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Calculating soft radiation at one loop

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Abstract: We present an efficient way to calculate the effect of soft QCD radiation at one loop, which is needed for predictions at next-to-next-to-leading logarithmic accuracy. We use rapidity coordinates and isolate the divergences in the integrand. By performing manipulations with cumulative variables, we avoid complications from plus distributions. We address rapidity divergences, divergences with an azimuthal dependence, complicated jet boundaries and multi-differential measurements. The process and measurements can be easily adjusted, as we demonstrate by reproducing many existing soft functions. The results for a general LHC process with multiple Wilson lines are obtained by treating Wilson lines that are not back-to-back using a boost. We also obtain, for the first time, the N-jettiness soft function for generic jet angularities, and the collinear-soft function for the measurement of two angularities.

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Contents

1 Introduction

LHC analyses involve restrictions on QCD radiation to increase their sensitivity. Restrictions can be imposed directly by e.g. requiring a specific number of jets, or indirectly through e.g. the transverse momentum of a Higgs boson. This leads to large logarithms in the cross section, requiring resummation to obtain reliable predictions. The origin of these large logarithms is the enhancement of collinear and soft radiation, which are treated as dynamic degrees of freedom in Soft-Collinear Effective Theory (SCET) [1–4]. SCET is an effective theory of QCD that achieves resummation through the factorization of hard, collinear and soft radiation at the level of the Lagrangian.

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In this paper we focus on soft radiation, which is encoded in the soft function in SCET. The soft function is (schematically) defined as the matrix element

$$\widehat{S}(m,\mu) = \Big\langle 0 \Big| \bar{\mathrm{T}} \Big[\prod_i \widehat{Y}_i^\dagger \Big] \, \delta(m - \hat{m}) \; \mathrm{T} \Big[\prod_i \widehat{Y}_i \Big] \Big| 0 \Big\rangle \,, \tag{1.1}$$

where Y_i is a soft Wilson line along the light-like direction of, and in the color representation of the i-th colored parton participating in the hard scattering. The T ($\bar{\mathrm{T}}$) denote (anti-) time ordering and the delta function encodes the measurement m through the operator $\hat{m}$.

We will present an efficient approach to calculate the one-loop soft function, which is an essential ingredient for resummation at next-to-next-to-leading logarithmic accuracy. We will not restrict ourselves to a specific process or measurement and demonstrate the versatility of our method by reproducing the one-loop soft function for thrust [5, 6], angularities [7, 8], transverse momentum [9] and transverse thrust [10]. Results for the double differential measurement of two angularities and of transverse momentum and beam thrust are also reproduced [11, 12]. These require an extension of SCET, called SCET$_+$ [12–15], with additional collinear-soft modes. The collinear-soft function is again a matrix element of eikonal Wilson lines and can be calculated in the same way. We present for the first time the calculation of N-jettiness with generic jet angularities and the collinear-soft function for the double angularity measurement.

Our approach involves a combination of several tricks: we use the coordinates transverse momentum k_T, rapidity y and azimuthal angle ϕ that make the symmetries of the soft matrix element manifest. By isolating divergences at the integrand level, the integrals are simplified. In particular, the integral for the finite terms can directly be written down and evaluated numerically, if desired. We work with the cumulative soft function, as this involves simple manipulations with logarithms rather than plus distributions. The soft function can be obtained by differentiating the final result. The N-jet soft function is given by the sum over emissions between all pairs of Wilson lines at one loop. We employ a boost to make the Wilson lines back-to-back, allowing us to recycle our dijet results. An extension of the hemisphere decomposition of ref. [16] is needed to handle more complicated boundaries between jets. Our approach is very general, as we also treat rapidity divergences and divergences with an azimuthal-angle dependence. In the latter case we find it convenient to use a version of dimensional regularization that has no ϵ-dependence associated with the azimuthal angle, and show that this is consistent.

The calculation also provides insight into the structure of the soft function at one loop. For example, rapidity divergences are simply the divergences as the rapidity of the soft gluon goes to infinity. The divergent structure near the Wilson lines is dominated by the asymptotic behavior of the measurement. On the other hand, the divergences away from the Wilson lines depend on the area in (y, ϕ)-space on which the measurement is defined, but are independent of the measurement itself.

The outline of the paper is as follows: in section 2 we present the setup of our calculation. We discuss detailed examples for dijets observables in section 3, generalized N-jettiness in section 4, and double differential measurements in section 5. The conclusions are in section 6, and additional details related to the Becher-Bell rapidity regulator, the calculation of the jet function for transverse thrust and the results for thrust-like N-jettiness are relegated to the appendices.

2 Calculational framework

In this section we develop our calculational framework. We first describe the measurements we consider and the rapidity coordinates we use to express them. In section 2.2 we consider the one-loop soft function for (back-to-back) dijets and present our master formula in eq. (2.5). We extend this to N-jet production in section 2.3, boosting to frames where Wilson lines are back-to-back. Multi-differential measurements are discussed in section 2.4.

2.1 Measurement function and rapidity coordinates

For two back-to-back jets, the corresponding soft radiation is emitted from back-to-back Wilson lines. Its boost invariance is made manifest by describing the emitted gluon using k_T, y and ϕ. We will denote the contribution of this soft gluon to a measurement by a function $F(k_T, y, \phi)$, and require that the measurement is additive when there are multiple emissions (avoiding clustering effects from jet algorithms, see e.g. [17]).

Collinear safety implies that for two partons in the collinear limit, $F(k_{1T}, y, \phi) + F(k_{2T}, y, \phi) = F(k_{1T} + k_{2T}, y, \phi)$. Consequently,

$$F(k_T, y, \phi) = k_T f(y, \phi)\,. \tag{2.1}$$

For a parton in the presence of a soft gluon $f(y', \phi') = f(y, \phi) + \mathcal{O}(k_T^{\rm soft}/k_T)$. In the soft limit $k_T^{\rm soft} \to 0$ the deflection $y' - y$ and $\phi' - \phi$ due to the soft gluon go to zero, from which we conclude that IR safety imposes that $f(y, \phi)$ is continuous. We will further assume that $f(y, \phi) \geq 0$, such that the measurement restricts the QCD radiation. To rewrite measurements in these coordinates, we use

$$k^\mu = k_T(\cosh y, \cos\phi, \sin\phi, \sinh y)\,, \quad k^- = k_T\, e^y\,, \quad k^+ = k_T\, e^{-y}\,. \tag{2.2}$$

Here $k^\mp = k^0 \pm k^3$ denote light-cone coordinates along the back-to-back jets, aligned with the z-axis.

2.2 Dijets

We find it convenient to calculate the cumulative distribution for the soft function in terms of the measurement m to avoid dealing with plus distributions

$$\begin{aligned} \delta[m - k_T f(y,\phi)] &\to \theta[m - k_T f(y,\phi)]\,, & \frac{1}{\mu}\frac{1}{(m/\mu)_+} &\to \theta(m)\,\ln\frac{m}{\mu}\,, \\ \delta(m) &\to \theta(m)\,, & \frac{1}{\mu}\left(\frac{\ln(m/\mu)}{m/\mu}\right)_+ &\to \frac{1}{2}\theta(m)\,\ln^2\frac{m}{\mu}\,, \quad \text{etc.} \end{aligned} \tag{2.3}$$

This simplifies intermediate steps, especially for multi-differential measurements. Of course, the distribution follows from differentiating the final expression with respect to m and typically does contain plus distributions. We assume that the measurements are positive and omit the explicit $\theta[m]$ in the following. Naturally, this overall theta function must be restored when taking the derivative of the cumulative soft functions in order to e.g. obtain the $\delta(m)$ terms in the distributions.

The calculation of the soft function will be carried out using dimensional regularization for both the UV and IR divergences, causing the virtual contributions to vanish ($1/\epsilon_{\rm UV} - 1/\epsilon_{\rm IR} = 0$) and avoiding complications [18–21] from the overlap with collinear radiation. Since our observable is IR safe the $1/\epsilon_{\rm IR}$ poles cancel, although this is not explicitly visible due to our choice of regulator. The real emission diagrams with the gluon attaching to Wilson lines 1 and 2 yield at this order (see also appendix C of ref. [22])

$$\begin{aligned} S_{12}^{(1)}(m,\mu) &= -\frac{\alpha_s}{\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2\,\frac{(e^{\gamma_E}\mu^2)^\epsilon}{\Gamma(1-\epsilon)}\,\nu^\eta \int_0^\infty \frac{\mathrm{d}k_T}{k_T^{1+\eta+2\epsilon}} \int_{-\infty}^\infty \frac{\mathrm{d}y}{|2\sinh y|^\eta} \int_0^{2\pi} \mathrm{d}\phi\,\theta[m-k_T f(y,\phi)] \\ &= \frac{\alpha_s}{\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2\,\frac{e^{\epsilon\gamma_E}}{(\eta+2\epsilon)\Gamma(1-\epsilon)}\,\frac{\nu^\eta\mu^{2\epsilon}}{m^{\eta+2\epsilon}} \int_{-\infty}^\infty \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\,\frac{\theta[f(y,\phi)]f(y,\phi)^{2\epsilon}}{|2\sinh y|^\eta}\,. \end{aligned} \tag{2.4}$$

Here $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the color charge of the emitted gluon in the representation of Wilson lines 1 and 2, respectively (in the notation of refs. [23, 24]). If there are only two Wilson lines, $\mathbf{T}_1\!\cdot\!\mathbf{T}_2 = -C_F$ for a quark-anti-quark and $-C_A$ for two gluons. From eq. (2.4) it is clear that the soft radiation is uniformly emitted in y and ϕ. Thus if $f(y,\phi)$ goes to a constant for $y \to \pm\infty$, the y integral diverges. We control these rapidity divergences in the soft function using the η regulator of refs. [9, 25]. Other regulators are possible [26–32], and the expression corresponding to eq. (2.4) for ref. [30] is given in appendix A. Note that at this order there is no distinction between outgoing and incoming Wilson lines, which is known to extend to two loops in certain cases [33].

We introduce a function $f_\infty(y,\phi)$ that captures the behavior of the measurement as $y \to \pm\infty$, such that $\ln(f/f_\infty)$ is integrable. In practice, f_∞ can be obtained by expanding $\ln f$ around $1/y = 0$. This allows us to already isolate the divergent behavior at the integrand level, resulting in our master formula

$$\begin{aligned} S_{12}^{(1)}(m,\mu) &= \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2 \int_{-\infty}^\infty \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\,\theta[f(y,\phi)]\,f_\infty(y,\phi)^{2\epsilon} e^{-\eta|y|} \\ &\quad\times \biggl[\frac{1}{\epsilon} + 2\ln\frac{\mu\, f(y,\phi)}{m\, f_\infty(y,\phi)} + 2\epsilon\biggl(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\biggr)\biggr]\biggl[1+\eta\biggl(-\frac{1}{2\epsilon}+\ln\frac{\nu}{m}\biggr)\biggr]. \end{aligned} \tag{2.5}$$

The UV divergences are fixed by f_∞ and the original measurement f only enters in the finite terms through $\ln(f/f_\infty)$. At this order, only the asymptotic behavior of the rapidity regulator enters, which is characterized by the (simpler) factor $e^{-\eta|y|}$.

An exception is when f vanishes in regions of phase-space (see eq. (2.7)). In these cases it is convenient to separate f into the measurement $f^M > 0$ and the theta function f^R defining the integration region. Eq. (2.5) now reads

$$\begin{aligned} S_{12}^{(1)}(m,\mu) &= \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2 \int_{-\infty}^\infty \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\, f^R(y,\phi)\,f_\infty(y,\phi)^{2\epsilon} e^{-\eta|y|} \\ &\quad\times \biggl[\frac{1}{\epsilon} + 2\ln\frac{\mu\, f^M(y,\phi)}{m\, f_\infty(y,\phi)} + 2\epsilon\biggl(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\biggr)\biggr]\biggl[1+\eta\biggl(-\frac{1}{2\epsilon}+\ln\frac{\nu}{m}\biggr)\biggr]. \end{aligned} \tag{2.6}$$

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Now f_∞ can be determined by only considering f^M (but is irrelevant if the integration is cut off by f^R). When the region described by f^R has a finite area A in (y,ϕ) space,

$$\begin{aligned} S_{12}^{(1)}(m,\mu) &= \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2 \int_{-\infty}^{\infty}\mathrm{d}y \int_0^{2\pi}\mathrm{d}\phi\, f^R(y,\phi)\left[\frac{1}{\epsilon}+2\ln\frac{\mu\, f^M(y,\phi)}{m}\right] \\ &= \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_1\!\cdot\!\mathbf{T}_2\,\frac{A}{\epsilon}+\mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{2.7}$$

Thus the divergence is independent of f_∞ and just proportional to this area. This is the motivation behind the hemisphere decomposition used in section 4. We will present several applications for dijet observables in section 3, demonstrating the efficiency of this approach.

2.3 N jets

To calculate the soft function for N Wilson lines, we can simply sum over the contribution from each pair of Wilson lines using eq. (2.5). However, we need to take into account that the Wilson lines are no longer back-to-back, which we address by boosting to a frame where they are back-to-back. Using primed coordinates for the former and unprimed coordinates for the latter, a momentum k^μ transforms as

$$k'^\mu = B(n_1', n_2')k^\mu\,. \tag{2.8}$$

with

$$B(n_1',n_2') = \begin{pmatrix} \gamma & -\gamma\vec{\beta}^T \\ -\gamma\vec{\beta} & \mathbf{1}+(\gamma-1)\vec{\beta}\vec{\beta}^T/\vec{\beta}^2 \end{pmatrix}, \quad \vec{\beta} = -\frac{1}{2}(\hat{n}_1'+\hat{n}_2')\,, \quad \gamma = \frac{\sqrt{2}}{\sqrt{n_1'\!\cdot n_2'}}\,, \tag{2.9}$$

where $n_i' = (1,\hat{n}_i')$ (i=1,2) denote the directions of the Wilson lines. The Wilson lines in the two frames simply transform into each other. Applying the reverse boost to n_1', n_2', $\bar{n}_1'$ and $\bar{n}_2'$,

$$\begin{aligned} \tilde{n}_1^\mu &= \Big(\gamma^{-1}, \frac{1}{2}(\hat{n}_1'-\hat{n}_2')\Big)\,, & \tilde{n}_2^\mu &= \Big(\gamma^{-1}, \frac{1}{2}(\hat{n}_2'-\hat{n}_1')\Big)\,, \\ \tilde{\bar{n}}_1^\mu &= -\tilde{n}_1^\mu + 2\gamma(1,\vec{\beta})\,, & \tilde{\bar{n}}_2^\mu &= -\tilde{n}_2^\mu + 2\gamma(1,\vec{\beta})\,, \end{aligned} \tag{2.10}$$

so $\tilde{n}_1$ and $\tilde{n}_2$ are indeed back-to-back, though $\tilde{n}_i$ and $\tilde{\bar{n}}_i$ are not. Because $\tilde{n}_i$ and $\tilde{\bar{n}}_i$ do not have the conventional $(1,\hat{n})$ normalization, we wrote a tilde on the n_i and $\bar{n}_i$, though this normalization is irrelevant for the Wilson lines. One can then convert the measurement between the two coordinates using Lorentz invariance of scalar products $n_i'\!\cdot k' = \tilde{n}_i\!\cdot k$. For $i=1,2$ this takes a particularly simple form

$$n_1'\!\cdot k' = \gamma^{-1} n_1\!\cdot k\,, \quad n_2'\!\cdot k' = \gamma^{-1} n_2\!\cdot k\,. \tag{2.11}$$

This approach requires modification in the presence of rapidity divergences, since the rapidity regulator is not boost invariant. For definiteness we first assume that only the

Wilson line in the n_1' direction requires rapidity regularization. For the exchange of a soft gluon between the Wilson lines in the n_1' and n_2' direction, the rapidity regulator is

$$\left(\frac{\nu}{|\bar{n}_1'\!\cdot\! k' - n_1'\!\cdot\! k'|}\right)^{\eta} \overset{y\to\infty}{=} \left(\frac{\nu}{2\gamma k_T \sinh y}\right)^{\eta}. \tag{2.12}$$

Although inserting eq. (2.10) leads to complicated expressions, the asymptotic behavior is simple and is the only thing that matters at one-loop order. The Wilson line requiring the rapidity regularization is at $y = \infty$, so this is the only relevant limit ($y \to -\infty$ is regulated by dimensional regularization). Note that if instead the Wilson line n_2 required rapidity regularization, the final expression would still be the same. From this we conclude that we may use our master formula eq. (2.5) by simply replacing $\nu \to \nu/\gamma$. In the presence of additional Wilson lines requiring rapidity regularization, we in principle need a copy of the rapidity regulator for each direction[1]

$$\prod_i \left(\frac{\nu_i}{|\bar{n}_i'\!\cdot\! k' - n_i'\!\cdot\! k'|}\right)^{\eta_i} \tag{2.13}$$

Ensuring that rapidity divergences corresponding to the n_i' direction are controlled by η_i, by taking the other η's to zero first, implies that eq. (2.12) still holds with $\nu \to \nu_i$ and $\eta \to \eta_i$. In particular, if at the end of the calculation we take all regulators equal, $\nu_i = \nu$ and $\eta_i = \eta$, we can simply do all calculations by replacing $\nu \to \nu/\gamma$ in our master formula.

We find our approach of boosting to back-to-back coordinates convenient as it allows us to recycle results, but it is not necessary. Direct calculations of soft functions with more than two Wilson lines and rapidity divergences have been carried out in e.g. refs. [10, 35].

2.4 Multi-differential measurements

We now consider multi-differential measurements, where large logarithms associated with additional scales arise and require resummation. The resummation can be achieved by an extension of SCET (SCET_+) with additional collinear-soft and/or soft-collinear degrees of freedom [12–15, 36–39]. These modes arise due to the different scales introduced by the multi-differential measurements, which will be discussed for an explicit example in section 5.1. Whereas the soft function defined in eq. (1.1) depends on one measurement m, multi-differential measurements give rise to a soft function depending on multiple measurements

$$\delta(m - \hat{m}) \to \prod_i \delta(m_i - \hat{m}_i)\,. \tag{2.14}$$

The collinear-soft radiation of SCET_+ is described by a collinear-soft function, which is also a matrix element of (collinear-soft) Wilson lines. It can be calculated in the same manner, as we will show in section 5.

[1] Even for Wilson lines in the n_1 and $\bar{n}_1$ directions we can have separate regulators, since the rapidity divergences should be cancelled by the collinear radiation in the n_1 and $\bar{n}_1$ direction, respectively [34].

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To incorporate the multiple measurements in the soft function, we extend the measurement to a vector

$$\vec{m} = k_T \vec{f}(y, \phi) \,. \tag{2.15}$$

allowing us to write eq. (2.14) for the cumulative soft function as

$$\prod_i \theta[m_i - k_T f_i(y,\phi)] = \theta[\max_i\{f_i(y,\phi)\}] \prod_i \theta[m_i/f_i(y,\phi) - k_T] \,. \tag{2.16}$$

For a given y and ϕ this is dominated by a single measurement m_I that imposes the strongest constraint on k_T. Regulating this dominant measurement for $y \to \pm\infty$ through f_∞, we arrive at following expression for the soft function

$$\begin{aligned} S_{12}^{(1)}(\vec{m},\mu) &= \frac{\alpha_s}{2\pi^2} \mathbf{T}_1 \cdot \mathbf{T}_2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\, \theta[\max_i\{f_i(y,\phi)\}]\, f_\infty(y,\phi)^{2\epsilon} e^{-\eta|y|} \\ &\times \left[\frac{1}{\epsilon} + 2\ln\frac{\mu\, f_I(y,\phi)}{m_I\, f_\infty(y,\phi)} + 2\epsilon\left(\ln^2\frac{\mu}{m_I} - \frac{\pi^2}{24}\right)\right]\left[1 + \eta\left(-\frac{1}{2\epsilon} + \ln\frac{\nu}{m_I}\right)\right]. \end{aligned} \tag{2.17}$$

We emphasize that the index I denoting the dominant measurement generally depends on y and ϕ. The corresponding division of phase-space provides a natural way to do the integration.

In section 5 we will apply this to several double-differential measurements. Specifically, the measurement of two angularities [11] and the simultaneous measurement of transverse momentum and beam thrust [40].

3 Dijet examples

We start by calculating the soft function for the thrust and angularity e^+e^- event shapes in sections 3.1 and 3.2. In section 3.3 we determine the transverse momentum soft function for $pp \to Z + X$ (or $pp \to H + X$), which contains rapidity divergences. For transverse thrust in e^+e^- collisions, discussed in section 3.4, the divergences depend on the azimuthal angle. We describe how to treat this in dimensional regularization without breaking the azimuthal symmetry.

3.1 Thrust

Thrust is an e^+e^- event shape defined through [41]

$$\tau = 1 - T = \min_{\hat{n}} \frac{1}{Q} \sum_i \min\left\{k_i^+, k_i^-\right\} \,. \tag{3.1}$$

Here i runs over the final-state particles, Q is the total invariant mass and we minimize over the dijet axis $\hat{n}$. The contribution of soft radiation to the measurement $m = Q\tau$, corresponds to

$$f(y,\phi) = e^{-|y|} \,. \tag{3.2}$$

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Since f is particularly simple, we choose $f_\infty = f$, leading to

$$
\begin{aligned}
S^{(1)}(m = Q\tau, \mu) &= -\frac{\alpha_s C_F}{\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-2\epsilon|y|} \left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right] \\
&= -\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon}\left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right]. \qquad (3.3)
\end{aligned}
$$

Differentiating this leads to the result of refs. [5, 6].

3.2 Angularities

The contribution of soft radiation to the measurement $m = Q\tau_a$ of the angularity [42]

$$
\tau_a = \frac{1}{Q}\sum_i k_{iT}\, e^{-|y_i|(1-a)}\,, \qquad (3.4)
$$

defined along a recoil-free axis [8], which is described by

$$
f(y,\phi) = e^{-|y|(1-a)}\,. \qquad (3.5)
$$

This family of event shapes is infrared safe for $a < 2$ and includes thrust ($a = 0$) and broadening ($a = 1$). For $a < 2$ and $a \neq 1$, with $f_\infty = f$, we obtain [7, 8]

$$
\begin{aligned}
S^{(1)}(m = Q\tau_a, \mu) &= -\frac{\alpha_s C_F}{\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-2\epsilon|y|(1-a)} \left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right] \\
&= \frac{\alpha_s C_F}{\pi}\frac{1}{a-1}\frac{1}{\epsilon}\left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right] \\
&= \frac{\alpha_s C_F}{\pi}\frac{1}{a-1}\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(\ln\frac{\mu^2}{Q^2} - 2\ln\tau_a\right)\right. \\
&\qquad \left. + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - 2\ln\frac{\mu^2}{Q^2}\ln\tau_a + 2\ln^2\tau_a - \frac{\pi^2}{12}\right]. \qquad (3.6)
\end{aligned}
$$

The case $a = 1$ is equivalent with the transverse momentum measurement discussed next.

3.3 Transverse momentum

When the transverse momentum, p_T, of the soft radiation is measured, f is trivial

$$
f(y,\phi) = f_\infty(y,\phi) = 1\,. \qquad (3.7)
$$

However, the calculation is slightly more complicated due to rapidity divergences arising from $y \to \pm\infty$ in the y integration,

$$
\begin{aligned}
S^{(1)}(m = p_T, \mu) &= -\frac{\alpha_s C_F}{\pi} \int_{-\infty}^{\infty} \mathrm{d}y \, e^{-\eta|y|} \left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right] \\
&\qquad \times \left[1 + \eta\left(-\frac{1}{2\epsilon} + \ln\frac{\nu}{m}\right)\right] \\
&= -\frac{\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right]\left(\frac{2}{\eta} - \frac{1}{\epsilon} + 2\ln\frac{\nu}{m}\right). \qquad (3.8)
\end{aligned}
$$

As the rapidity regulator η should not regulate UV divergences, it must be taken to zero before ϵ. For Wilson lines in the adjoint representation (gluons), $C_F \to C_A$. This agrees with the calculation in ref. [9], when converting their $\vec{p}_T$ measurement.

3.4 Transverse thrust in e^+e^-

The transverse thrust event shape $T_\perp$ [43], is designed for hadron collisions, but has also been calculated for $e^+e^- \to 2$ jets [10],

$$\begin{aligned}\tau_\perp = 1 - T_\perp &= \max_{\vec{n}_\perp} \frac{\sum_i |\vec{k}_{i\perp}| - |\vec{k}_{i\perp} \cdot \vec{n}_\perp|}{\sum_i |\vec{k}_{i\perp}|} \\ &= \frac{\sum_i |\vec{k}_{i\perp}| - |\vec{k}_{i\perp} \cdot \vec{n}_\perp|}{Q|\sin\theta|} \,. \end{aligned} \tag{3.9}$$

Here the sum i runs over the final-state particles and the transverse ($\perp$) is with respect to the electron-positron beam axis. In the second line, power suppressed contributions have been neglected in order to write it in terms of the angle θ between the beam and the thrust axis, and the transverse orientation of the thrust axis $\vec{n}_\perp$. The contribution to $\tau_\perp$ from one soft particle with momentum k is thus described by

$$\begin{aligned} f(y,\phi) &= \frac{1}{Q|\sin\theta|}\left[\sqrt{(\cos\phi\cos\theta + \sinh y \sin\theta)^2 + \sin^2\phi} - |\cos\phi\cos\theta + \sinh y\sin\theta|\right], \\ f_\infty(y,\phi) &= \frac{\sin^2\phi}{Q\sin^2\theta} e^{-|y|} \,, \end{aligned} \tag{3.10}$$

where we have expressed $k_\perp$ and $n_\perp$ in eq. (3.9) in terms of the variables k_T, y and ϕ in the frame where the thrust axis is along the $\hat{z}$ direction. Interestingly, the structure of the divergence as $y \to \pm\infty$ in eq. (3.10) has an azimuthal angle dependence. This results in

$$\begin{aligned} S^{(1)}(m = \tau_\perp, \mu) &= -\frac{\alpha_s C_F}{2\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi \\ &\quad \times \left\{ f_\infty(y,\phi)^{2\epsilon}\left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right] + 2\ln\frac{f(y,\phi)}{f_\infty(y,\phi)}\right\} \\ &= -\frac{\alpha_s C_F}{\pi}\left[\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{\mu}{4mQ\sin^2\theta} + 2\ln^2\frac{\mu}{4mQ\sin^2\theta} + \frac{7\pi^2}{12} + A(\theta)\right], \end{aligned} \tag{3.11}$$

where the finite term $A(\theta)$ is given by

$$A(\theta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi \ln\frac{f(y,\phi)}{f_\infty(y,\phi)} \,. \tag{3.12}$$

We remind the reader that these y and ϕ are defined in the frame where the thrust axis is along the $\hat{z}$ axis, while the *transverse* in transverse thrust means perpendicular to the beam axis. The results have been cross checked with ref. [10], and agree once the different scheme for dimensional regularization is taken into account, as discussed in detail below.

The transverse part of the d-dimensional integration measure can be written

$$\int \mathrm{d}^{2-2\epsilon}k_\perp = \frac{\Omega_{1-2\epsilon}}{2}\frac{1}{2}\int \mathrm{d}k_T^2 (k_T^2)^{-\epsilon} \int_0^{2\pi} \mathrm{d}\phi \left[\sin^2(\phi - \phi_0)\right]^{-\epsilon}, \tag{3.13}$$

where $\phi - \phi_0$ is the azimuthal angle between the momentum $k_\perp$ and an arbitrary reference axis. With the choice $\phi_0 = 0$ we obtain the integration measure used in ref. [10]. We prefer to preserve the azimuthal symmetry, and integrate over the choice of this reference axis

$$\begin{aligned}\int \mathrm{d}^{2-2\epsilon}k_\perp &= \frac{\Omega_{1-2\epsilon}}{2}\frac{1}{2}\int \mathrm{d}k_T^2 (k_T^2)^{-\epsilon}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi_0\int_0^{2\pi}\mathrm{d}\phi\big[\sin^2(\phi-\phi_0)\big]^{-\epsilon} \\ &= \frac{\Omega_{2-2\epsilon}}{2\pi}\frac{1}{2}\int \mathrm{d}k_T^2 (k_T^2)^{-\epsilon}\int_0^{2\pi}\mathrm{d}\phi. \end{aligned} \tag{3.14}$$

When the measurement does not depend on ϕ the two ways gives the same results since

$$\frac{\Omega_{1-2\epsilon}}{2}\int_0^{2\pi}\mathrm{d}\phi\big(\sin^2\phi\big)^{-\epsilon} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} = \Omega_{2-2\epsilon}. \tag{3.15}$$

However, for transverse thrust, which does depend on the azimuthal angle, the two schemes give different results for the cumulative soft function. With $f_\infty^{2\epsilon} \propto (\sin^2\phi)^{2\epsilon}$, the two measures lead to contributions to the cumulative soft function that are related through

$$\begin{aligned}&-\frac{\alpha_s C_F}{2\pi^2}\frac{1}{\epsilon^2}\frac{\Omega_{2-2\epsilon}}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\,(\sin^2\phi)^{2\epsilon} \\ &\qquad = -\frac{\alpha_s C_F}{2\pi^2}\frac{1}{\epsilon^2}\frac{\Omega_{1-2\epsilon}}{2}\int_0^{2\pi}\mathrm{d}\phi\,(\sin^2\phi)^{\epsilon} - \frac{\alpha_s C_F}{\pi}\frac{2\pi^2}{3} + \mathcal{O}(\epsilon). \end{aligned} \tag{3.16}$$

The extra π^2 term in the finite part of the cumulative soft function is cancelled by corresponding terms in the two jet functions, calculated in appendix B.

4 N-jettiness with generic jet angularities

We extend the thrust-like N-jettiness definition [16, 44], by considering the measurement of a different angularity for each jet[2]

$$\mathcal{T}_N = \sum_h \min_\ell \left\{\frac{2\omega_\ell}{Q_\ell}(n'_\ell\!\cdot\! k'_h)^{1-\alpha_\ell/2}(\bar{n}'_\ell\!\cdot\! k'_h)^{\alpha_\ell/2}\right\} \equiv \sum_\ell \mathcal{T}_N^{\ell,\alpha_\ell}\,, \tag{4.1}$$

where h runs over the hadronic final-state particles and ℓ over the jets in the event with (label) momenta

$$q'_\ell = \omega_\ell n'_\ell = \omega_\ell(1,\hat{n}'_\ell)\,. \tag{4.2}$$

The primed variables indicate that the momenta are defined in generic coordinates. We will later boost to (unprimed) coordinates where two of the Wilson lines are back to back, as discussed in section 2.3. The ω_ℓ in eqs. (4.1) and (4.2) is considered a parameter which does not transform between frames (i.e. no ω'_ℓ). The minimization of eq. (4.1) assigns each particle to a jet region, and $\mathcal{T}_N^{\ell,\alpha_\ell}$ is the total contribution from jet region ℓ. The 'standard' thrust-like N-jettiness definition is recovered if all α_ℓ are zero, $\mathcal{T}_N^\ell \equiv \mathcal{T}_N^{\ell,\alpha_\ell=0}$. We show how our results reduce to the expressions in ref. [16] in appendix C. We will assume $\alpha_\ell \neq 1$ to avoid rapidity divergences.

[2]Here we use the term 'jets' to refer to both final-state and beam jets.

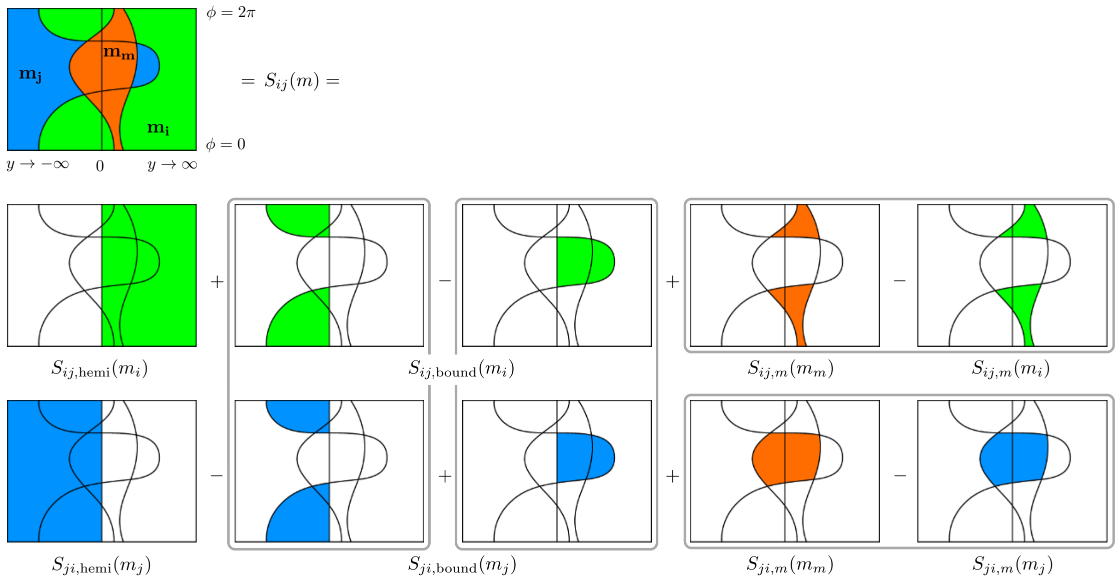

Figure 1. Separation of the soft function, S_{ij}, with a gluon emitted between the ith and jth Wilson line, into hemisphere, boundary and non-hemisphere contributions. The contributions surrounded by a gray box are together finite.

The one-loop soft function is the sum over contributions from gluons exchanged between Wilson lines corresponding to the jets i and j

$$S^{(1)}(m,\mu) = \sum_{i<j} S_{ij}^{(1)}(m,\mu)\,. \tag{4.3}$$

To simplify the discussion we consider 1-jettiness in pp collisions (or equivalently 3-jettiness in e^+e^- collisions). We label the three jets by $\ell = i,j,m$ to make the extension to N jets straightforward. The contribution of a soft gluon to $\mathcal{T}_1^{i,\alpha_i}$ is given by[3]

$$k_T\, k_i \theta\big(k_j - k_i\big)\theta\big(k_m - k_i\big)\,, \tag{4.4}$$

and similarly for $\mathcal{T}_1^{j,\alpha_j}$ and $\mathcal{T}_1^{m,\alpha_m}$, where we introduced

$$k_\ell = \frac{2\omega_\ell}{Q_\ell k_T}(n'_\ell{\cdot}k')^{1-\alpha_\ell/2}(\bar{n}'_\ell{\cdot}k')^{\alpha_\ell/2}\,. \tag{4.5}$$

We extend the hemisphere decomposition [16] to handle the azimuthally dependent phase-space boundaries between regions arising from the more general N-jettiness measurement. This approach is discussed in detail in ref. [45]. The decomposition of $S_{ij}^{(1)}$ into hemisphere, boundary and non-hemisphere contributions is depicted in figure 1 and will be discussed below. The soft function involves three regions associated with the measurements: $\theta(k_j - k_i)\theta(k_m - k_i)$ for m_i, $\theta(k_i - k_j)\theta(k_m - k_j)$ for m_j and $\theta(k_i - k_m)\theta(k_j - k_m)$ for m_m. With the purpose of making the analytical calculation of the divergent parts, as well

[3] To simplify the expressions for the measurement functions f^M, we already pull out a factor of the transverse momentum k_T in the unprimed coordinates (where Wilson lines are back-to-back).

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as the extension to N jets as easy as possible, we first allow the measurements of m_i and m_j to extend over the region of m_m. This is then compensated for by the non-hemisphere contributions $S_{ij,m}$ and $S_{ji,m}$ to the soft function. For a generic measurement, such as the one considered here, the separation between the regions for m_i and m_j is a non-trivial contour in $(y,\, \phi)$-space, but the divergencies of the soft function do not depend on the exact form of the contour. We therefore split the $(y,\, \phi)$-space into the two hemispheres $y > 0$ for m_i and $y < 0$ for m_j. To compensate for the difference between cutting the phase space along $y = 0$ compared to the original contour between m_i and m_j, we introduce the boundary contribution $S_{ij,\mathrm{bound}} + S_{ji,\mathrm{bound}}$. Adding up these contributions,

$$\begin{aligned}
S_{ij}^{(1)}(m = \{\mathcal{T}_1^{i,\alpha_i}, \mathcal{T}_1^{j,\alpha_j}, \mathcal{T}_1^{m,\alpha_m}\}, \mu) = {}& S_{ij,\mathrm{hemi}}^{(1)}(m_i = \mathcal{T}_1^{i,\alpha_i}, \mu) + S_{ij,\mathrm{bound}}^{(1)}(m_i = \mathcal{T}_1^{i,\alpha_i}\}, \mu) \\
&+ S_{ij,m}^{(1)}(m_m = \mathcal{T}_1^{m,\alpha_m}, \mu) - S_{ij,m}^{(1)}(m_i = \mathcal{T}_1^{i,\alpha_i}, \mu) \\
&+ (j \leftrightarrow i)\,.
\end{aligned} \tag{4.6}$$

As we will see, the hemisphere contributions contain all divergencies, whereas the boundary and non-hemisphere contributions are UV and IR finite. When there are additional jets, the hemisphere and boundary contributions are of course the same, but there will be additional non-hemisphere contributions.

We now boost such that the Wilson lines i and j become back-to-back, allowing us to use section 2.3 to perform the calculation. Using eqs. (2.10) and (2.11), this leads to the following expressions for the k_ℓ in the back-to-back frame

$$\begin{aligned}
k_i &= \frac{2\omega_i}{Q_i}\gamma^{-1} e^{-y\,(1-\alpha_i/2)} \left(a e^{-y} + b e^{y} + c\cos(\phi - \phi_0)\right)^{\alpha_i/2}\,, \\
k_j &= \frac{2\omega_j}{Q_j}\gamma^{-1} e^{y\,(1-\alpha_j/2)} \left(b e^{-y} + a e^{y} + c\cos(\phi - \phi_0)\right)^{\alpha_j/2}\,, \\
k_m &= \frac{2\omega_m}{Q_m}\left(\frac{1}{2}e^{y}(\tilde{n}_m^0 - \tilde{n}_m^3) + \frac{1}{2}e^{-y}(\tilde{n}_m^0 + \tilde{n}_m^3) - \tilde{n}_m^1\cos\phi - \tilde{n}_m^2 \sin\phi\right)^{1-\alpha_m/2} \\
&\qquad \times \left(\frac{1}{2}e^{y}(\tilde{\tilde{n}}_m^0 - \tilde{\tilde{n}}_m^3) + \frac{1}{2}e^{-y}(\tilde{\tilde{n}}_m^0 + \tilde{\tilde{n}}_m^3) - \tilde{\tilde{n}}_m^1\cos\phi - \tilde{\tilde{n}}_m^2 \sin\phi\right)^{\alpha_m/2}\,,
\end{aligned} \tag{4.7}$$

with

$$a = \gamma^2 - 1\,, \qquad b = \gamma^2\,, \qquad c = 2\gamma\sqrt{(\gamma^2-1)}\,. \tag{4.8}$$

Here we have explicitly chosen the z-axis through $\hat{z} = \frac{1}{2}\gamma(\hat{n}_i' - \hat{n}_j')$. The azimuthal angle ϕ_0 of the boost $-\vec{\beta}$ in eq. (2.9) plays no role in the rest of the calculation.

The measurement functions for the different jets are defined as

$$f_i^M = k_i\,, \qquad f_j^M = k_j\,, \qquad f_m^M = k_m\,. \tag{4.9}$$

Starting with $S_{ij,\mathrm{hemi}}^{(1)}$, we have

$$f_{\mathrm{hemi,i}}^R = \theta(y)\,. \tag{4.10}$$

The form of the measurement simplifies considerably in the limit of $y \to \infty$, in particular the dependence on the azimuthal angle vanishes

$$f_{\infty,i} = 2\frac{\omega_i}{Q_i}\gamma^{-1}b^{\alpha_i/2}e^{-(1-\alpha_i)y}\,. \tag{4.11}$$

The hemisphere contribution is now

$$\begin{aligned} S^{(1)}_{ij,\text{hemi}}(m_i = \mathcal{T}_1^{i,\alpha_i}, \mu) &= \frac{\alpha_s}{\pi}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j \int_{-\infty}^{\infty} \mathrm{d}y\, f^R_{\text{hemi,i}}\, f^{2\epsilon}_{\infty,i} \\ &\qquad\qquad \times \left[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m_i} + 2\epsilon\left(\ln^2\frac{\mu}{m_i} - \frac{\pi^2}{24}\right)\right] + I_{\text{hemi,i}} \\ &= \frac{\alpha_s}{2\pi}\frac{1}{(1-\alpha_i)}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j\left(\frac{1}{\epsilon^2} + \frac{2}{\epsilon}\ln\frac{B_i\mu}{m_i} + 2\ln^2\frac{B_i\mu}{m_i} - \frac{\pi^2}{12}\right) + I_{\text{hemi,i}}\,, \end{aligned} \tag{4.12}$$

where

$$B_i = 2\frac{\omega_i}{Q_i}\gamma^{-1}b^{\alpha_i/2}\,, \tag{4.13}$$

and the remaining finite integral is

$$I_{\text{hemi,i}} = \frac{\alpha_s}{\pi^2}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j \int_{-\infty}^{\infty}\mathrm{d}y \int_0^{2\pi}\mathrm{d}\phi\, f^R_{\text{hemi,i}}\,\ln\frac{f_i^M}{f_{\infty,i}}\,. \tag{4.14}$$

The second hemisphere contribution $S^{(1)}_{ji,\text{hemi}}(m_j = \mathcal{T}_1^{j,\alpha_j}, \mu)$, describing the region $y < 0$, is given by replacing $i \to j$ in the final line of eq. (4.12).

Next we calculate the boundary contribution, shown in the second and third column of figure 1. The integration over y and ϕ is finite and we can use eq. (2.7) to write

$$S^{(1)}_{ij,\text{bound}}(m_i,\mu) = \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j\int_{-\infty}^{\infty}\mathrm{d}y\int_0^{2\pi}\mathrm{d}\phi\, f^R_{ij,\text{bound}}\left(\frac{1}{\epsilon} + 2\ln\frac{\mu}{m_i} + 2\ln f_i^M\right), \tag{4.15}$$

with

$$f^R_{ij,\text{bound}} = \theta(-y)\theta(k_j - k_i) - \theta(y)\theta(k_i - k_j)\,, \tag{4.16}$$

The region for $S^{(1)}_{ji,\text{bound}}(m_j,\mu)$ is given by $f^R_{ji,\text{bound}} = -f^R_{ij,\text{bound}}$. Therefore, the area of the two contributions are equal but enter with different signs, such that the poles cancel in the combination. The total boundary contribution is thus

$$\begin{aligned} &S^{(1)}_{ij,\text{bound}}(m_i,\mu) + S^{(1)}_{ji,\text{bound}}(m_j,\mu) \\ &\qquad\qquad = \frac{\alpha_s}{2\pi^2}\,\mathbf{T}_i\!\cdot\!\mathbf{T}_j\int_{-\infty}^{\infty}\mathrm{d}y\int_0^{2\pi}\mathrm{d}\phi\, f^R_{ij,\text{bound}}\left(2\ln\frac{m_j}{m_i} + 2\ln\frac{f_i^M}{f_j^M}\right). \end{aligned} \tag{4.17}$$

The measurement functions for the non-hemisphere contributions, $S^{(1)}_{ij,m}(m_m,\mu)$ and $S^{(1)}_{ij,m}(m_i,\mu)$, shown in the last two columns of figure 1, are defined on the same region,

$$f^R_{ij,m} = \theta(k_j - k_i)\theta(k_i - k_m)\,. \tag{4.18}$$

Application of eq. (2.7) gives

$$S^{(1)}_{ij,m}(m_m = \mathcal{T}_1^{m,\alpha_m}, \mu) = \frac{\alpha_s}{2\pi^2}\, \mathbf{T}_i\!\cdot\!\mathbf{T}_j \int_{-\infty}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\, f^R_{ij,m} \biggl(\frac{1}{\epsilon} + 2\ln\frac{\mu}{m_m} + 2\ln f^M_m\biggr)\,, \tag{4.19}$$

and similarly for $S^{(1)}_{ij,m}(m_i,\mu)$ with the replacement $m \to i$. Subtracting the non-hemisphere i contribution from the non-hemisphere m contribution, the $1/\epsilon$ poles cancel and for the full non-hemisphere contribution we find

$$S^{(1)}_{ij,m}(m_m) - S^{(1)}_{ij,m}(m_i) = \frac{\alpha_s}{\pi}\, \mathbf{T}_i\!\cdot\!\mathbf{T}_j \biggl(\tilde{I}_0 \ln\frac{m_i}{m_m} + \tilde{I}_1\biggr)\,, \tag{4.20}$$

with

$$\begin{aligned}
\tilde{I}_0 &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}y \int_0^{2\pi}\mathrm{d}\phi\, f^R_m\,, \\
\tilde{I}_1 &= \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}y \int_0^{2\pi}\mathrm{d}\phi\, f^R_m \ln\frac{f^M_m}{f^M_i}\,.
\end{aligned} \tag{4.21}$$

Note that $\tilde{I}_0$ is simply the area of region m with $k_i < k_j$. The result for the second non-hemisphere contribution $S^{(1)}_{ji,m}(m_m) - S^{(1)}_{ji,m}(m_j)$ is obtained by the replacement $i \leftrightarrow j$ in eq. (4.20) and eq. (4.21). We show in appendix C how for $\alpha_\ell = 0$ these expressions reduce to those in ref. [16].

5 Multi-differential measurements

We present results for the soft function and the collinear-soft function for double differential measurements. In section 5.1 we consider the simultaneous measurement of (beam) thrust and transverse momentum, and in section 5.2 the measurement of two angularities.

5.1 Thrust and transverse momentum

Following ref. [12], we combine the (beam) thrust and transverse momentum measurements of sections 3.1 and 3.3. The soft radiation in the regime $p_T \sim \tau Q$ is described by a double differential soft function with

$$\vec{f}(y,\phi) = (e^{-|y|}, 1)\,. \tag{5.1}$$

When $y \to \pm\infty$ the transverse momentum measurement dominates,

$$f_\infty(y,\phi) = 1\,, \tag{5.2}$$

leading to [12]

$$\begin{aligned}
&S^{(1)}(\vec{m} = (Q\tau, p_T),\mu) \\
&\quad = -\frac{\alpha_s C_F}{\pi}\int_{-\infty}^{\infty}\mathrm{d}y\, e^{-\eta|y|}\biggl[\frac{1}{\epsilon} + 2\ln\frac{\mu}{\min(m_1 e^{|y|}, m_2)} + 2\epsilon\biggl(\ln^2\frac{\mu}{m_2} - \frac{\pi^2}{24}\biggr)\biggr] \\
&\qquad\qquad \times \biggl[1 + \eta\biggl(-\frac{1}{2\epsilon} + \ln\frac{\nu}{m_2}\biggr)\biggr]
\end{aligned}$$

Mode:	Scaling $(-,+,\perp)$
n-collinear	$Q\big(1, (\frac{p_T}{Q})^2, \frac{p_T}{Q}\big)$
$\bar{n}$-collinear	$Q\big((\frac{p_T}{Q})^2, 1, \frac{p_T}{Q}\big)$
n-collinear-soft	$\frac{p_T^2}{\tau Q}\big(1, (\frac{\tau Q}{p_T})^2, \frac{\tau Q}{p_T}\big)$
$\bar{n}$-collinear-soft	$\frac{p_T^2}{\tau Q}\big((\frac{\tau Q}{p_T})^2, 1, \frac{\tau Q}{p_T}\big)$
soft	$\tau Q(1,1,1)$

Table 1. SCET$_+$ modes describing the measurement of τ and p_T in the region $\tau^2 Q^2 \ll p_T^2 \ll \tau Q^2$.

$$
\begin{aligned}
&= S^{(1)}(m = p_T, \mu) - \theta(m_2 - m_1)\, \frac{2\alpha_s C_F}{\pi} \int_0^{\ln(m_2/m_1)} \mathrm{d}y\, 2\ln \frac{m_2}{m_1 e^y} \\
&= S^{(1)}(m = p_T, \mu) - \theta(m_2 - m_1)\, \frac{2\alpha_s C_F}{\pi}\, \ln^2 \frac{m_2}{m_1}\,. \qquad (5.3)
\end{aligned}
$$

In the second step we first assumed that $\min(m_1 e^{|y|}, m_2) = m_2$, leading to the transverse momentum soft function, and corrected for this through the second term.

When $\tau^2 Q^2 \ll p_T^2 \ll \tau Q^2$, the cross section is described by SCET$_+$, for which the modes are listed in table 1. The collinear-soft modes are less collimated than the collinear modes and more energetic than the soft mode. They are described by collinear-soft functions in the factorization formula. These are matrix elements of (collinear-soft) Wilson lines and therefore give rise to the same amplitude as in eq. (2.5). However, due to the collinear nature of this radiation, we use the measurement function for the hemisphere it goes into.[4] For collinear-soft radiation going into the $y < 0$ hemisphere,

$$
\vec{f}(y,\phi) = (e^y, 1)\,, \quad f_\infty(y,\phi) = \theta(-y) + \theta(y) e^y\,. \qquad (5.4)
$$

We thus find

$$
\begin{aligned}
\mathscr{S}^{(1)}(\vec{m} = (p^-, p_T), \mu) &= \frac{1}{2} S^{(1)}(\vec{m} = (Q\tau, p_T), \mu) \\
&\quad - \frac{\alpha_s C_F}{\pi} \int_0^\infty \mathrm{d}y\, e^{2\epsilon y} \bigg[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m_1} + 2\epsilon\Big(\ln^2\frac{\mu}{m_1} - \frac{\pi^2}{24}\Big)\bigg] \\
&\quad - \theta(m_1 - m_2) \frac{\alpha_s C_F}{\pi} \int_0^{\ln(m_1/m_2)} \mathrm{d}y\, 2\ln\frac{m_1}{m_2 e^y} \\
&= \frac{1}{2} S^{(1)}(\vec{m} = (Q\tau, p_T), \mu) - \frac{\alpha_s C_F}{\pi} \bigg[-\frac{1}{2\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu}{m_1} \\
&\qquad - \ln^2\frac{\mu}{m_1} + \frac{\pi^2}{24} - \theta(m_1 - m_2)\ln^2\frac{m_1}{m_2}\bigg]\,, \qquad (5.5)
\end{aligned}
$$

[4] In the calculation one also integrates over the other hemisphere. This is corrected for through zero-bin subtractions [19] that remove the overlap with soft radiation, but vanish in pure dimensional regularization.

exploiting that the measurement is identical to the soft function in eq. (5.3) for $y < 0$. Our result agrees with ref. [12].[5] Note that the collinear-soft function for the hemisphere $y > 0$ has an identical expression.

5.2 Two angularities

We now extend section 3.2 to consider the measurement of two angularities τ_a and τ_b as in refs. [11, 12, 46]. Taking $2 > b > a$ (and $a, b \neq 1$) implies $\tau_b > \tau_a$ and

$$\vec{f}(y,\phi) = (e^{-|y|(1-a)}, e^{-|y|(1-b)})\,, \quad f_\infty(y,\phi) = e^{-|y|(1-b)}\,. \tag{5.6}$$

Writing $m_a = Q\tau^a$ and $m_b = Q\tau^b$, this leads to

$$\begin{aligned} S^{(1)}(\vec{m} = (m_a, m_b), \mu) &= S^{(1)}(m_b, \mu) \\ &\quad - \theta(m_b - m_a)\,\frac{2\alpha_s C_F}{\pi} \int_0^{\frac{1}{(b-a)}\ln\frac{m_b}{m_a}} \mathrm{d}y\, 2\Big(\ln\frac{m_b}{m_a} + (a-b)y\Big) \\ &= S^{(1)}(m_b, \mu) - \theta(m_b - m_a)\,\frac{2\alpha_s C_F}{\pi}\,\frac{1}{b-a}\ln^2\frac{m_b}{m_a}\,. \end{aligned} \tag{5.7}$$

This agrees with the expression in ref. [11], when converting their angular exponents α, β to our (current) conventions, $\alpha = 2 - a$, $\beta = 2 - b$, and taking into account that they consider only one jet which halves the result.

The corresponding collinear-soft function has again the same amplitude but a modified measurement. For collinear-soft radiation going into the $y < 0$ hemisphere,

$$\vec{f}(y,\phi) = (e^{y(1-a)}, e^{y(1-b)})\,, \quad f_\infty(y,\phi) = \theta(-y)e^{y(1-b)} + \theta(y)e^{y(1-a)}, \tag{5.8}$$

which is identical to eq. (5.6) for $y < 0$ but not for $y > 0$. This leads to

$$\begin{aligned} \mathscr{S}^{(1)}(\vec{m} = (m_a, m_b), \mu) &= \frac{1}{2}S^{(1)}(\vec{m} = (m_a, m_b), \mu) \\ &\quad - \frac{\alpha_s C_F}{\pi}\int_0^\infty \mathrm{d}y\, e^{2\epsilon y(1-a)}\Big[\frac{1}{\epsilon} + 2\ln\frac{\mu}{m_a} + 2\epsilon\Big(\ln^2\frac{\mu}{m_a} - \frac{\pi^2}{24}\Big)\Big] \\ &\quad - \theta(m_a - m_b)\frac{\alpha_s C_F}{\pi}\int_0^{\frac{1}{(b-a)}\ln\frac{m_a}{m_b}} \mathrm{d}y\, 2\Big(\ln\frac{m_a}{m_b} + (a-b)y\Big) \\ &= \frac{1}{2}S^{(1)}(\vec{m} = (m_a, m_b), \mu) - \frac{1}{2}S^{(1)}(m_a, \mu) \\ &\quad - \theta(m_a - m_b)\,\frac{\alpha_s C_F}{\pi}\,\frac{1}{b-a}\ln^2\frac{m_b}{m_a} \\ &= \frac{1}{2}S^{(1)}(m_b, \mu) - \frac{1}{2}S^{(1)}(m_a, \mu) - \frac{\alpha_s C_F}{\pi}\,\frac{1}{b-a}\ln^2\frac{m_b}{m_a}\,. \end{aligned} \tag{5.9}$$

This is consistent with matching the SCET_+ factorization theorem in the bulk with the $\mathrm{SCET_I}$ factorization on the boundary, discussed in section 4 of ref. [12], since the last term in the second-to-last line of eq. (5.9) drops out due to $m_b > m_a$.

[5] In the second-to-last expression in eq. (3.17) of ref. [12], the $\delta(k^+ - |\vec{k}_\perp|)$ term is equal to zero. Due to a typo, the π^2 term is a factor 2 too big there. However, the final result is correct.

It may not be a priori obvious that the collinear-soft function satisfies the kinematic constraint $m_a < m_b$. However, inserting the collinear-soft scale

$$\mu_{\mathscr{S}} = \left(m_a^{b-1} m_b^{1-a}\right)^{1/(b-a)} \tag{5.10}$$

in the finite terms gives,

$$\begin{aligned}\mathscr{S}^{(1)}(\vec{m} = (m_a, m_b), \mu_{\mathscr{S}}) &= \frac{\alpha_s C_F}{\pi}\left(\frac{1}{b-1}\ln^2\frac{\mu}{m_b} + \frac{1}{1-a}\ln^2\frac{\mu}{m_a} - \frac{1}{b-a}\ln^2\frac{m_b}{m_a}\right)\\ &= 0\,.\end{aligned} \tag{5.11}$$

6 Conclusions

We have presented a convenient method for calculating the effect of soft QCD radiation at one-loop order, for generic N-jet processes and measurements. This exploits that soft emissions are uniform in rapidity and azimuthal angle. Through an isolation of the divergent parts, we are able to perform a partial expansion in the regulators already before the integration, simplifying the calculation of the poles and directly leading to an integral for the finite terms. By working with cumulative distributions, complications from plus distributions in intermediate expressions are avoided. As a demonstration of the ease of the calculational framework, soft functions for a range of processes and measurements are computed. We obtain original results for the soft function for N-jettiness with generic jet angularities, which required an extension of the hemisphere decomposition [16] to make the complicated boundaries between regions tractable, see also ref. [45]. We also determine the collinear-soft function for the measurement of two angularities for the first time. Many of the techniques developed in this paper can also be extended to higher order in perturbation theory. There are new contributions involving gluon exchanges between multiple Wilson lines, and the divergence structure becomes much more challenging. The NNLO calculation of the thrust-like N-jettiness soft function was outlined in ref. [47], which seems amenable to generic $\mathrm{SCET_I}$ observables. An automated approach to the two-loop soft function for $\mathrm{SCET_I}$ observables for dijets is underway [48]. Both of these NNLO calculations rely on sector decomposition [49, 50] to handle the divergences.

In summary, our method reduces the work required for calculating one-loop soft functions, and can for example be applied to calculate the soft functions for the recently introduced XCone class of jet algorithms [51].

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A Becher-Bell rapidity regulator

One may also use the regulator in [30] to regulate rapidity divergences. This amounts to the substitution

$$\int \mathrm{d}^d k\, \delta(k^2)\, \theta(k^0) \to \int \mathrm{d}^d k\, \delta(k^2)\, \theta(k^0) \left(\frac{\nu^-}{k^-}\right)^\alpha \tag{A.1}$$

in the integration over the soft radiation. With $k^- = k_T e^y$ this leads to the replacements

$$|2\sinh y|^\eta \to e^{\alpha y}\,, \quad \eta \to \alpha\,, \quad \nu \to \nu^- \tag{A.2}$$

in eq. (2.4). With this regulator, eq. (2.5) gets modified to

$$\begin{aligned} S_{12}^{(1)}(m,\mu) = \frac{\alpha_s}{2\pi^2}\, \mathbf{T}_1\!\cdot\!\mathbf{T}_2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^{2\pi} \mathrm{d}\phi\, \theta[f(y,\phi)]\, f_\infty(y,\phi)^{2\epsilon} e^{-\alpha y} \\ \times \left[\frac{1}{\epsilon} + 2\ln\frac{\mu\, f(y,\phi)}{m\, f_\infty(y,\phi)} + 2\epsilon\left(\ln^2\frac{\mu}{m} - \frac{\pi^2}{24}\right)\right]\left[1 + \alpha\left(-\frac{1}{2\epsilon} + \ln\frac{\nu^-}{m}\right)\right]. \end{aligned} \tag{A.3}$$

Note that in order to regulate the integrals, α has to take opposite signs for $y \to \infty$ and $y \to -\infty$. This is similar to the opposite sign of ϵ for UV and IR divergencies in dimensional regularization.

B Jet function for transverse thrust

At one-loop order the jet function contains two emissions. Their contribution to transverse thrust is given by

$$\tau_\perp = \frac{1}{Q\sin^2\theta} \sum_i \frac{k_{i\perp T}^2}{2E_i} = \frac{1}{Q\sin^2\theta}\left(\frac{k_T^2\sin^2\phi}{zQ} + \frac{k_T^2\sin^2\phi}{(1-z)Q}\right) = \frac{s\sin^2\phi}{Q^2\sin^2\theta} \tag{B.1}$$

Here $k_{i\perp T}^2$ is the momentum component perpendicular to the beam *and* thrust axis, k_T is the momentum transverse to the thrust axis (equal and opposite for the two emissions), ϕ the azimuthal angle around the thrust axis and s the invariant mass. Calculating the quark jet function in the approach of ref. [52] with no ϵ-dependence in the ϕ integral,

$$\begin{aligned} J_q^{(1)}(\tau_\perp) &= \int_0^\infty \mathrm{d}s \int_0^1 \mathrm{d}z \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \frac{[z(1-z)s]^{-\epsilon}}{(4\pi)^{2-\epsilon}\Gamma(1-\epsilon)} \frac{2g^2\, C_F}{s}\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] \\ &\quad \times \delta\left(\tau_\perp - \frac{s\sin^2\phi}{Q^2\sin^2\theta}\right) \\ &= -\frac{\alpha_s C_F}{\pi}\left(\frac{\mu^2 e^{\gamma_E}}{Q^2\sin^2\theta}\right)^\epsilon \frac{(1-\epsilon/4)\Gamma(1/2+\epsilon)\,\Gamma(2-\epsilon)}{\sqrt{\pi}\,\epsilon\,\Gamma(1+\epsilon)\,\Gamma(2-2\epsilon)} \frac{1}{\tau_\perp^{1+\epsilon}}\,. \end{aligned} \tag{B.2}$$

Expanding

$$\frac{1}{\tau_\perp^{1+\epsilon}} = -\frac{1}{\epsilon}\,\delta(\tau_\perp) + \left(\frac{1}{\tau_\perp}\right)_+ - \epsilon\left(\frac{\ln\tau_\perp}{\tau_\perp}\right)_+ + \mathcal{O}(\epsilon^2)\,, \tag{B.3}$$

the finite terms in the one-loop jet function differ from the result in ref. [10] by

$$J_q(\tau_\perp) = J_q^{[10]}(\tau_\perp) + \frac{\alpha_s C_F}{\pi}\,\frac{\pi^2}{3}\,\delta(\tau_\perp) + \mathcal{O}(\alpha_s^2,\epsilon)\,. \tag{B.4}$$

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C Thrust-like N-jettiness

When all $\alpha_\ell = 0$, the expressions for the soft function contributions given in section 4 simplify and the results of ref. [16] are reproduced, as we will show now. Starting with the hemisphere and the boundary contributions, we solve remaining integrals analytically and the sum of eq. (4.12) and eq. (4.15) reduces to the expression [16]

$$S^{(1)}_{ij,\mathrm{hemi}}(m_i = \mathcal{T}_1^i, \mu) = \frac{\alpha_s}{2\pi}\,\mathbf{T}_i\cdot\mathbf{T}_j\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\hat{s}_{ij}\mu^2}{m_i^2} + \frac{1}{2}\ln^2\frac{\hat{s}_{ij}\mu^2}{m_i^2} - \frac{\pi^2}{12}\right], \tag{C.1}$$

with

$$\hat{s}_{ij} = \frac{2q_i'\cdot q_j'}{Q_iQ_j} = \frac{4\omega_i\omega_j}{Q_iQ_j\gamma^2}\,. \tag{C.2}$$

Note that for the thrust-like N-jettiness the measurement regions are ϕ independent and the same result can be obtained without the trick of simplifying the hemisphere contributions by splitting off the boundary contributions.

For the non-hemisphere contributions, the integrals in eq. (4.21) are simplified by performing the substitutions

$$\tilde{y} = \sqrt{\frac{\tilde{n}_m^0 - \tilde{n}_m^3}{\tilde{n}_m^0 + \tilde{n}_m^3}}\,e^y\,, \qquad \tilde{\phi} = \phi - \arccos\sqrt{\frac{(\tilde{n}_m^1)^2}{(\tilde{n}_m^1)^2 + (\tilde{n}_m^2)^2}}\,, \tag{C.3}$$

which leads to

$$\begin{aligned}
\tilde{I}_0(\alpha_\ell = 0) &= \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\tilde{y}}{\tilde{y}}\int_{-\pi}^{\pi}\mathrm{d}\tilde{\phi}\,\theta\!\left(\tilde{y}^2 - \frac{\hat{s}_{im}}{\hat{s}_{jm}}\right)\theta\!\left(\frac{\hat{s}_{ij}}{\hat{s}_{jm}} - 1 - \tilde{y}^2 + 2\tilde{y}\cos\tilde{\phi}\right),\\
\tilde{I}_1(\alpha_\ell = 0) &= \tilde{I}_0(\alpha_l = 0)\ln\left(\frac{\hat{s}_{jm}}{\hat{s}_{ij}}\right) + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}\tilde{y}}{\tilde{y}}\int_{-\pi}^{\pi}\mathrm{d}\tilde{\phi}\\
&\quad\times\theta\!\left(\tilde{y}^2 - \frac{\hat{s}_{im}}{\hat{s}_{jm}}\right)\theta\!\left(\frac{\hat{s}_{ij}}{\hat{s}_{jm}} - 1 - \tilde{y}^2 + 2\tilde{y}\cos\tilde{\phi}\right)\ln(\tilde{y}^2 + 1 - 2\tilde{y}\cos\tilde{\phi})\,,
\end{aligned} \tag{C.4}$$

with

$$\hat{s}_{im} = \frac{2q_i'\cdot q_m'}{Q_iQ_m} = \frac{2\omega_i\omega_m}{Q_iQ_m\gamma}(\tilde{n}_m^0 - \tilde{n}_m^3)\,, \qquad \hat{s}_{jm} = \frac{2q_j'\cdot q_m'}{Q_jQ_m} = \frac{2\omega_j\omega_m}{Q_jQ_m\gamma}(\tilde{n}_m^0 + \tilde{n}_m^3)\,. \tag{C.5}$$

This is in agreement with the non-hemisphere expression of ref. [16]. There the remaining integrals have been further simplified to one-dimensional integrals.

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