \right| F_T].$$ \hfill (4.69)

First define,

$$\Lambda := \exp \left( iu \left[ z(T_i) - z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_i(T_{i-1}, T_i) \right] \right) \hfill (4.70)$$

hence by using the tower law for conditional expectations and the (conditional) characteristic function of our model (4.61), we obtain the following expression for the characteristic function...
where the characteristic function (4.71) is of the following Gaussian-quadratic form:

\[
\phi_{T_{i-1}, T_i}(u) = \mathbb{E}_n^{T_i} \{ \Lambda | \mathcal{F}_i \} = \mathbb{E}_n^{T_i} \{ \mathbb{E}_n^{T_i} \{ \Lambda | \mathcal{F}_{T_{i-1}} \} | \mathcal{F}_i \} 
\]

\[
= \mathbb{E}_n^{T_i} \{ \exp \{ iu[-z(T_{i-1}) - \log \rho_n(T_{i-1}, T_i) + \log \rho_j(T_{i-1}, T_i)] \} \} 
\]

\[
\times \mathbb{E}_n^{T_i} \{ \exp \{ iu(z(T_{i-1})) | \mathcal{F}_{T_{i-1}} \} | \mathcal{F}_i \} 
\]

\[
= \exp \left( iu \left[ A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i) \right] + A(u, T_{i-1}, T_i) \right) 
\times \mathbb{E}_n^{T_i} \{ \exp \left( iu \left[ B_n(T_{i-1}, T_i)X_n(T_{i-1}) - B_r(T_{i-1}, T_i)X_r(T_{i-1}) \right] \right. 
\]

\[
\left. + C(u, T_{i-1}, T_i)\nu(T_{i-1}) + \frac{1}{2} D(u, T_{i-1}, T_i)\nu^2(T_{i-1}) \right) | \mathcal{F}_i \} 
\]

(4.71)

Though the latter expectation depends only on the (correlated) Gaussian variates \( x^j(T_{i-1}), x^j(T_{i-1}), \nu(T_{i-1}) \), we also have that the integrated volatility process \( \int_{t}^{T_{i-1}} \nu(u) du \) arises in the real rate processes \( x^j(T_{i-1}) \), e.g. see Proposition 4.2.2. To this end, we decompose \( x^j(T_{i-1}) \) into

\[
x^j(T_{i-1}) = V^j(T_{i-1}) + \tilde{x}^j(T_{i-1}) 
\]

(4.72)

\[
V^j(T_{i-1}) := \rho_{j; r} \sigma^j \int_{t}^{T_{i-1}} e^{-\sigma^j(T_{i-1} - u)} \nu(u) du 
\]

\[
\sim N \left( \mu^j(t, T_{i-1}), \sigma^j(T_{i-1}) \right) 
\]

(4.73)

\[
\tilde{x}^j(T_{i-1}) = \mu^j(t, T_{i-1}) + \sigma^j \int_{t}^{T_{i-1}} e^{-\sigma^j(T_{i-1} - u)} dW^j(u) 
\]

\[
\sim N \left( \mu^j(t, T_{i-1}), \sigma^j(t, T_{i-1}) \right) 
\]

(4.74)

where \( \mu^j(t, T_{i-1}), \sigma^j(t, T_{i-1}), \mu^j(t, T_{i-1}) \) and \( \sigma^j(t, T_{i-1}) \) as defined in (4.136), (4.137), (4.145) and (4.146), see appendix 4.7.2.

Hence we find that the characteristic function (4.71) is of the following Gaussian-quadratic form:

\[
\exp \left( iu \left[ A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i) \right] + A(u, T_{i-1}, T_i) \right) 
\times \mathbb{E}_n^{T_i} \{ \exp \left( iu \left[ B_n(T_{i-1}, T_i)X_n(T_{i-1}) - B_r(T_{i-1}, T_i)X_r(T_{i-1}) \right] \right. 
\]

\[
\left. + C(u, T_{i-1}, T_i)\nu(T_{i-1}) + \frac{1}{2} D(u, T_{i-1}, T_i)\nu^2(T_{i-1}) \right) | \mathcal{F}_i \} 
\]

\[
= \mathbb{E}_n^{T_i} \{ \exp \left[ a_0 + a^\prime Z + Z^\prime BZ \right] \}, 
\]

(4.75)
4.4. Characteristic function of the model

with \( a_0 \) a constant, \( a' \) a row-vector, \( B \) a matrix and where \( Z \) follows a multivariate standard normal distribution with correlation matrix \( S \). Thus the random vector \( Z \) consists of the \( 1 + K + 2M \) driving elements \( \nu, x^1_n, \ldots, x^K_n, x^1_r, \ldots, x^M_r, V^1, \ldots, V^M \). Note that since we are only dealing with one quadratic term (i.e. \( \nu^2(T_{t-1}) \)), we can reduce the quadratic form (4.75) of the random vector \( X \) to

\[
E_n^{T_{t_i}} \left\{ \exp \left[ a_0 + a'Z + b_0 Z^{(1)} \right] \right\},
\]

(4.76)

where the constants \( a_0, b_0 \), the column-vector \( a \) and the correlation matrix \( S \) of the standard Gaussian vector \( Z \), can be easily be deduced from (4.75) and are explicitly defined in appendix 4.7.2.

Using standard theory on Gaussian-quadratic forms (e.g. see Glasserman (2003) or Feuerverger and Wong (2000)) we can now easily find an explicit solution for (4.75). Recalling that (4.75) is equivalent to the characteristic function (4.71) of the forward return on the log inflation index, we come to the following proposition.

**Proposition 4.4.2** Let \( C \) be a matrix (with typical element \( c_{i,j} \)) satisfying \( C'C = S \) (e.g. by a Cholesky decomposition), define

\[
p_j := \sum_{i=1}^{1+K+2M} c_{i,j} a^{(i)} \quad \text{and} \quad q_1 := \sum_{i=1}^{1+K+2M} c_{i,1}^2 b_0,
\]

(4.77) \hspace{1cm} (4.78)

with correlation matrix \( S \), column-vector \( a \) and constant \( b_0 \) as defined in Appendix 4.7.2. The characteristic function of the forward log return \( y(T_{t-1}, T_i) \) (4.67) under the \( T_i \)-forward measure is given by the following closed-form solution:

\[
\phi_{T_{t-1}, T_i}(u) = \exp \left( a_0 + \frac{p_j^2}{2q_1(1 - 2q_1)} - \frac{p_j^2}{4q_1} + \frac{1 + K + 2M}{2} \frac{p_j^2}{\pi} \right) \frac{1}{\sqrt{1 - 2q_1}}.
\]

(4.79)

**Proof** Since (4.75) is equivalent to (4.71), the characteristic function of the forward return on the log inflation index is given by an explicit solution of the Gaussian-quadratic form (4.75), which is given by standard theory on quadratic forms, e.g. see Glasserman (2003) or Feuerverger and Wong (2000). □

Equipped with the characteristic function of the log-inflation index return, the prices of year-on-year inflation-indexed swaps and inflation-indexed caps/floors are directly obtained by the formulas (4.44) and (4.45).
4.4.2 Heston stochastic volatility

The characteristic function-based pricing method in our model with Heston (1993) stochastic volatility will turn out to be somewhat more involved than under Schöbel and Zhu (1999) stochastic volatility. In fact for the general model Heston (1993) stochastic volatility we need to resort to approximate methods for the pricing of inflation-indexed options.

Recall from (4.17) and (4.20) that the general model dynamics with Heston (1993) volatility under the $T$-forward measure $Q^n_T$ are given by

\[
\frac{dI_f(t)}{I_f(t)} = \left(\nu(t) + \Sigma_n(t, T) - \Sigma_r(t, T)\right)dW^T_n(t),
\]

\[
d\nu^2(t) = \kappa\left[\zeta(t) - \nu^2(t)\right]dt + \xi\nu(t)dW^T_v(t).
\]

To derive the characteristic function of the log-inflation rate, one can in principle then pursue the same steps as in the model with Schöbel and Zhu (1999) volatility, that is solving the Kolmogorov backward equation for the log-inflation rate with a certain boundary condition. However, due to the square-root volatility process, the Heston partial differential equation in combination with correlated Gaussian rates is unfortunately not affine any more. Hence, contrary to the previous model, there is (as far as we know) no exact closed-form expression for the characteristic function for this model. Nevertheless, in case we make the simplifying assumption that the rate processes are perpendicular to the stochastic volatility and the asset price processes, one can easily find an closed-form solution for its characteristic function. For the general case, we consider two alternative pricing methods

1. A projection of the characteristic function in the general model onto the uncorrelated case.

2. A control variate based Monte Carlo pricing technique that uses an exact result from the uncorrelated model.

The setup of the following section is therefore as follows: we first discuss the pricing for the log-inflation rate and the log-inflation index return in the model with uncorrelated Heston (1993) stochastic volatility. Then we show a projection technique of the general case onto the uncorrelated model. Finally, though the projection already works quite well, we also discuss the use of the approximate model as control variate in a Monte Carlo pricing procedure of the exact model.

Characteristic function of the log-index price: uncorrelated case

For the derivation of the characteristic function of the uncorrelated model (i.e. with rate processes perpendicular to the variance and asset price process), we will use two propositions. First of all, let $z(t) = \log \frac{P(t, T)}{P_0(t, T)}$ denote the $T$-forward log-asset price, with dynamics

\[
dz(t) = \frac{1}{2}\nu^2(t) + \nu(t)dW^T(t),
\]

\[
d\nu^2(t) = \kappa\left[\theta - \nu^2(t)\right]dt + \xi\nu(t)dW^T_v(t).
\]
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i.e. with stochastic interest rate dynamics. One then has following proposition regarding the characteristic function of \( z(t) \).

**Proposition 4.4.3** starting from time \( t \), the characteristic function \( \phi_{HE}(u) \) of the \( T \)-forward log-asset price \( z(T) \) of the classical Heston (1993) model is given by

\[
\phi_{HE}(u) := \exp \left[ iuz(t) + A_{HE}(u, t, T) + B_{HE}(u, t, T)\nu^2(t) \right]
\]

(4.84)

where:

\[
A_{HE}(u, t, T) := \theta \kappa \xi^{-2} \left( (\kappa - \rho \xi iu - d)T - 2 \log \left( \frac{1 - g_2 e^{-dT}}{1 - g_2} \right) \right),
\]

(4.85)

\[
B_{HE}(u, t, T) := \xi^{-2} (\kappa - \rho \xi iu - d) \frac{1 - e^{-dT}}{1 - g_2 e^{-dT}}
\]

(4.86)

and with:

\[
d := \sqrt{(\rho \xi iu - \kappa)^2 + \xi^2 (iu + u^2)},
\]

(4.87)

\[
g_2 := \frac{\kappa - \rho \xi iu - d}{\kappa - \rho \xi iu + d}.
\]

(4.88)

**Proof** For the proof we refer to Heston (1993) or Gatheral (2005). □

Note that in the literature one can find two (mathematically) equivalent formulations for the Heston characteristic function: the one presented above can for example be found in Albrecher et al. (2005) or Gatheral (2005) and is free of a numerical difficulty called branch cutting, while another representation can be found in the original Heston paper Heston (1993) or Kahl and Jäckel (2005), which may cause some numerical difficulties for certain model parameters, see Albrecher et al. (2005).

The second proposition concerns the interest rates part of the inflation dynamics. To this end, define

\[
R_{K,M}(t, T) := -\frac{1}{2} V_{K,M}(t, T) + \int_t^T \left[ \Sigma_i(u, T) dW^T_i(u) - \Sigma_r(u, T) dW^T_r(u) \right] du,
\]

(4.89)

we then come to the following proposition of the characteristic function of \( R_{K,M}(t, T) \).

**Proposition 4.4.4** The characteristic function of \( \phi_R(u) \) of \( R_{K,M}(t, T) \) (4.89) is given by

\[
\phi_{K,M}(u) := \exp \left[ -\frac{1}{2} u(i + u) V_{K,M}(t, T) \right].
\]

(4.90)

**Proof** As \( \int_i^T \Sigma_i(u, T) du, \ i \in n, r \) follows a Gaussian distribution with mean 0, \( R_{K,M}(t, T) \) as sum of Gaussian variates is also Gaussian with mean \( -\frac{1}{2} V_{K,M}(t, T) \). From Fubini and Itô’s isometry,
it follows that $R_{K,M}(t, T)$ is normally distributed with mean $-\frac{1}{2} V_{K,M}(t, T)$ and variance $V_{K,M}(t, T)$ as explicitly given by (4.51). Moreover, the characteristic function $\phi_{K,M}(u)$ of $R_{K,M}(t, T)$ follows directly as consequence of this normality. □

With the results from Propositions 4.4.3 and 4.4.4, we can now easily determine the characteristic function of the log-inflation index in the uncorrelated model, which results in the following proposition.

**Proposition 4.4.5** The characteristic function $\phi_F(u)$ for the log-inflation index $\log I_F(t)$ of the uncorrelated Heston dynamics (4.80) is given by the following closed-form expression:

$$
\phi_F(u) = \phi_{HE}(u) \cdot \phi_{K,M}(u) \quad (4.91)
$$

**Proof** Since the Brownian motions driving the Heston dynamics $z(t)$, i.e. $W^T_I(t)$ and $W^T_\nu(t)$, are uncorrelated with the Brownian motions that drive the rate process $R_{K,M}(t, T)$, i.e. $W^T_i(u)$ and $W^T_j(u)$, we have that we can write for the log-inflation index dynamics $\log I_F(t)$ of the dynamics of (4.23) (or equivalently of (4.80)) as the sum of the above two processes, i.e.

$$
\log I_F(t) = z(t) + R_{K,M}(t, T).
$$

Since the driving Brownian motions are uncorrelated, we then have that $z(t)$ is independent of $R_{K,M}(t, T)$ and furthermore that the characteristic function $\phi_F(u)$ of $\log I_F(t)$ is given by the product of the characteristic functions of $z(t)$ and $R_{K,M}(t, T)$. □

**Characteristic function of the log index return: uncorrelated case**

We will now derive the (forward-starting) characteristic function of the log-inflation index return. Just as in our model from Section 4.4.1, we follow Hong (2004) and Chapter 3; that is, we consider the characteristic function $\phi_{T_{i-1}, T_i}(u)$ of the log-inflation index return

$$
y(T_{i-1}, T_i) := \log \left( \frac{I(T_i)}{I(T_{i-1})} \right) = z(T_i) - z(T_{i-1}) - \log P_i(T_{i-1}, T_i) + \log P_i(T_{i-1}, T_i). \quad (4.92)
$$

In particular we want to solve the characteristic function $\phi_{T_{i-1}, T_i}(u)$ of $y(T_{i-1}, T_i)$ under the $T_i$-forward measure; using similar arguments (e.g. the tower law for conditional expectations) as in (4.71), we can obtain the following expression of the forward-starting characteristic function in
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our (uncorrelated) model:

\[
\phi_{T_{i-1}, T_i}(u) = \mathbb{E}_{T_i}^{T_i} \left\{ \exp \left( iu \left[ -z(T_{i-1}) - \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) \right] \right) \right\} \\
\cdot \mathbb{E}_{T_i}^{T_i} \left\{ \exp [iu z(T_i)] \left| \mathcal{F}_{T_{i-1}} \right| / \mathcal{F}_T \right\} \\
= \exp \left( A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2} u(i + u) V_{K,M}(T_{i-1}, T_i) \right) \\
\cdot \mathbb{E}_{T_i}^{T_i} \left\{ \exp \left( iu \left[ -\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) \right] \right) \right\} \\
\cdot \mathbb{E}_{T_i}^{T_i} \left\{ \exp \left[ B_{HE}(u, T_{i-1}, T_i) \nu^2(T_{i-1}) \right] \right\}. \quad (4.93)
\]

Hence since the rate processes \( x_n^i(T_{i-1}) \) and \( x_r^j(T_{i-1}) \) are independent of the variance process \( \nu^2(T_{i-1}) \), we have

\[
\phi_{T_{i-1}, T_i}(u) = \exp \left( A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2} u(i + u) V_{K,M}(T_{i-1}, T_i) \right) \\
\cdot \mathbb{E}_{T_i}^{T_i} \left\{ \exp \left( iu \left[ -\log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) \right] \right) \right\} \\
\cdot \mathbb{E}_{T_i}^{T_i} \left\{ \exp \left[ B_{HE}(u, T_{i-1}, T_i) \nu^2(T_{i-1}) \right] \right\}. \quad (4.94)
\]

Hence its remains to evaluate the expectations in the latter expression; since the first expectation can be seen as the characteristic function of the log-bond prices, we have the following proposition.

**Proposition 4.4.6** The characteristic function \( \phi_{K,M}(u) \) of the log bond prices in (4.94) under the \( T_i \)-forward measure is given by

\[
\phi_{K,M}(u) = \exp \left[ iuh_0 - \frac{u^2}{2} h' S_R h \right], \quad (4.95)
\]

with the constant \( h_0 \), column vector \( h \) and correlation matrix \( S_R \) respectively as defined in (4.159), (4.160) and (4.161).

**Proof** Note that one can write

\[
- \log P_n(T_{i-1}, T_i) + \log P_r(T_{i-1}, T_i) =: h_0 + h' Z_R, \quad (4.96)
\]

with \( Z_R \) the random Gaussian vector consisting of the normalized stochastic parts of the Gaussian factors \( x_n^1, \ldots, x_n^K, x_r^1, \ldots, x_r^M \). Therefore (4.96) is nothing more than the characteristic function of the normal distribution \( h_0 + h' Z_R \), which is given by expression (4.95). Alternatively, one can see this expectation as a special case of the Gaussian-quadratic form (4.75) of the model in Proposition 4.4.2, i.e. without the volatility components \( \nu(t) \) and \( V^j(t) \). \( \square \)
For the calculation of the second expectation of (4.94) we will use the following property of the square root process $\nu^2(T_{i-1})$.

**Proposition 4.4.7** The characteristic function $\phi_{\nu^2}(y)$ of $\nu^2(T_{i-1})$ is given by

$$\phi_{\nu^2}(y) = \mathbb{E}\left[\exp(iy\nu^2(T_{i-1}))\right] = \frac{\exp\left(\frac{cy \sqrt{\lambda_1}}{1-2cy}\right)}{(1-2cy)^{\frac{2\kappa}{\xi^2}}},$$  \hspace{1cm} (4.97)

where

$$c := \frac{\xi^2(1-e^{-\kappa(T_{i-1}-t)})}{4\kappa},$$  \hspace{1cm} (4.98)

$$\lambda := \frac{4\kappa e^{-\kappa(T_{i-1}-t)}\psi(s)}{\xi^2(1-e^{-\kappa(T_{i-1}-t)})}.$$  \hspace{1cm} (4.99)

**Proof** The proposition follows directly from the fact that variance process $\nu^2(T_{i-1})$ is distributed as a constant $c$ times a non-central chi-squared distribution with $\frac{4\kappa}{\xi^2}$ degrees of freedom and non-centrality parameter $\lambda$, e.g. see Cox et al. (1985). □

Hence we come to the following proposition for the characteristic function $\phi_{T_{i-1},T_i}(u)$ as in expression (4.94).

**Proposition 4.4.8** The forward-starting characteristic function $\phi_{T_{i-1},T_i}(u)$ of the model (4.8) with uncorrelated Heston (1993) stochastic volatility is given by the following closed-form expression:

$$\phi_{T_{i-1},T_i}(u) = \exp\left(A_{HE}(u, T_{i-1}, T_i) - \frac{1}{2}u(i+u)V_{K,M}(T_{i-1}, T_i)\right)\phi_{K,M}(u)\phi_{\nu^2}(-iB_{HE}(u, T_{i-1}, T_i)),$$  \hspace{1cm} (4.100)

with $A_{HE}(u, t, T_{i-1})$ and $B_{HE}(u, t, T_{i-1})$ as defined in equations (4.85) and (4.86), and with $\phi_{K,M}(u)$ and $\phi_{\nu^2}(u)$ as in proposition (4.4.6) and 4.4.7.

**Proof** The characteristic function (4.100) of the forward log-inflation index return follows directly by evaluating the two expectations of (4.94). The first expectation of (4.94) equals the characteristic-generating function $\phi_{K,M}(u)$ of the log bond prices (4.96). The second expectation equals the moment-generating function $\phi_{\nu^2}$ of the shifted non-central chi-squared distributed random variable $\nu^2(T_{i-1})$, evaluated in the point $B(u, t, T_{i-1})$. □

**Projection of the general case onto the uncorrelated model**

Since in the general Heston model setup (i.e. with a full correlation structure) the affine structure is destroyed, it is challenging to find the characteristic function of the log-inflation index; we are not aware of an exact closed-form expression for characteristic function in the Heston model with correlated Gaussian rates. Nevertheless one can try to approximate the general dynamics by a simpler process for which a closed-form pricing expression does exists. Where a heuristic approach based on moment-matching techniques was suggested by van Haastrecht (2007), a
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more rigorous projection method was recently suggested by Antonov et al. (2008), which uses a Markovian projection technique of the general model onto the (affine) uncorrelated model. After the projected parameters are determined, one can just use the uncorrelated model and corresponding pricing formulas to price stock, foreign exchange and inflation derivatives. Though the Markovian projection technique is fast and works well for mild parameter settings and short maturities (i.e. when the 'distance' between the models is relatively small), the projection is rather involved and deteriorates for longer maturities and more extreme model parameters (i.e. when the 'distance' is relatively large), in particular for a large index-rate correlation in combination with a high volatility of the rates. For details on the Markovian projection and numerical results of the approximation, we refer the reader to Antonov et al. (2008).

Monte Carlo pricing method for the general model

Instead of approximating the prices of vanilla options in the general Heston setup, e.g. by a projection technique as touched upon in subsection 4.4.2, one can also entail a Monte Carlo procedure to price these options. Where the approximation formulas can be rather biased for certain model settings (e.g. see the discussion in subsection 4.4.2), a Monte Carlo estimate has the advantage that it converges to the true option price in the limit for the number of sample paths. Moreover a Monte Carlo procedure is generic (i.e. is suitable for a whole range of exotic options) and is straightforward to implement (if not already implemented for exotic option pricing). The main practical disadvantage of a Monte Carlo calibration procedure, is the speed with which vanilla option prices can be calculated within some error measure. Since one repeatedly needs to update an error function of the 'distance' between model and market vanilla prices, the speed to calculate these model option prices is rather important. Even though one can price multiple options (e.g. on different times) with one Monte Carlo run, the use of closed-form option pricing formulas is often much faster. Nevertheless, with the use modern-day variance reduction techniques and the ever-growing computational power (in particular the fact that Monte Carlo procedure can be easily parallelized over multiple processors), we expect Monte Carlo techniques to become even more popular in the near future.

In this section we present an very effective control variate estimator for the pricing of vanilla options the general Heston dynamics. To demonstrate its efficiency, we take the pricing of a vanilla call option as example. To benchmark the numerical results against the Markovian projection, we consider the same hybrid equity-interest rate (stock) example as in Antonov et al. (2008). The setup of this section is as follows: we first discuss the control variate technique for the general model, after which we demonstrate which variance reductions can be expected and discuss its numerical efficiency.

Uncorrelated price as control variate estimator

As discussed in Section 4.4.2, Monte Carlo pricing procedures might be easy to implement and quite generic, but often lack of speed and are hence sometimes being considered as 'brute-force'. Nowadays, however, a whole variety of variance reductions techniques are available to boost the computational efficiency of the Monte Carlo run, e.g. see Glasserman (2003) or Jäckel (2002)
for an overview of such methods. One of these variance reductions techniques is the control variate estimator. The key idea behind this technique is that we can use the error in estimating a similar quantity (from which we know the theoretical value) as a control to correct for the Monte Carlo error for the unknown quantity, see Glasserman (2003). The effectiveness of such a control variate depends explicitly on the correlation between the control and the to be estimated price. Thus if the control contains many information of the estimated price, it can correct quite a lot of Monte Carlo noise in the resulting estimator (and vice versa). Mathematically, it can be shown that, if the correlation between control and the standard Monte Carlo estimator are correlated with correlation coefficient $\rho$ in combination with an optimal control parameter, one obtains (on average) a variance reduction of

$$ VR(\rho) = \frac{1}{1 - \rho^2}, $$

(4.101)

which is enormous for $\rho$ close to one, e.g. see Glasserman (2003).

Before turning to the control variate estimator, we first introduce some notation. Let $\overline{C^0}, C^v$ and $\overline{C^0}_\rho, C^v_\rho$ respectively denote the expected (European) call option price and the simulated call option prices for the general (superscript $\rho$) and the uncorrelated (superscript $0$) dynamics. Since we know the call option price $\bar{C}^0$ of the uncorrelated price in closed-form by inverting (4.100), and usually this price is largely correlated with the call option price $C^\rho$ of the general model, we propose to use $\bar{C}^0$ as a control for $C^\rho$. Since the prices are highly correlated, we expect to see large variance reductions of the control variate estimator $\overline{C^v}(b)$ over the ordinary estimator $C^v$, i.e. from formula (4.101). The resulting control variate estimator $\overline{C^v}(b)$ is given by

$$ \overline{C^v}(b) = \frac{1}{n} \sum_{i=1}^{n} (C^v_i - b(C^0_i - \mathbb{E}[C^0])), $$

(4.102)

where $b$ is a real coefficient. The optimal coefficient $b^*$ that minimizes the variance of (4.102) can easily be calculated and is explicitly given by

$$ b^* = \frac{\sigma C^\rho}{\rho C^\rho C^0} = \frac{\text{Cov}[C^0, C^\rho]}{\text{Var}[C^0]}. $$

(4.103)

Note that one often also needs to estimate $b^*$ from the simulations and this might induce some bias in the effectiveness (4.101) of the control variate. However, as discussed in Glasserman (2003), this bias is often very small; in case $\rho_{C^0,C^\rho}$ is close to one and $\sigma_{C^\rho} \approx \sigma_C$ (which more than often is the case), it might even be a more efficient to just set $b^*$ equal to one (since one does not have to estimate $b^*$, see Glasserman (2003). In Section 4.5.1 the quality of the control variate estimator is investigated.
4.5 Applications and Numerical Results

In this section, we look at two applications of the model; first, for an equity example and with Heston (1993) stochastic volatility, we test the quality of the control variate estimator \( \tilde{C} \rho \) of (4.102), compare it to the Markovian projection technique of Antonov et al. (2008) and discuss its practical applicability in a Monte Carlo calibration and/or pricing procedure. Secondly, we consider two applications (one with Schöbel and Zhu (1999) and one with Heston (1993) stochastic volatility) in which we calibrate our model to FX (option) market data. The example explicitly takes into account the pronounced long-term FX implied volatility skew/smile that is present in the markets. Finally the results are compared and analyzed.

4.5.1 Quality of the control variate estimator

To test the numerical quality of the control variate estimator \( \tilde{C} \rho \) of (4.102), we turn to the pricing of (European) call options under the general hybrid Heston dynamics. To this end we consider two different parameter settings, listed in Table 1 below.

| Example | \( \kappa \) | \( \xi \) | \( \rho_{1,\nu^2} \) | \( v(0) \) | \( \theta \) | \( y_r \) | \( y_q \) | \( \sigma_n \) | \( \rho_{1,\nu^1} \) | \( \rho_{\nu^2,\nu^1} \) |
|---------|---|---|---|---|---|---|---|---|---|---|---|
| case I  | 2.0 | 1.0 | -0.3 | 0.09 | 0.09 | 0.04 | 0.0 | 0.03 | 0.007 | \( \ast \) | 0.0 |
| case II | 0.25 | 0.625 | -0.4 | 0.0625 | 0.0625 | 0.05 | 0.02 | 0.05 | 0.01 | 0.30 | 0.15 |

Table 1: Numerical test cases for the Control Variate estimator (4.102). \( y_r \) denotes the continuous (constant) interest rate yield, \( y_q \) the continuous (constant) dividend yield, the \( \ast \) indicates that we vary this parameter during the experiments and in all cases \( I(0) = 100 \).

Both test cases roughly correspond to parameter settings that are likely to be encountered in medium to long maturity equity markets. The first test case is prevalent in the existing literature: similar Heston parameter settings, in a pure equity context, are considered in Broadie and Kaya (2006), Lord et al. (2008) and Andersen (2008). The second test case is taken from Antonov et al. (2008) wherein it serves to test their Markovian projection approximation, i.e. as touched upon in Section 4.4.2. Using these test cases, we first look at the quality of the control as function of the equity rates correlation coefficient and secondly we investigate the efficiency the control variate estimator (4.102) as function of the option maturity and compare it with the Markovian Projection technique of Antonov et al. (2008).

Results for case I

Though the uncorrelated price is often highly correlated with the price of the general model, the efficiency is dependent on the specific model parameters. For example notice that for \( \rho_{1,\nu^2} = \rho_{\nu^2,\nu^1} = 0\% \), the control variate estimator is exact, because in that case the uncorrelated price equals the required estimate. Though the effectiveness depends on both correlation parameters, the impact of the correlation rate-vol is usually much smaller than the impact of the rate-stock correlation, e.g. see Antonov et al. (2008) or Chapter 3. Moreover, from a practical point of
view, the rate-stock parameter is most important for the pricing and hedging of hybrid equity-interest rate securities. We therefore restrict ourselves to investigate the impact of the rate-stock parameter on the quality of the control variate estimator: we look at the (empirical) variance reductions for a three year call option with an ATMF (at-the-money-forward) strike level of 100% for different $\rho_{1,x^*_1}$. The results can be found in Table 2 below.

<table>
<thead>
<tr>
<th>$\rho_{1,x^*_1}$</th>
<th>$\hat{\rho}_{C,C'}$</th>
<th>$b$</th>
<th>Var. Red.</th>
<th>$\rho_{1,x^*_1}$</th>
<th>$\hat{\rho}_{C,C'}$</th>
<th>$b$</th>
<th>Var. Red.</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>99.859%</td>
<td>0.960</td>
<td>356</td>
<td>0.9</td>
<td>99.864%</td>
<td>1.034</td>
<td>367</td>
</tr>
<tr>
<td>-0.8</td>
<td>99.911%</td>
<td>0.965</td>
<td>562</td>
<td>0.8</td>
<td>99.913%</td>
<td>1.031</td>
<td>574</td>
</tr>
<tr>
<td>-0.7</td>
<td>99.940%</td>
<td>0.970</td>
<td>839</td>
<td>0.7</td>
<td>99.941%</td>
<td>1.027</td>
<td>852</td>
</tr>
<tr>
<td>-0.6</td>
<td>99.960%</td>
<td>0.974</td>
<td>1254</td>
<td>0.6</td>
<td>99.961%</td>
<td>1.024</td>
<td>1268</td>
</tr>
<tr>
<td>-0.5</td>
<td>99.974%</td>
<td>0.979</td>
<td>1937</td>
<td>0.5</td>
<td>99.974%</td>
<td>1.020</td>
<td>1950</td>
</tr>
<tr>
<td>-0.4</td>
<td>99.984%</td>
<td>0.983</td>
<td>3188</td>
<td>0.4</td>
<td>99.984%</td>
<td>1.016</td>
<td>3202</td>
</tr>
<tr>
<td>-0.3</td>
<td>99.992%</td>
<td>0.987</td>
<td>5888</td>
<td>0.3</td>
<td>99.992%</td>
<td>1.012</td>
<td>5902</td>
</tr>
<tr>
<td>-0.2</td>
<td>99.996%</td>
<td>0.992</td>
<td>13597</td>
<td>0.2</td>
<td>99.996%</td>
<td>1.008</td>
<td>13614</td>
</tr>
<tr>
<td>-0.1</td>
<td>99.999%</td>
<td>0.996</td>
<td>55209</td>
<td>0.1</td>
<td>99.999%</td>
<td>1.004</td>
<td>55252</td>
</tr>
<tr>
<td>0</td>
<td>100%</td>
<td>1</td>
<td>$\infty$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2: Expected variance reductions when using the Control variate estimator of (4.102) instead of the standard Monte Carlo estimator. For various values of $\rho_{1,x^*_1}$ the expected reduction for a three-year call option with an at-the-money strike is calculated using the estimates $\hat{b}$ and $\hat{\rho}_{C,C'}$, respectively for the optimal control coefficient and correlation between the control and the estimated quantity. Parameter settings from case I of Table 1. Results based on 50,000 pseudo-random paths.

From the above table (the case $\rho_{1,x^*_1} = 1$ does not constitute in a valid correlation matrix and is hence omitted), we can see that the control is in all cases very effective, i.e. resulting in large to huge variance reductions. As expected, the variance reductions become larger for smaller absolute values of $\rho_{1,x^*_1}$; for the case $\rho_{1,x^*_1} = 0$, the control is perfect and results in a zero variance control variate estimator, whereas for larger values of $|\rho_{1,x^*_1}|$ the correlation between the 'uncorrelated' and 'correlated' option prices is smaller and therefore reduces the effectiveness of the control, as is being theoretically underpinned by formula (4.101). Thus from Table 2 we can see that the effectiveness of the control, i.e. the resulting variance reduction, depends to a large extent on the absolute value of the correlation 'between' interest rates and equity underlying. Finally, it is worthwhile mentioning that because the $\rho_{C,C'}$ and $\hat{\sigma}_{C',C}$ the (estimated) optimal coefficients $\hat{b}$ are also close to one. In such a situation it might be more efficient to just set $b^* = 1$ and consequently save the computational effort in estimating $\hat{\rho}_{C,C'}$, see Glasserman (2003).

Results for case II

The second test case of Table 1, consists of an experiment where investigate the variance reductions of (4.102) over the standard Monte Carlo estimator for European call options of different maturities and strikes. Furthermore, since the same parameters are being used in Antonov et al.
4.5. Applications and Numerical Results

(2008), we can use these results to draw a comparison between the Monte Carlo control variate estimator and the Markovian projection technique. The numerical results can be found in Table 3.

From that table, we can see that the control variate estimator by far outperforms the ordinary Monte Carlo estimator; for short to moderate maturity options the control variate shows large to huge variance reduction factors varying from 629 to 7938. For middle to long term option options, the variance reductions are smaller, but still quite reasonable with reductions from 54 to 371. If we look at the variance reductions over different strike levels, the differences are somewhat smaller. It is worthwhile to notice that, for a fixed maturity, the control variate is most effective for out-of-money options, which are usually the hardest options to value by (plain) Monte Carlo.

We conclude the analysis, by comparing the Monte Carlo control variate estimator (4.102) with the Markovian Projection technique. The results of the best projection technique of Antonov et al. (2008) is denoted by Heston DV (displaced volatility) and can be found in the fourth column of Table 3.

A crucial difference between the simulation and MP method is that the MP technique is in principle a biased approximation, whereas the control variate is unbiased and converges to the true price. However, in practice one often only has a limited available computational budget and one will also note bias in the Monte estimates as a consequence of the limited number of simulations; this bias could be larger than the error in the approximation. Essentially the choice between both methods therefore constitutes of a tradeoff between speed and accuracy, which might differ across applications and products.
Table 3: Variance reductions for case I of Table 1 using 50000 pseudo-random paths. Reported is the variance reduction factor ('Var. Red.'), i.e. the fraction between the variance of the control variate (4.102) and the standard Monte Carlo estimator. The starred results, i.e. from the simulated volatility and standard deviations ('Sim. Vol. (std. dev.)') and the errors of the Heston DV Markovian projection ('MP error'), were taken from Antonov et al. (2008).

Using the numerical results of Table 3, let us consider the concrete example of pricing a ten-year ATMF call option. For the sake of the argument we assume here that the Monte Carlo volatility of 18.01 is in fact the true volatility and hence the Markovian Projection error is 0.10. We can then ask ourselves how many simulations are needed to improve the error of this approximation in at least 90% of the cases. By definition, 90% of all the spanned confidence intervals should contain the 'true' price of 18.01, hence to improve the MP error, we should aim to get the standard deviation of the Monte Carlo estimated volatility smaller than $\sqrt{\frac{0.10}{\Psi^{-1}(0.955)}} = 0.061$ with $\Psi^{-1}$ the inverse of a standard Gaussian distribution. Using the fact that the Black and Scholes (1973) ATMF price is close to linear as a function of the volatility, taking the standard deviation 0.08
of the simulated volatility and the variance reduction factor 108 of the above table and assuming a convergence rate of the Monte Carlo of one over square root of the number of simulations $N$, one can find that one needs

$$M = \frac{\text{Var}_N}{\text{Var}_{\text{REQ}}} \frac{N}{VR} = \frac{0.08^2 \times 50000}{0.061^2 \times 108} = 802$$

simulations to improve upon the MP error in 90% of the cases, with VR the variance reduction factor and where Var_{REQ} represents the required variance corresponding to a confidence level $1 - \alpha = 90\%$. Should we for example take $\alpha = 50\%$, one can find that on average one only has to use 134 simulations to perform ‘equally well’ as the MP projection. Hence due to the large variance reductions, only a very moderate amount of simulations is needed to come up with a good estimate. Though the above analysis is too small to draw very strong conclusions about the comparison between the MP projection technique and the control variate, the main conclusion we do like to draw is that only a moderate amount of simulations is required to obtain reliable price/volatility estimates for the above call options: in most situations a couple of thousand paths will suffice to obtain prices that lie within typical bid-ask spreads.

Finally we would also like to point out that the MP projection might also be used in conjunction with the control variate estimator (4.102) in a model calibration procedure; a first point (in future research) could be to investigate the quality of the MP projection as control for the exact dynamics. Secondly, in a practical implementation one might first use the MP approximation to calibrate the model (which consists of most of the iterations) and consecutively use the control variate to refine the (near) optimal parameters found in the previous minimization. Please note hereby that (for each new parameter guess) one only needs a single Monte Carlo run to price all options simultaneously. In this way (assuming one uses a sufficiently large number of paths in the last few optimization steps using the Monte Carlo) one can get the best of both worlds, i.e. the speed of an approximating formula combined with the accuracy of the control variate estimator.

### 4.5.2 Calibration to FX market

We will test our model by calibrating it to FX option market data. To this end, we consider the same vanilla FX data (see appendix 4.7.3) as is considered in Piterbarg (2005) who uses this set for the calibration of his local volatility model. In an elegant paper, Piterbarg (2005) concludes that for the pricing and managing of exotic FX derivatives (i.e. PRDCs), it is essential to take the FX implied volatility skew/smile into account; hence though FX model setups may differ, i.e. local volatility in Piterbarg (2005), Heston (1993) stochastic volatility with independent stochastic interest rate drivers in Andreasen (2006) and our stochastic volatility model with multi-factor Gaussian rates and Heston (1993) or Schöbel and Zhu (1999) volatility under a full correlation structure, all these models share the essential feature of explicitly accounting for the FX skew/smile.

For the calibration results of our model we consider the same interest rate and correlation parameters as in Piterbarg (2005); that is, the interest curves in the domestic (Japanese yen) and foreign...
(US dollar) economies are given by

\[
P_n(0, T) = \exp(-0.02 \cdot T),
\]

\[
P_r(0, T) = \exp(-0.05 \cdot T),
\]

and the one-factor Hull and White (1993) interest rate parameters for the interest rate evolutions in both currencies are given by

\[
a_n(t) := 0.0\%, \quad \sigma_n(t) := 0.70\%, \]

\[
a_r(t) := 5.0\%, \quad \sigma_r(t) := 1.2\%.
\]

The correlation parameters are given by

\[
\rho_{n,r} = 25.00\%, \quad \rho_{I,n} = \rho_{I,r} = -15.00\%, \quad \rho_{n,v} = \rho_{r,v} = 0.00\%.
\]

Note that our stochastic volatility model has the additional flexibility of correlating the domestic of foreign exchanges with the volatility drivers (i.e. through \(\rho_{n,v}\) or \(\rho_{r,v}\)), however for simplicity we fix them on zero here. The initial spot FX rate (yen per dollar) is set at 105.00. The ten expiry dates and the seven strikes that are considered for the calibration, are given in Table 4 of appendix 4.7.3. For each maturity \(T_n\), the strikes \(K_i(T_n)\) are being computed using the formula

\[
K_i(T_n) = F(0, T_n) \exp(0.1 \cdot \delta_i \sqrt{T_n}), \quad \delta_i \in \{-1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5\}. \quad (4.104)
\]

In particular, note that the fourth strike level corresponds to the forward FX rate for that date. The implied volatilities corresponding to the above strikes and maturities can be found in Table 5 of appendix 4.7.3. With the above setup, we consider in the next section how well the models (4.8), i.e. with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility, fit the market implied volatilities of Table 5.

**Calibration results**

We calibrate the models (4.8) with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility to the various maturities by minimising the differences between model and market implied volatilities by using a local optimization method. The differences are reported in Table 6 and 7 below of Appendix 4.7.3. For visual comparison, we represent the calibration results for a few maturities in the graphs below.
4.5. Applications and Numerical Results

Figure 1: Calibration results for the model with Schöbel-Zhu and Heston stochastic volatility. For the maturities 0.5, 5, 20 and 30 the implied volatilities (vertical axis) are plotted against the corresponding strikes (horizontal axis). The market data is represented in blue, the model (4.8) with Schöbel-Zhu volatility in red and the model (4.8) with Heston volatility in green.

We first consider the model (4.8) with Schöbel and Zhu (1999) stochastic volatility. The model produces a good fit to the market, as can be seen from Table 6 and Figure 1, with differences smaller than 0.50% in most points and with a good fit around the at-the-money-forward volatilities and the slope of the volatility skews for each maturity. The model produces similar calibration results as the models of Piterbarg (2005) and Andreasen (2006). The low-strike (in-the-money call) options are underestimated by the model, which seems to have slight difficulties in fitting the tails of the implied volatility structure, suggesting the addition of an extra factor, e.g. a trivial extension including Poisson-type jumps. Nonetheless, the smiles produced by the model are much closer to the market than a log-normal model would indicate, in particular for in- and out-the-money options.

Secondly, we consider the model (4.8) with Heston (1993) stochastic volatility. For simplicity
we have considered uncorrelated stochastic volatility, as we can then directly price the required FX options in closed form. Nonetheless, the calibration results to call option prices should be very similar as it is shown in Antonov et al. (2008), that the parameters of the general model can often be projected onto parameters of the uncorrelated model, while to a large extent preserving option prices and model characteristics. The calibration results can be found in Figure 1 and in Table 7 of appendix 4.7.3. From this, we can see that the model again produces a very good fit to the market, with differences now even smaller than 0.30% in most points and with excellent fits across moneyness and maturities. It seems that Heston (1993) model is slightly better in fitting extreme/convex FX skew we calibrating against; in a way is able to capture both the volatility part of the at-the-money prices, as well as the extremes of the in- and out-the-money prices. Alternatively, one can argue that the addition of an extra factor is still needed for the pricing of certain exotic options (e.g. see van der Ploeg (2006) and Fouque et al. (2000)), which discussion is however beyond the scope of this article.

As shown in Piterbarg (2005) and Andreasen (2006), it is of crucial importance to take the FX skew into account for the pricing and managing of exotic FX structures like PRDCs (power reverse dual contracts) or cliquets. Therefore, since the skews/smiles generated by our stochastic volatility models are much closer to the market than produced by a log-normal model, we can conclude our stochastic volatility model(s) (4.8) is better suited to price and manage these exotic FX structures. Finally, though the models of Piterbarg (2005) and Andreasen (2006) account for the FX skew, our model stands out as we model stochastic volatility (versus local volatility used in Piterbarg (2005)) and stochastic interest rates, whilst we allow all driving model factors to be instantaneously correlated with each other (versus independent Gaussian rates used in Andreasen (2006)). Having this flexibility yields a realistic model, which is of practical importance for the pricing and hedging of options with a long-term FX exposure.

Given data on FX prices our model can also be used to examine the pricing and hedging performance of products which explicitly depend on future volatility smiles, such as barrier FX options. An empirical study on the relative performance of the stochastic volatility models discussed here versus other models for FX, as well as the calibration and pricing of liquid vol-sensitive instruments such as double-no-touch options (e.g. see Kainth and Saravanamuttu (2007)), is beyond the scope of this thesis, and is left for future research.

### 4.5.3 Calibration to Inflation Markets

In a recent paper, Mercurio and Moreni (2009) consider the pricing of inflation-indexed year-on-year and zero-coupon caps/floors, using a market model with SABR Hagan et al. (2002) stochastic volatility dynamics for year-on-year inflation rates and a lognormal Libor Market model for Nominal interest rates. Other market model approaches for inflation can for instance be found in Belgrade et al. (2004), Kenyon (2008) and Brigo and Mercurio (2006). Compared to the latter models, the approach considered in Mercurio and Moreni (2009) stands out by reconciling both zero-coupon and year-on-year quotes. Similar to the framework considered in this paper, these authors consider a full correlation structure between the stochastic quantities underlying the model, whilst preserving closed-form and flexible calibration methods for calibration to market option data.
4.5. Applications and Numerical Results

Differences between market models and low dimensional Markov models, as considered in this chapter, have been described by several authors, e.g. see Brigo and Mercurio (2006), Pelsser (2000); market models explicitly model observable quantities (e.g. Year-on-Year inflation rates), and due to their dimensionality provide a larger calibration flexibility compared to low-dimensional Markov models. On the other hand, the dimensionality of such models can also be disadvantageous. For instance, due to a lack of calibration instruments in less liquid markets (such as inflation options), hedges and calibrations may become unstable when using market models, e.g. see Jäckel and Bonneton (2010). These models can also be relatively slow compared to low-dimensional Markov models, e.g. see Glasserman (2003). In this sense, both market and low dimensional market models show (dis)advantages and the model choice ultimately depends on the exotic product one wants to price.

One clear objective to judge the quality of a model is of course its calibration flexibility. We will therefore look at calibrations of the model (4.8) with Schöbel and Zhu (1999) stochastic volatility, with one-factor rates for the nominal economies, to inflation-indexed caplets and floorlets. To test the calibration of this model, we use the same market data and zero correlation assumptions as in the first case of Mercurio and Moreni (2009). Adopting this setup has the additional advantage that it enables us to draw a comparison between these methods, for a further description of the market data we refer the reader to that paper. Calibration results are shown in Figure 2.

![Figure 2: Calibration results: market and model implied volatilities for caplets/floorlets maturing in 3, 5, 7, 10, 15 years. Calibration results for the model the model (4.8) with Schöbel-Zhu volatility. Data for September 4th 2008, corresponding with the uncorrelated case of Figure 1 in Mercurio and Moreni (2009).](image)

We can see from Figure 2, in which market and model implied volatilities are reported, that the fit is accurate. We note that the market data displays small non-smooth behaviours where cap and
floor quotes meet (strikes 2% – 2.5%) or on single strikes (e.g. the 7-years 0%-floorlet). Similar to Mercurio and Moreni (2009), we consider these discrepancies as being essentially bound to liquidity reasons and stress that a parameterized models also provide useful smoothing tools for such market data. The calibration results of the model with Schöbel-Zhu and SABR stochastic volatility are very similar: both models are well able to qualitatively fit the shape of the implied volatility, whilst they are also capable to detect small market anomalies. Because the liquidity of inflation options is not that large, as can be seen from wide bid/ask spreads for inflation caps and floors, such smoothing of market data might be very useful, as indicated Mercurio and Moreni (2009) and Jäckel and Bonneton (2010). We therefore conclude that whilst the Schöbel-Zhu stochastic volatility model has all advantages of a low dimensional Markov model, it is also flexible enough to fit prices of vanilla inflation-indexed options in an accurate way.

4.6 Conclusion

We have introduced a generic model incorporating stochastic interest rates and stochastic volatility under a full correlation structure of all driving model factors, with closed-form pricing formulas for vanilla options and which is able to incorporate the markets implied volatility structures. Having the flexibility to correlate the underlying FX/Inflation/Stock-index with both the stochastic volatility and the stochastic interest rates yields a realistic model, which is of practical importance for the pricing and hedging of options with a long-term exposure. Furthermore, closed-form pricing of vanilla FX, Inflation and stock options is a big advantage for the calibration (and sensitivity analysis) of the model; using Fourier methods, we have shown how vanilla call/put options, forward starting options, year-on-year inflation-indexed swaps and inflation-indexed caps/floors can be valued in closed-form. Hereby it must be noted that our model can cover Poisson type jumps with a trivial extension. Under Schöbel and Zhu (1999) stochastic volatility, using its affine properties, we were able to derive the corresponding characteristic functions in closed-form. Under Heston (1993) stochastic volatility, the characteristic functions can only be derived explicitly under special zero correlation assumptions. Nonetheless, we have demonstrated that one can still use these pricing formulas: either by using a projection of the general model onto the uncorrelated case, or by using it as a control variate for the general model. The latter method even results in such large variance reductions that its incorporation in the calibration becomes a more than viable option. Our model can be used for multi-asset purposes (e.g. interest rates, FX, inflation, equity, commodities) and is fast enough for the real life risk management of big portfolios of such products. We think it is particularly suitable for the pricing and hedging of long-dated multi-currency structures (e.g. hybrid TARN options, variable annuities, inflation LPI options and PRDC FX swaps) which are sensitive to both future interest rates evolutions as well as movements from the underlying index and/or corresponding volatility smiles.
4.7 Appendix

4.7.1 Deriving the characteristic function of the log ‘Schöbel-Zhu’ inflation rate

In this appendix we will prove that the partial differential equation (4.60), i.e.

\[
0 = f_t - \frac{1}{2}v_t^2(t)f_{zz} + \kappa(\xi(t) - \nu(t))f_\nu + \frac{1}{2}v_{zz}^2(t)f_{zz}
\]

subject to the boundary condition \(f(T, y, \sigma) = \exp(iuT))\) has a solution given by (4.61)-(4.66); to solve this differential equation, we use the ansatz (4.61), find the corresponding partial derivatives and substitute these in (4.60). We then obtain a system of ordinary differential equations that is similar to the one-factor model as in van Haastrecht et al. (2009) and which can be solved using similar methods.

Expanding \(v_t^2(t)\) according to (4.49) and collecting the terms for \(y(t), \nu(t)\) and \(\frac{1}{2}v_t^2(t)\) yields the following system of ordinary differential equations for the functions \(A(u, t, T), \ldots, D(u, t, T)\):

\[
0 = \frac{\partial B(u, t, T)}{\partial t}, \quad (4.106)
\]

\[
0 = \frac{\partial D(u, t, T)}{\partial t} - 2(\kappa - \rho_{\nu \nu}T)D(u, t, T) + \tau^2D^2(t) + (B^2 - B), \quad (4.107)
\]

\[
0 = \frac{\partial C(u, t, T)}{\partial t} + (\rho_{\nu \nu}T - \kappa + \tau^2D)C(u, t, T)
\]

\[
+ \left( \sum_{i=1}^{K} [\rho_{\nu \nu_{i}}\sigma_{i}v_{i}B_{i}(t, T)] - \sum_{j=1}^{M} [\rho_{\nu \nu_{j}}\sigma_{j}v_{j}B_{j}(t, T)] \right) (B^2 - B)
\]

\[
+ \left\{ \kappa \xi(t) + \left( \sum_{i=1}^{K} \rho_{\nu \nu_{i}}\sigma_{i}v_{i}B_{i}(t, T) - \sum_{j=1}^{M} \rho_{\nu \nu_{j}}\sigma_{j}v_{j}B_{j}(t, T) \right) B \right\} D(u, t, T),
\]

\[
0 = \frac{\partial A(u, t, T)}{\partial t} + \left[ \kappa \xi(t) + \sum_{i=1}^{K} \rho_{\nu \nu_{i}}\sigma_{i}v_{i}B_{i}(t, T)B - \sum_{j=1}^{M} \rho_{\nu \nu_{j}}\sigma_{j}v_{j}B_{j}(t, T)B \right] C(u, t, T)
\]

\[
+ \frac{1}{2}\tau^2(C^2(u, t, T) + D(u, t, T)) + \frac{1}{2}(B^2 - B)\nu_{k, m}(t, T),
\]

with \(\nu_{k, m}(t, T)\) the instantaneous variance of the Gaussian rate processes, see (4.50). It immediately that follows \(B(u, t, T) = B\) equals a constant since its derivative is zero, subject to the boundary condition (4.54) we have

\[
B = iu.
\]

(4.110)
The second equation (4.107) yields a Riccati equation with constant coefficients and boundary condition $D(u, T) = 0$ which is equivalent to the PDE for the $D$-term in the SZHW model (see Chapter 3) and has the following solution:

$$D(u, t, T) = -u(i + u) \frac{1 - e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \quad (4.111)$$

with: $\gamma = \sqrt{(k - \rho_{\nu \tau}B)^2 - \tau^2(B^2 - B)}, \quad (4.112)$

$$\gamma_1 = \gamma + \frac{1}{2}q_1 = \gamma + (\kappa - \rho_{\nu \tau}B), \quad (4.113)$$

$$\gamma_2 = \gamma - \frac{1}{2}q_1 = \gamma - (\kappa - \rho_{\nu \tau}B). \quad (4.114)$$

The third equation (4.108) for $C(u, t, T)$ is a first order linear differential equation of the form $\frac{dC(u, t, T)}{dt} + g(t)C(u, t, T) + h(t) = 0$, with associated boundary condition $C(u, T) = 0$. Hence we can represent a solution for $C(u, t, T)$ as:

$$C(u, t, T) = \int_t^T h(s) \exp\left[ \int_t^s g(v)dv \right] ds, \quad (4.115)$$

with: $g(v) = -(k - \rho_{\nu \tau}B) + \tau^2 D(u, v, T), \quad (4.116)$

$$h(s) = \left( \kappa \xi(u) + \sum_{i=1}^K \left( \rho_{\nu i \nu} \tau \sigma_i^j B_n^i(s, T) - \sum_{j=1}^M \rho_{\nu j i} \tau \sigma_i^j B_j^i(s, T) \right) \right) D(u, s, T)$$

$$\quad + \left\{ \sum_{i=1}^K \rho_{\nu i \nu} \sigma_i^j B_n^i(s, T) - \sum_{j=1}^M \rho_{\nu j i} \sigma_i^j B_j^i(s, T) \right\} (B^2 - B)$$

$$\quad + \sum_{i=1}^K \left\{ \rho_{\nu i \nu} \sigma_n^i B_n^i(s, T) (B^2 - B) + \left[ \rho_{\nu i \nu} (B - 1) \tau \sigma_n^i B_n^i(s, T) \right] D(u, s, T) \right\}$$

$$\quad - \sum_{j=1}^M \left\{ \rho_{\nu j i} \sigma_i^j B_j^i(s, T) (B^2 - B) + \left[ \rho_{\nu j i} B \tau \sigma_i^j B_j^i(s, T) \right] D(u, s, T) \right\}. \quad (4.117)$$

We first consider the integral over $g$: dividing equation (4.116) by $D(u, t, T)$, rearranging terms and integrating we find the surprisingly simple solution:

$$\int g(v) dv = \int -(k - \rho_{\nu \tau}B) + \tau^2 D(u, v, T) dv$$

$$\quad = \int (k - \rho_{\nu \tau}B) - \frac{(B^2 - B)}{D(u, v, T)} - \frac{\partial D(u, v, T)}{\partial v} \frac{1}{D(u, v, T)} dv$$

$$\quad = \log(\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}) + c, \quad (4.118)$$

where $\gamma, \gamma_1$ and $\gamma_2$ are defined in (4.66) and with $c$ the integration constant. Hence taking the
exponent and filling in the required integration boundaries yields

$$\exp\left[\int_0^T g(v)dv\right] = \frac{\gamma_1 e^{\gamma(T-t)} + \gamma_2 e^{-\gamma(T-t)}}{\gamma_1 e^{\gamma T} + \gamma_2 e^{-\gamma T}},$$ (4.119)

Substituting this expression into (4.115) we find (after a long but straightforward calculation) for $C(u, t, T)$:

$$C(u, t, T) = -\frac{u(i + u)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}\left\{\gamma_0\left(1 + e^{-2\gamma(T-t)}\right)ight.$$ (4.120)

$$+ \sum_{i=1}^{K} \left[\left(\gamma_i^2 - \gamma_4 e^{-2\gamma(T-t)}\right) - \left(\gamma_i e^{-\gamma(T-t)} - \gamma_i^2 e^{-(2\gamma u)(T-t)}\right) - \gamma_i e^{-\gamma(T-t)}\right]$$

$$- \sum_{j=1}^{M} \left[\left(\gamma_j^2 - \gamma_9 e^{-2\gamma(T-t)}\right) - \left(\gamma_j e^{-\gamma(T-t)} - \gamma_j^2 e^{-(2\gamma u)(T-t)}\right) - \gamma_j e^{-\gamma(T-t)}\right]\right\}$$

with the constants $\gamma, \gamma_0, \ldots, \gamma_{12}$ as defined in (4.66).

Finally, by integration equation (4.110), we find the following expression for $A(u, t, T)$:

$$A(u, t, T) = \int_t^T \left[\frac{1}{2} (B^2 - B) \gamma_{K,M}^2(t, T) + \kappa \xi(s) C(u, s, T) + \frac{1}{2} \tau^2 (C^2(u, s, T) + D(u, s, T))\right] ds$$

$$= -\frac{1}{2} u(i + u) V_{K,M}(t, T)$$

$$+ \int_t^T \left[\left(\kappa \psi + (iu - 1) \sum_{i=1}^{K} \rho_{\sigma_i} \tau \sigma_i^2 B_i^0(t, T) - iu \sum_{j=1}^{M} \rho_{\sigma_j} \tau \sigma_j^2 B_j^0(t, T)\right) C(u, s, T)\right.$$ (4.121)

$$\left. + \frac{1}{2} \tau^2 \left(C^2(u, s, T) + D(u, s, T)\right)\right] ds,$$

where $V_{K,M}(t, T)$ is the integrated variance of the multi-factor Gaussian rates which can found by simple integration, see (4.51). It is possible to write a closed-form expression for the remaining integral in (4.121). As the ordinary differential equation for $D(u, s, T)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2007) in order to avoid any discontinuities. The main problem however lies in the integrals over $C(u, s, T)$ and $C^2(u, s, T)$, which will involve the Gaussian hypergeometric $2F_1(a, b; c; z)$. The most efficient way to evaluate this hypergeometric function (according to Numerical Recipes, Press and Flannery (1992)) is to integrate the defining differential equation. Since all of the terms involved in $D(u, s, T)$ are also required in $C(u, s, T)$, numerical integration of the second part of (4.121) seems to be the most efficient method for evaluating $A(u, t, T)$. Hereby we conveniently avoid any issues regarding
complex discontinuities altogether.

### 4.7.2 Analytical properties of the Gaussian factors driving the asset price process

In this section we will discuss some properties of the processes that drive the asset price dynamics. That is, we discuss the pricing of bonds under multi-factor Gaussian interest rates (Section 4.7.2) and the moments of the Gaussian interest rates processes and the Ornstein-Uhlenbeck distributed volatility process under the $T$-forward measure (Section 4.7.2).

#### Zero-coupon bond prices under multi-factor Gaussian rates

In this appendix we briefly review zero-coupon bond prices of the Gaussian multi-factor rate model, i.e. one has the following analytical formulas for the zero-coupon bond prices (e.g. see Brigo and Mercurio (2006) for the two-factor model, which easily extends to the multi-factor case):

$$P_n(t, T) = \mathbb{E}_n \left\{ e^{-\int_t^T r(u) du} \right\} = A_n(t, T) = A_n(t, T) e^{-\sum_{i=0}^{K-1} \int_t^T B_i(t, T) \sigma_i^2(t)},$$

(4.122)

$$A_n(t, T) = \frac{P_n(0, T)}{P_n(0, t)} \exp \left\{ \frac{1}{2} \left[ V_n(t, T) - V_n(0, T) + V_n(0, t) \right] \right\},$$

(4.123)

where $B_i(t, T) = \frac{1-e^{-a(t-T)}}{a}$. It is straightforward to generalize this to the case of time-dependent model parameters, i.e. in that case $B_i(t, T) := \int_t^T e^{-a(u)}(T-u) du$. Expressions for the real bond prices $P_r(t, T)$ and affine terms $A_r(t, T), B_r(t, T)$ are completely analogous.

For the integrated rate variances $V_i(t, T)$, one also has closed-form expressions. To this end we let (just as in Section 4.4) $C^{i,j}$ and $R^{i,j}$ respectively denote the integrated covariance and correlation between the $i$-th and $j$-th element of the vector of rate volatilities $\Sigma(t)$ of (4.48). One can then express the integrated rate variances as

$$V_n(t, T) = \sum_{i=2}^{K+1} C^{(i,i)} + 2 \sum_{i=2}^{K+1} \sum_{j=i+1}^{K+1} R^{(i,j)} C^{(i,j)},$$

(4.124)

$$V_r(t, T) = \sum_{i=k+2}^{K+M+1} C^{(i,i)} + 2 \sum_{i=k+2}^{K+M+1} \sum_{j=i+1}^{K+M+1} R^{(i,j)} C^{(i,j)}.$$

(4.125)

Expressions for these covariances are provided in Section 4.4.
4.7. Appendix

Moments of the interest rate and volatility processes

In this appendix, we will derive the moments of the stochastic factors that drive the nominal, real and volatility rate. Since all factors follow Ornstein-Uhlenbeck processes, the moments can be found relatively easy.

Moments of the volatility process

By integrating the $\text{forward dynamics of (4.18) conditional on } \nu$, we obtain

\[ \nu(T_{i-1}) = \nu(t)e^{-\kappa(T_{i-1}-t)} + \int_{t}^{T_{i-1}} \kappa \xi(u)e^{-\kappa(T_{i-1}-u)} du + \tau \int_{t}^{T_{i-1}} e^{-\kappa(T_{i-1}-u)} dW_{\nu}^{T}(u), \quad (4.126) \]

where $\xi(u) := \psi - \sum_{i=1}^{K} \frac{\rho_{n}^{i} \sigma_{n}^{2}}{a_{n}^{i}} [1 - e^{-a_{n}(T_{i-1}-u)}]$. From Itô’s isometry, we then have that the mean and variance of $\nu$ under the $T_{i}$-forward measure are given by:

\begin{align*}
\mu_{\nu} &= \nu(t)e^{-\kappa(T_{i-1}-t)} + \left( \psi \sum_{i=1}^{K} \frac{\rho_{n}^{i} \sigma_{n}^{2} \tau}{a_{n}^{i}} \right) \left( 1 - e^{-\kappa(T_{i-1}-t)} \right) \\
&\quad - \sum_{i=1}^{K} \frac{\rho_{n}^{i} \sigma_{n}^{2} \tau}{a_{n}^{i}(\kappa + a_{n}^{i})} \left[ e^{-a_{n}(T_{i-1}-T_{i}-t)} - e^{-a_{n}(T_{i-1}-t)} \right], \\
\sigma_{\nu}^{2} &= \frac{\tau^{2}}{2\kappa} \left( 1 - e^{-2\kappa(T_{i-1}-t)} \right), \quad (4.127) \end{align*}

Moments of the rate processes

Starting from time $t$, one can integrate the rate dynamics of $x_{n}^{i}(T_{i-1})$ and $x_{j}^{j}(T_{i-1})$, from time $t$ to $T_{i-1}$, to obtain the following following explicit solutions (see also Pelsser (2000) or Brigo and Mercurio (2006))

\begin{align*}
x_{n}^{i}(T_{i-1}) &= x_{n}^{i}(t)e^{-a_{n}^{i}(T_{i-1}-t)} - M_{n}^{T_{i}}(t, T_{i-1}) + \sigma_{n}^{i} \int_{t}^{T_{i-1}} e^{-a_{n}^{i}(T_{i-1}-u)} dW_{n}^{T_{i}}(u), \quad (4.129) \\
x_{j}^{j}(T_{i-1}) &= x_{j}^{j}(t)e^{-a_{j}^{j}(T_{i-1}-t)} - M_{j}^{T_{i}}(t, T_{i-1}) + \sigma_{j}^{j} \int_{t}^{T_{i-1}} e^{-a_{j}^{j}(T_{i-1}-u)} dW_{j}^{T_{i}}(u), \quad (4.130) \end{align*}
where

\[ M_{n_k}^T(t, T_{t-1}) = \int_T^{T_{t-1}} \left[ \sigma_n^k \sum_{i=1}^K \rho_{x_n,t_n}^i \sigma_n^i B_n^i(u, T_i) \right] e^{-\sigma_n^i(T_{t-1} - u)} du \]

\[ = \sigma_n^k \frac{1 - e^{-\sigma_n^k(T_{t-1} - t)}}{\alpha_n^k} \sum_{i=1}^K \rho_{x_n,t_n}^i \sigma_n^i \\
- \sigma_n^k \sum_{i=1}^K \rho_{x_n,t_n}^i \frac{\sigma_n^i}{\alpha_n^i} \left[ e^{-\sigma_n^i(T_{t-1} - u)} - e^{-\sigma_n^i(T_{t-1} - t)} \right]. \tag{4.131} \]

\[ M_{n_i}^T(t, T_{t-1}) = \int_T^{T_{t-1}} \rho_{x_n,t_n}^i v(u) \sigma_n^i \sum_{i=1}^K \rho_{x_n,t_n}^i \sigma_n^i B_n^i(u, T_i) e^{-\sigma_n^i(T_{t-1} - u)} du \]

\[ =: M_{\nu}^T(t, T_{t-1}) = M_{\nu}^{\tilde{T}}(t, T_{t-1}). \tag{4.132} \]

In the last step we decompose \( M_{n_k}^T(t, T_{t-1}) \) into a deterministic part, denoted by \( M_{n_i}^T(t, T_{t-1}) \) and a stochastic part depending on \( v(u) \), denoted by \( M_{\nu}^T(t, T_{t-1}) \). The calculation of the \( M_{n_i}^T(t, T_{t-1}) \)-term is similar to the nominal interest rate case and results in the following expression:

\[ \tilde{M}_{n_i}^T(t, T_{t-1}) = \sigma_n^i \frac{1 - e^{-\sigma_n^i(T_{t-1} - t)}}{\alpha_n^i} \sum_{i=1}^K \rho_{x_n,t_n}^i \sigma_n^i \\
- \sigma_n^i \sum_{i=1}^K \rho_{x_n,t_n}^i \frac{\sigma_n^i}{\alpha_n^i} \left[ e^{-\sigma_n^i(T_{t-1} - u)} - e^{-\sigma_n^i(T_{t-1} - t)} \right]. \tag{4.133} \]

Hence from Itô’s isometry we then have that the mean and variance of \( x_n^k(T_{t-1}) \) and \( \tilde{x}_n^i(T_{t-1}) \) (starting from time \( t \)) are respectively given by

\[ \mu_n^k(t, T_{t-1}) = x_n^k(t) e^{-\sigma_n^k(T_{t-1} - t)} - M_{n_k}^T(t, T_{t-1}) \tag{4.134} \]

\[ \left( \sigma_n^k(t, T_{t-1}) \right)^2 = \frac{\left(\sigma_n^k \right)^2}{2\alpha_n^k} \left(1 - e^{-2\sigma_n^k(T_{t-1} - t)} \right) \tag{4.135} \]

\[ \mu_n^i(t, T_{t-1}) = x_n^i(t) e^{-\sigma_n^i(T_{t-1} - t)} - \tilde{M}_{n_i}^T(t, T_{t-1}) \tag{4.136} \]

\[ \left( \sigma_n^i(t, T_{t-1}) \right)^2 = \frac{\left(\sigma_n^i \right)^2}{2\alpha_n^i} \left(1 - e^{-2\sigma_n^i(T_{t-1} - t)} \right). \tag{4.137} \]

It remains to determine the moments of \( \tilde{M}_n^T(t, T_{t-1}) \), i.e. of

\[ \tilde{M}_n^T(t, T_{t-1}) = \sigma_n^i \rho_{x_n,t_n}^i \int_T^{T_{t-1}} v(u) e^{-\sigma_n^i(T_{t-1} - u)} du. \tag{4.138} \]
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By substituting the explicit solution (4.126) for \( \nu(u) \) one obtains the following three integrals:

\[
\sigma_i^j \rho_{L,i}^j \nu(t) \int_t^{T_{i-1}} e^{-\kappa(u-t)} e^{-\alpha_j^i(T_{i-1}-u)} \, du
\]  
(4.139)

\[
\sigma_i^j \rho_{L,i}^j \int_t^{T_{i-1}} \int_t^u \xi(s) e^{-\kappa(u-s)} \, ds \, e^{-\alpha_j^i(T_{i-1}-u)} \, du
\]  
(4.140)

\[
\sigma_i^j \rho_{L,i}^j \tau \int_t^{T_{i-1}} \int_t^u \nu(s) \, dW_T^\nu(s) \, e^{-\alpha_j^i(T_{i-1}-u)} \, du.
\]  
(4.141)

The integral of (4.139) can be calculated exactly and becomes

\[
\nu(t) \frac{\sigma_i^j \rho_{L,i}^j}{(\alpha_j^i - \kappa)} \left[ e^{-\kappa(T_{i-1}-t)} - e^{-\alpha_j^i(T_{i-1}-t)} \right].
\]  
(4.142)

By using Fubini’s theorem to interchange the order of integration, one can find that the integral of (4.140) becomes

\[
\sigma_i^j \rho_{L,i}^j \tau \frac{\kappa e^{-\alpha_j^i(T_{i-1}-t)}}{(\alpha_j^i - \kappa) \kappa a_j^i} \left[ \psi - \sum_{i=1}^K \frac{\rho_{L,i}^j \sigma_i^j \tau}{\kappa a_j^i} \right] 
- \frac{\sigma_i^j \rho_{L,i}^j}{\kappa(\alpha_j^i - \kappa)} \sum_{i=1}^K \frac{\rho_{L,i}^j \sigma_i^j \tau}{(\kappa + a_j^i) a_j^i (\alpha_j^i + a_j^i)} \left\{ (\alpha_j^i + a_j^i) e^{-\kappa(T_{i-1}-1)-\alpha_j^i(T_{i-1})} 
- (\kappa + a_j^i) e^{-\alpha_j^i(T_{i-1}-1)-\alpha_j^i(T_{i-1})} 
- (\alpha_j^i - \kappa) e^{-\alpha_j^i(T_{i-1}-1)} 
\right\}.
\]  
(4.143)

Again by changing the integration order, we find that the following expression holds for the stochastic integral of (4.141):

\[
\sigma_i^j \rho_{L,i}^j \tau \frac{1}{(\alpha_j^i - \kappa)} \int_t^{T_{i-1}} \left[ e^{-\kappa(u-t)} - e^{-\alpha_j^i(T_{i-1}-u)} \right] dW_T^\nu(s).
\]  
(4.144)

Hence from Itô’s isometry, we have that \( \tilde{M}_i^j(t, T_{i-1}) \) of (4.138) is normally distributed with mean
All of the above processes can be written in the form
\[ y_m(T_{i-1}) = O(dt) + c_m \int_t^{T_{i-1}} a_m(u) dW_m(u), \]

\[ \mu^j(t, T_{i-1}) \text{ and variance } \left( \sigma^j(t, T_{i-1}) \right)^2 \text{ given by} \]

\[
\mu^j(t, T_{i-1}) = \sigma^j \rho_{j,i} \nu(t) \left[ e^{-\kappa(T_{i-1} - t)} - e^{-a_j^i(T_{i-1} - t)} \right] \\
+ \sigma^j \rho_{j,i} \kappa e^{-a_j^i(T_{i-1} - t)} + (a_j^i - \kappa) - a_j^i e^{-\kappa(T_{i-1} - t)} + \frac{\psi - \sum_{i=1}^K \rho_{j,i} a_k^i}{a_j^i \kappa} \\
- \frac{\sigma^j \rho_{j,i} \kappa}{\kappa(a_j^i - \kappa)} \sum_{i=1}^K \frac{\rho_{j,i} \sigma_n^2 \tau}{(\kappa + a_j^i)(a_j^i + a_n)} \left\{ (a_j^i + a_n) e^{-\kappa(T_{i-1} - t) - d_n^i(T_{i-1})} - (a_j^i - \kappa) e^{-d_n^i(T_{i-1})} \right\}.
\]

\[
\left( \sigma^j(t, T_{i-1}) \right)^2 = \frac{\sigma^j \rho_{j,i} \kappa}{(a_j^i - \kappa)} \left[ \frac{1}{2 \kappa} + \frac{1}{2 a_j^i} - \frac{2}{(\kappa + a_j^i)} \right] \\
- \frac{e^{-2 \kappa(T_{i-1} - t)}}{2 \kappa} - \frac{e^{-2a_j^i(T_{i-1} - t)}}{2 a_j^i} + \frac{2 e^{-\kappa(a_j^i)(T_{i-1} - t)}}{\kappa + a_j^i}.
\]

**Terminal correlations between the driving factors**

In this section we provide simple analytical expressions for the (terminal) correlations between the driving model factors, \( \nu, x_1^1, \ldots, x_n^1, x_1^k, \ldots, x_n^k, V^1, \ldots, V^M \), from the current time \( t \) to time \( T_{i-1} \). To this end, we consider the following explicit solutions for these Gaussian processes:

\[
\nu(T_{i-1}) = O(dt) + \tau \int_t^{T_{i-1}} e^{-\kappa(u-t)} dW^T_{\nu}(u),
\]

\[
x_n^k(T_{i-1}) = O(dt) + \sigma_n \int_t^{T_{i-1}} e^{-a_n^k(u-t)} dW^T_m(u),
\]

\[
x_j^l(T_{i-1}) = O(dt) + \sigma_j^l \int_t^{T_{i-1}} e^{-a_j^l(u-t)} dW^T_j(u),
\]

\[
V_j(T_{i-1}) = O(dt) + \frac{\sigma_j \rho_{j,v} \tau}{(a_j^i - \kappa)} \int_t^{T_{i-1}} \left[ e^{-\kappa(u-t)} - e^{-a_j^i(u-t)} \right] dW^T_{\nu}(u).
\]
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hence by Itô’s isometry the correlation can be easily calculated; in general, we have that the correlation between, say \( y_1(T_{i-1}) \) and \( y_2(T_{i-1}) \), is given by

\[
\rho_{y_1y_2}(t, T_{i-1}) = \frac{\text{Cov}(y_1(T_{i-1}), y_2(T_{i-1}))}{\sqrt{\text{Var}(y_1(T_{i-1})) \cdot \text{Var}(y_2(T_{i-1}))}} \quad \text{(4.151)}
\]

\[
= \rho_{y_1y_2} \int_t^{T_{i-1}} c_1a_1(u)c_2a_2(u)du \left[ \int_t^{T_{i-1}} [c_1a_1(u)]^2 du \cdot \int_t^{T_{i-1}} [c_2a_2(u)]^2 du \right].
\]

After identification in (4.147)-(4.150), one has that \( a_m(u) \) takes two particular forms

\[
a_m(u) = \begin{cases} 
  e^{-b_m(T_{i-1}-u)} & \text{for } \nu, x^i_m, \ldots, x^K_m, \\
  e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)} & \text{for } V^1, \ldots, V^M, \quad b_m \in \{a^1, \ldots, a^M\}, \quad b_m \in \{a^1, \ldots, a^M\}.
\end{cases}
\]

Hence by combining the above two forms and using formula (4.151), one has that the resulting correlations take one of the three forms below; to ease notation, we first define the following two integral expressions:

\[
I_1(b_m) = \int_t^{T_{i-1}} c_m^2 \left[ e^{-b_m(T_{i-1}-u)} \right]^2 du
\]

\[
= c_m^2 \left\{ \frac{1 - e^{-2b_m(T_{i-1}-t)}}{2b_m} \right\}
\]

\[
I_2(b_m) = \int_t^{T_{i-1}} c_m^2 \left[ e^{-\kappa(T_{i-1}-u)} - e^{-b_m(T_{i-1}-u)} \right]^2 du
\]

\[
= c_m^2 \left\{ \frac{1}{2\kappa} + \frac{1}{2b_m} - \frac{2}{(\kappa + b_m)} - \frac{e^{-2\kappa(T_{i-1}-t)}}{2\kappa} - \frac{e^{-2b_m(T_{i-1}-t)}}{2b_m} + \frac{2e^{-(\kappa+b_m)(T_{i-1}-t)}}{(\kappa + b_m)} \right\}.
\]

If \( a_1(u) \) and \( a_2(u) \) are both of the first form, then the correlation between \( y_1(T_{i-1}) \) and \( y_2(T_{i-1}) \) is given by

\[
\rho_{y_1y_2} = \frac{c_1c_2}{\sqrt{I_1(b_1)I_1(b_2)}} \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)}. \quad \text{(4.152)}
\]

If \( a_1(u) \) is of the first form and \( a_2(u) \) of the second, then the correlation between \( y_1(T_{i-1}) \) and \( y_2(T_{i-1}) \) is given by

\[
\rho_{y_1y_2} = \frac{c_1c_2}{\sqrt{I_1(b_1)I_2(b_2)}} \left[ \frac{1 - e^{-(b_1+b)(T_{i-1}-t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_1+b_2)(T_{i-1}-t)}}{(b_1 + b_2)} \right]. \quad \text{(4.153)}
\]
Finally, if \( a_1 (u) \) and \( a_2 (u) \) are both of the second form then the correlation between \( y_1 (T_{i-1}) \) and \( y_2 (T_{i-1}) \) is given by

\[
\rho_{y_1 y_2} = \frac{c_1 c_2}{\sqrt{I_2 (b_1) I_2 (b_2)}} \left[ \frac{1 - e^{-2\kappa(T_{i-1} - t)}}{2\kappa} + \frac{1 - e^{-(b_1 + b_2)(T_{i-1} - t)}}{(b_1 + b_2)} \right. \\
\left. - \frac{1 - e^{-(b_1 + \kappa)(T_{i-1} - t)}}{(b_1 + \kappa)} - \frac{1 - e^{-(b_2 + \kappa)(T_{i-1} - t)}}{(b_2 + \kappa)} \right].
\] (4.154)

**Constants in the Quadratic form (4.76)**

The constants \( a_0, b_0 \) and vector \( a \) of the quadratic form (4.76) can be directly extracted from equation (4.75) and are given by

\[
a_0 := iu \left[ A_r (T_{i-1}, T_i) - A_n (T_{i-1}, T_i) \right] + A(u, T_{i-1}, T_i) + C(T_{i-1}) \mu_r (t, T_{i-1}) + \frac{1}{2} D(T_{i-1}) \mu_r^2 (t, T_{i-1})
\]

\[
+ iu \sum_{k=1}^{K} B_k^r (T_{i-1}, T_i) \mu_k (t, T_{i-1})
\]

\[
- iu \sum_{j=1}^{M} B_j^r (T_{i-1}, T_i) \left[ \mu_j (t, T_{i-1}) + \mu_j^* (t, T_{i-1}) \right],
\]

\[
b_0 := \frac{1}{2} D(u, T_{i-1}, T_i) \sigma_r^2 (t, T_{i-1}),
\] (4.156)

\[
a := \left[ \begin{array}{c}
\sigma_r (t, T_{i-1}) \left[ C(T_{i-1}) + D(T_{i-1}) \mu_r (t, T_{i-1}) \right] \\
\sigma_n^2 (t, T_{i-1}) \mu_n (T_{i-1}, T_i) \\
\vdots \\
\sigma_n^K (t, T_{i-1}) \mu_n^K (T_{i-1}, T_i) \\
- \sigma_n^1 (t, T_{i-1}) \mu_n^1 (T_{i-1}, T_i) \\
\vdots \\
- \sigma_r^M (t, T_{i-1}) B_r^M (T_{i-1}, T_i) \\
\sigma_r^1 (t, T_{i-1}) B_r^1 (T_{i-1}, T_i) \\
\vdots \\
\sigma_r^M (t, T_{i-1}) B_r^M (T_{i-1}, T_i)
\end{array} \right],
\] (4.157)

and with the \((1 + K + 2M) \times (1 + K + 2M)\) correlation matrix \( S \) given by

\[
S := \begin{pmatrix}
1 & \rho_{\omega,\omega} (t, T_{i-1}) & \cdots & \rho_{\omega,\psi} (t, T_{i-1}) \\
\rho_{\omega,\psi} (t, T_{i-1}) & 1 & \cdots & \rho_{\omega,\omega} (t, T_{i-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{\psi,\omega} (t, T_{i-1}) & \rho_{\psi,\psi} (t, T_{i-1}) & \cdots & 1
\end{pmatrix}.
\] (4.158)
4.7. Appendix

**Constants in proposition 4.4.6**

The constant $h_0$ and vector $h$ and correlation matrix $S_R$ be extracted from equation (4.96) and are given by:

$$h_0 := \left[ A_r(T_{i-1}, T_i) - A_n(T_{i-1}, T_i) \right]$$

$$+ \sum_{k=1}^{K} B_k^n(T_{i-1}, T_i) \mu_k^n(t, T_{i-1}) - \sum_{j=1}^{M} B_j^l(T_{i-1}, T_i) \mu_j^l(t, T_{i-1}),$$

$$h := \begin{bmatrix} \sigma_n^1(t, T_{i-1}) B_1^n(T_{i-1}, T_i) \\ \vdots \\ \sigma_n^K(t, T_{i-1}) B_K^n(T_{i-1}, T_i) \\ -\sigma_r^1(t, T_{i-1}) B_1^l(T_{i-1}, T_i) \\ \vdots \\ -\sigma_r^M(t, T_{i-1}) B_M^l(T_{i-1}, T_i) \end{bmatrix},$$

with $(K + M) \times (K + M)$ correlation matrix $S_R$ given by

$$S_R := \begin{pmatrix} 1 & \cdots & \rho_{x_1^1,x_1^2}(t, T_{i-1}) & \rho_{x_1^1,x_1^3}(t, T_{i-1}) & \cdots & \rho_{x_1^1,x_1^M}(t, T_{i-1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{x_1^1,x_1^2}(t, T_{i-1}) & \cdots & 1 & \rho_{x_1^2,x_1^3}(t, T_{i-1}) & \cdots & \rho_{x_1^2,x_1^M}(t, T_{i-1}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{x_1^1,x_1^M}(t, T_{i-1}) & \cdots & \rho_{x_1^M,x_1^2}(t, T_{i-1}) & \rho_{x_1^M,x_1^3}(t, T_{i-1}) & \cdots & 1 \end{pmatrix}.$$  

(4.161)

**4.7.3 FX Calibration**

We briefly describe the used FX market data set, which can be found in Piterbarg (2005): the set consists of ten maturities, each with seven strikes. The strikes are computed according to formula (4.104). These strikes and corresponding Black and Scholes (1973) implied volatilities can be found in Table 4 and 5 below. Note in Table 5 the increasing term structure of implied volatility and the pronounced implied volatility skew/smile, which both do not die out for long-maturities.
Chapter 4. Generic pricing of FX, Inflation and Stock Options

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Table 4: Strikes.

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Table 5: Market implied vols.

We then report the detailed calibration results of the model (4.8) to the above market data. In Table 6 and 7, we report the calibration differences, in implied volatilities for the model (4.8), respectively with Schöbel and Zhu (1999) and Heston (1993) stochastic volatility. For an analysis of these results, see Section 4.5.2.
4.7. Appendix

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**Table 6:** Differences, in implied Black volatilities, between market and model values using Schöbel-Zhu stochastic volatility.

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**Table 7:** Differences, in implied Black volatilities, between market and model values using Heston stochastic volatility.

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